The ISS Philosophy as a Unifying Framework for Stability-Like Behavior

Eduardo D. Sontag

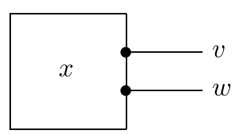
Department of Mathematics
Rutgers University
New Brunswick, NJ 08903
sontag@control.rutgers.edu
http://www.math.rutgers.edu/~sontag

Abstract. The input to state stability (ISS) paradigm is motivated as a generalization of classical linear systems concepts under coordinate changes. A summary is provided of the main theoretical results concerning ISS and related notions of input/output stability and detectability. A bibliography is also included, listing extensions, applications, and other current work.

1 Introduction

In this talk, I discuss the "input to state stability" *way of thinking* about nonlinear stability questions. I will be very informal — the expository paper [68] should be consulted for more details and precise statements of results as of 1998, and several more recent citations are provided later.

Consider the general "port" picture

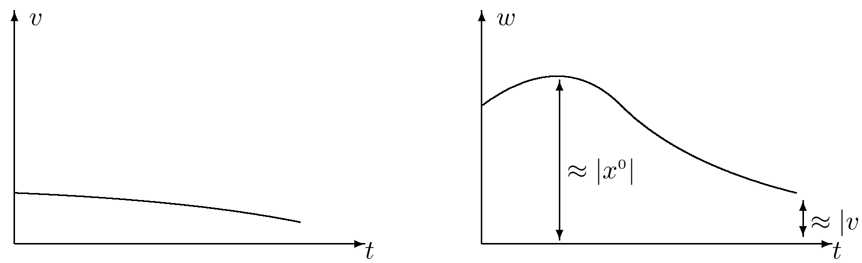


here v and w are external signals, x the internal state. We study *conditional (asymptotic) stability* of v, w. There are two desirable, and complementary, features of stability:

- *asymptotic:* "v small $\Rightarrow$ w small" — where "small" may be interpreted as "$\to 0$ when $t \to +\infty$", "bounded", or via an $\varepsilon - \delta$ definition.
- *transient:* "overshoot depends on initial state" — with fading effect of $x(0) = x^0$.

Our definitions attempt to capture these two aspects.

Pictorially:



The "magnitude" of a signal might be e.g.:

- norm: $|w(t)|$
- error: $|w(t) - w_{\text{desired}}(t)|$
- distance to a set $\mathcal{A}$: $|w(t)|_{\mathcal{A}} = \operatorname{dist}(w(t), \mathcal{A})$ — e.g. $\mathcal{A}$ = periodic orbit, ask $w(t) \to \mathcal{A}$ as $t \to \infty$

but in this presentation, we restrict ourselves to norms. (The literature usually deals with more general cases. For instance, results on internal stability are often given for $|w(t)|_{\mathcal{A}}$. This generality allows considering issues such as full-state observer design, in which the relevant concepts concern stability with respect to the "diagonal" set $\mathcal{A} = \{(x, x)\}$ where the states of the plant and observer coincide.)

Specifically, let us consider i/o systems

$$u(\cdot) \;\to \boxed{\;x(\cdot)\;} \to\; y(\cdot)$$

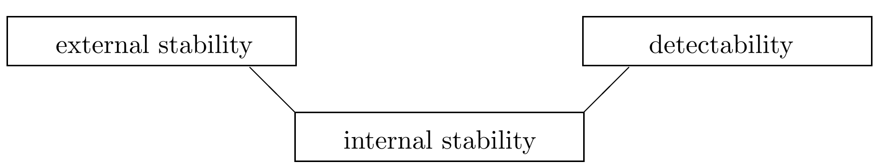
and various choices of v and w. Three central theoretical concepts for linear systems

$$\dot{x} = Ax + Bu\,, \quad y = Cx$$

(to be generalized) are as follows:

1. Internal Stability (*input to state*): $v = u$, $w = x$.
2. External Stability (*input to output*): $v = u$, $w = y$.
3. Detectability (*input* **and** *output* **to** *state*): $v = (u, y)$, $w = x$.

We will refer to them as the *fundamental triad*.



Internal Stability means that A is a Hurwitz matrix, i.e.: $x(t) \to 0$ for all solutions of $\dot{x} = Ax$, or equivalently, that $x(t) \to 0$ whenever $u(t) \to 0$, and moreover one has the explicit estimate

$$|x(t)| \;\leq\; \beta(t)|x^0| + \gamma \left\|u\right\|_\infty$$

where

$$\beta(t) = \left\|e^{tA}\right\| \to 0 \;\text{ and }\; \gamma = \|B\| \int_0^\infty \left\|e^{sA}\right\| ds$$

and $\left\|u\right\|_\infty$ = (essential) sup norm of u restricted to $[0,t]$. For t large, $x(t)$ is bounded by $\gamma \left\|u\right\|_\infty$, independently of initial conditions; for small t, the effect of initial states may dominate. Note the superposition of transient and asymptotic effects. Internal stability will be generalized to "ISS" later, with the linear functions of $|x^0|$ and $\left\|u\right\|_\infty$ replaced by nonlinear ones.

External Stability means that the transfer function is stable or, in terms of a state-space realization, that an estimate as follows holds:

$$|y(t)| \;\leq\; \beta(t)|x^0| + \gamma \left\|u\right\|_\infty$$

where γ is a constant and β converges to zero (β may be obtained from the restriction of A to a minimal subsystem). Note that even though we only require that y, not x, be "small" (relative to $\left\|u\right\|_\infty$), the initial internal states still affect the estimate in a "fading memory" manner, via the β term. (For example, in PID control, when considering the combination of plant, exosystem and controller, the overshoot of the regulated variable will be determined by the magnitude of the constant disturbance, and the initial state of the integrator.) External stability will generalize to "IOS".

(Zero-)Detectability means that the unobservable part is stable i.e.,

$$y(t) = Cx(t) \equiv 0 \;\&\; u(t) \equiv 0 \;\Rightarrow\; x(t) \to 0 \;\text{ as } t \to \infty$$

or equivalently:

$$u(t) \to 0 \;\&\; y(t) \to 0 \;\Rightarrow\; x(t) \to 0$$

and can be also expressed by means of an estimate of the following form:

$$|x(t)| \;\leq\; \beta(t)|x^0| + \gamma_1 \left\|u\right\|_\infty + \gamma_2 \left\|y\right\|_\infty$$

where γ_i's are constants and β converges to zero (now β is obtained from a suitable matrix $A - LC$, where L is an observer gain). Zero-detectability's nonlinear version will be "IOSS".

The components of this triad are interrelated:

$$\textbf{external stability} \;\;\&\;\; \textbf{detectability} \;\;\Longleftrightarrow\;\; \textbf{internal stability}$$

— this is a routine exercise in linear systems theory and obvious intuitively:

- If internally stable, then $x \to 0$ for all $u \to 0$, so in particular this happens when $Cx(t) \to 0$ (detectability), and it always holds that $y(t) = Cx(t) \to 0$ (i/o stability).
- Conversely, if $u \to 0$ then $y \to 0$, (by external stability) and this then implies $x \to 0$ (by detectability).

Let us turn to the nonlinear generalizations. These generalizations will be so that, in particular, the above equivalence still holds true.

2 Input-to-State Stability

We consider systems of the form

$$\dot{x} \;=\; f(x,u)\,, \quad y \;=\; h(x)$$

evolving in finite-dimensional spaces $\mathbb{R}^n$, and we suppose that inputs u take values in $\mathbb{R}^m$ and outputs y are $\mathbb{R}^p$-valued. An *input* is a measurable locally essentially bounded $u(\cdot) : [0,\infty) \to \mathbb{R}^m$. We employ the notation $|x|$ for Euclidean norms, and use $\|u\|$, or $\|u\|_\infty$ for emphasis, to indicate the essential supremum of a function $u(\cdot)$. The map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz and satisfies $f(0,0) = 0$. The map $h : \mathbb{R}^n \to \mathbb{R}^p$ is locally Lipschitz and satisfies $h(0) = 0$.

The internal stability property for linear systems amounts to the "$L^\infty \to L^\infty$ finite-gain condition" that

$$|x(t)| \;\leq\; c|x^0|e^{-\lambda t} \,+\, c \sup_{s\in[0,t]} |u(s)|$$

holds for all solutions (assumed defined for all $t > 0$), where c and $\lambda > 0$ and appropriate constants. What is a reasonable nonlinear version of this?

Two central characteristic of the ISS philosophy are: (1) using nonlinear gains rather than linear estimates, (2) not asking about exact values of gains but instead asking *qualitative* questions of *existence*: a "topological" vs. a "metric" point of view. (The linear analogy would be to ask "is the gain $< \infty$?" "is an operator bounded?")

Our general guiding principle may be formulated thus:

notions of stability should be invariant
under (nonlinear) changes of variables.

By a *change of variables* in $\mathbb{R}^\ell$, let us mean here any transformation $z = T(x)$ with $T(0) = 0$, where $T : \mathbb{R}^\ell \to \mathbb{R}^\ell$ is a homeomorphism whose restriction $T|_{\mathbb{R}^\ell \setminus \{0\}}$ is a diffeomorphism. (We allow less differentiability at the origin in order to state elegantly a certain converse result later.)

Let us see where this principle leads us, starting from the "$L^\infty \to L^\infty$ finite-gain condition"

$$|x(t)| \;\leq\; c|x^0|e^{-\lambda t} \;+\; c \sup_{s\in[0,t]} |u(s)|$$

and taking both state and input coordinates changes $x = T(z)$, $u = S(v)$. For any input u and initial state x^0, and corresponding trajectory $x(t) = x(t, x^0, u)$, we let $x(t) = T(z(t))$, $u(t) = S(v(t))$, $z^0 = z(0) = T^{-1}(x^0)$.

For suitable functions $\underline{\alpha}, \overline{\alpha}, \overline{\gamma} \in \mathcal{K}_\infty$, we have:

$$\underline{\alpha}(|z|) \;\leq\; |T(z)| \;\leq\; \overline{\alpha}(|z|) \;\; \forall\, z \in \mathbb{R}^n$$

$$|S(v)| \;\leq\; \overline{\gamma}(|v|) \;\; \forall\, v \in \mathbb{R}^m\,.$$

The condition $|x(t)| \leq c|x^0|e^{-\lambda t} + c\sup_{s\in[0,t]} |u(s)|$ becomes, in terms of z, v:

$$\underline{\alpha}(|z(t)|) \;\leq\; c\,e^{-\lambda t}\overline{\alpha}(|z^0|) \;+\; c \sup_{s\in[0,t]} \overline{\gamma}(|v(s)|) \quad \forall\, t \geq 0\,.$$

Using again "x" and "u" and letting $\beta(s,t) := ce^{-\lambda t}\overline{\alpha}(s)$ and $\gamma(s) := c\overline{\gamma}(s)$, we arrive to this estimate, with $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$:

$$\underline{\alpha}\left(|x(t)|\right) \;\leq\; \beta(|x^0|, t) \;+\; \gamma\left(\|u\|_\infty\right)\,.$$

(For any $\mathcal{KL}$ function β, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ with ([67])

$$\beta(r,t) \;\leq\; \alpha_2\left(\alpha_1(r)e^{-t}\right) \;\; \forall\, s, t$$

so the special form of β adds no extra information.) Equivalently, one may write (for different β, γ)

$$|x(t)| \;\leq\; \beta(|x^0|, t) \;+\; \gamma\left(\|u\|_\infty\right)$$

or one may use "max" instead of "+" in the bound.

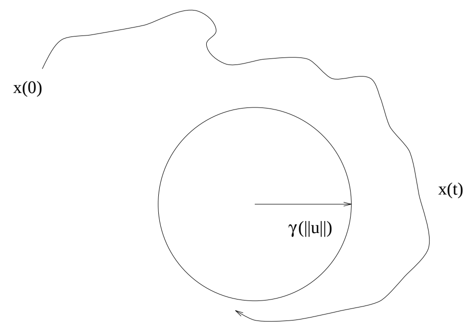
A system is *input to state stable (ISS)* if such an estimate holds, for some $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$. More precisely, for each x^0, u, the solution $x(t) = x(t, x^0, u)$ is defined for all $t \geq 0$, and the estimate holds.

2.1 Asymptotic Gain Characterization

For $u \equiv 0$, the estimate reduces to $|x(t)| \leq \beta(|x^0|, t)$, so ISS implies that the unforced system $\dot{x} = f(x, 0)$ is (asymptotically) stable (with respect to $x = 0$).

An ISS system has a well-defined *asymptotic gain*: there is some $\gamma \in \mathcal{K}_\infty$ so that, for all x^0 and u:

$$\overline{\lim_{t\to+\infty}}\, |x(t, x^0, u)| \;\leq\; \gamma\left(\|u\|_\infty\right) .$$



A far less obvious converse holds:

Theorem. *("Superposition principle for ISS") A system is ISS if and only if it admits an asymptotic gain and the unforced system is stable.*

This result is nontrivial, and constitutes the main contribution of the paper [73], which establishes as well many other fundamental characterizations of the ISS property. The proof hinges upon a relaxation theorem for differential inclusions, shown in that paper, which relates global asymptotic stability of an inclusion $\dot{x} \in F(x)$ to global asymptotic stability of its convexification.

2.2 Dissipation Characterization of ISS

A smooth, proper, and positive definite $V : \mathbb{R}^n \to \mathbb{R}$ is an *ISS-Lyapunov function* for $\dot{x} = f(x, u)$ if, for some $\gamma, \alpha \in \mathcal{K}_\infty$,

$$\dot{V}(x, u) = \nabla V(x)\, f(x, u) \;\leq\; -\alpha(|x|) + \gamma(|u|) \quad \forall\, x, u$$

i.e., one has the dissipation inequality

$$V(x(t_2)) - V(x(t_1)) \;\leq\; \int_{t_1}^{t_2} w(u(s), x(s))\, ds$$

along all trajectories of the system, with "supply" function $w(u,x) = \gamma(|u|) - \alpha(|x|)$.

The following is a fundamental result in ISS theory:

Theorem. [69] *A system is ISS if and only if it admits an ISS-Lyapunov function.*

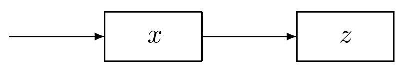
(Sufficiency is easy: a differential inequality for V provides an estimate on $V(x(t))$, and hence on $|x(t)|$. Necessity follows by applying a converse Lyapunov theorem for uniform GAS ([45]) over all $\|d\|_\infty \leq 1$, to a system of the form $\dot{x} = g(x,d) = f(x, d\rho(|x|))$, for an appropriate "robustness margin" $\rho \in \mathcal{K}_\infty$. This is in effect a smooth converse Lyapunov theorem for locally Lipschitz differential inclusions.)

2.3 ISS is Natural for Series Connections

Consider a cascade connection of ISS systems

$$\begin{aligned} \dot{z} &= f(z,x) \\ \dot{x} &= g(x,u) \end{aligned}$$

(the z system is ISS with x as input).



Pick *matching* (cf. [83]) ISS-Lyapunov functions for each subsystem:

$$\begin{aligned} &\dot{V}_1(z,x) \leq \theta(|x|) - \alpha(|z|) \\ &\dot{V}_2(x,u) \leq \tilde{\theta}(|u|) - 2\theta(|x|)\,. \end{aligned}$$

Then, $W(x,z) := V_1(z) + V_2(x)$ is an ISS-Lyapunov function:

$$\dot{W}(x,z) \;\leq\; \tilde{\theta}(|u|) - \theta(|x|) - \alpha(|z|)$$

and so a cascade of ISS systems is ISS.

2.4 Generalization to Small Gains

In particular, when $u = 0$, one obtains that a cascade of a GAS and an ISS system is again GAS. More generally, one may allow inputs u fed-back with "small gain": if $u = k(z)$ is so that $|k(z)| \leq \tilde{\theta}^{-1}((1-\varepsilon)\alpha(|z|))$, i.e.

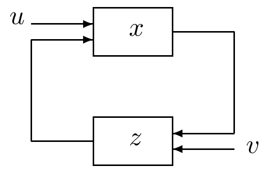
$$\tilde{\theta}(|u|) \leq (1-\varepsilon)\alpha(|z|)$$

then

$$\dot{W}(x,z) \leq -\theta(|x|) - \varepsilon\alpha(|z|)$$

and the closed-loop system is still GAS.

Even more generally, under suitable conditions on gains (Small-Gain Theorem [27] of Jiang, Praly, and Teel) the closed loop system obtained from an interconnection of two ISS systems $\dot{x} = f(x, z, u)$ and $\dot{z} = g(z, x, v)$, is itself ISS with respect to (u, v).



2.5 Series Connections: An Example

As a simple illustration of the cascade technique, consider the angular momentum stabilization of a rigid body controlled by two torques acting along principal axes (for instance, a satellite controlled by two opposing jet pairs). If $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity of a body-attached frame with respect to inertial coordinates, and $I = \mathrm{diag}(I_1, I_2, I_3)$ are the principal moments of inertia, we obtain the equations:

$$I\dot{\omega} \;=\; \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} I\omega + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} v\,.$$

We assume $I_2 \neq I_3$; then, introducing new state and input coordinates via $(I_2 - I_3)x_1 = I_1\omega_1$, $x_2 = \omega_2$, $x_3 = \omega_3$, $I_2 u_1 = (I_3 - I_1)\omega_1\omega_3 + v_1$, and $I_3 u_2 = (I_1 - I_2)\omega_1\omega_2 + v_2$, we obtain a system on $\mathbb{R}^3$, with controls in $\mathbb{R}^2$:

$$\begin{aligned} \dot{x}_1 &= x_2 x_3 \\ \dot{x}_2 &= u_1 \\ \dot{x}_3 &= u_2\,. \end{aligned}$$

Then the following feedback law globally stabilizes the system:

$$\begin{aligned} u_1 &= -x_1 - x_2 - x_2 x_3 + v_1 \\ u_2 &= -x_3 + x_1^2 + 2x_1 x_2 x_3 + v_2 \end{aligned}$$

when $v_1 = v_2 \equiv 0$. The feedback was obtained arguing in this way: with $z_2 := x_1 + x_2$, $z_3 := x_3 - x_1^2$, the system becomes:

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + \alpha(x_1, z_2, z_3) \\ \dot{z}_2 &= -z_2 + v_1 \\ \dot{z}_3 &= -z_3 + v_2\,. \end{aligned}$$

The x_1-subsystem is easily seen to be ISS, because $\deg_{x_1} \alpha \leq 2$ and hence the cubic term dominates, for large x_1. Thus the cascade is also ISS; in particular, it is GAS if $v_1 = v_2 \equiv 0$. (We also proved a stronger result: ISS implies a global robustness result with respect to actuator noise.)

2.6 Generalizations of Other Gains

ISS generalizes finite $L^\infty \to L^\infty$ gains ("L^1 stability") but other classical norms often considered are induced $L^2 \to L^2$ ("H_∞") or $L^2 \to L^\infty$ ("H_2").

Nonlinear transformations starting from "H_∞"

$$\int_0^t |x(s)|^2 \, ds \;\leq\; c|x^0|^2 + c\int_0^t |u(s)|^2 \, ds \quad \forall\, t \geq 0$$

lead to (for appropriate comparison functions):

$$\int_0^t \underline{\alpha}(|x(s)|)\, ds \;\leq\; \kappa(|x^0|) + \int_0^t \gamma(|u(s)|)\, ds \quad \forall\, t \geq 0\,.$$

Theorem. *There is such an "integral to integral" estimate if and only if the system is ISS.*

The proof of this unexpected result is based upon certain known (and nontrivial) characterizations of the ISS property; see [67].

On the other hand, "$L^2 \to L^\infty$" stability:

$$|x(t)| \;\leq\; c|x^0|e^{-\lambda t} + c\int_0^t |u(s)|^2 \, ds \quad \text{for all } t \geq 0$$

leads to (for appropriate comparison functions):

$$\underline{\alpha}\,(|x(t)|) \;\leq\; \beta(|x^0|, t) + \int_0^t \gamma(|u(s)|)\, ds \quad \text{for all } t \geq 0\,.$$

This is the *iISS* (*integral* ISS) property to which we'll return later.

2.7 Remark: Reversing Coordinate Changes

The "integral to integral" version of ISS arose, in the above discussion, from coordinate changes when starting from L^2-induced operator norms. Interestingly, this result from [16] shows that the reasoning can be reversed:

Theorem. *Assume $n \neq 4, 5$. If $\dot{x} = f(x,u)$ is ISS, then, under a coordinate change, for all solutions one has:*

$$\int_0^t |x(s)|^2 \, ds \;\leq\; |x^0|^2 + \int_0^t |u(s)|^2 \, ds\,.$$

Similarly, global exponential stability is equivalent to global asymptotic stability. (Center manifold dimensions are not invariant, since coordinate changes are not necessarily $\mathcal{C}^\infty$ at 0.) The cases $n = 4, 5$ are still open.

A sketch of proof is as follows. Suppose $\dot{x} = f(x, u)$ is ISS. Pick a robustness margin $\rho \in \mathcal{K}_\infty$, so that $\dot{x} = f(x, d\rho(|x|))$ is uniformly GAS over all $\|d\|_\infty \leq 1$ and let V be $\mathcal{C}^\infty$, proper, positive definite, so that

$$\nabla V(x) \cdot f(x, d\rho(|x|)) \;\leq\; -V(x) \quad \forall\, x, d\,.$$

Suppose (see below) that we have been able to change coordinates so that $V(x) = |x|^2$. So, $W(z) := V(T^{-1}(z)) = |z|^2$ with $z = T(x)$. Then, whenever $|u| \leq \rho(|x|)$, we have

$$d\,|z|^2 / dt = \dot{W}(z) = \dot{V}(x) \leq -V(x) = -\,|z|^2\;.$$

So, if $\chi \in \mathcal{K}_\infty$ is so that $|T(x)| \leq \chi(\rho(|x|))$, and

$$\alpha(r) \;:=\; \max_{|u| \leq r, |z| \leq \chi(r)} d\,|z|^2 / dt$$

then:

$$\frac{d\,|z|^2}{dt} \;\leq\; -\,|z|^2 + \alpha(|u|) = -\,|z|^2 + v$$

(v is the input in new coordinates) and integrating, one obtains $\int |z|^2 \leq |z^0|^2 + \int |v|^2$. This gives the L^2 estimate as wanted.

The critical technical step, thus, is to show that, up to coordinate changes, *every Lyapunov function V is quadratic* — let us provide a sketch of the proof.

First notice that the level set $S := \{V(x) = 1\}$ is homotopically equivalent to $\mathbb{S}^{n-1}$ (this is well-known: $S \times \mathbb{R} \simeq S$ because $\mathbb{R}$ is contractible, and $S \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^n \setminus \{0\} \simeq \mathbb{S}^{n-1}$ via the flow of $\dot{x} = f(x, 0)$). Thus, $\{V(x) = 1\}$ is diffeomorphic to $\mathbb{S}^{n-1}$, provided $n \neq 4, 5$ (h-cobordism theory of Smale and Milnor; Poincaré would give a homeomorphism, for $n \neq 4$).

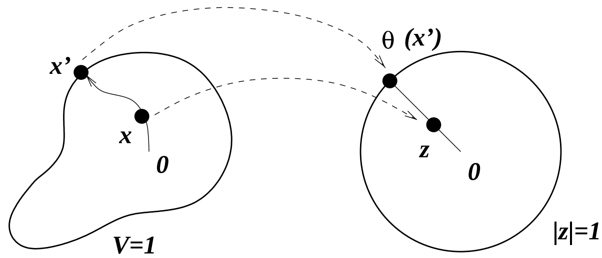
Finally, consider the normed gradient flow

$$\dot{x} \;=\; \frac{\nabla V(x)'}{|\nabla V(x)|^2}$$

and take the new variable

$$z \;:=\; \sqrt{V(x)}\;\theta(x')$$

where x' is the translate via the flow back into the level set, and $\theta : \{V = 1\} \simeq \{|z| = 1\}$ is the given diffeomorphism. The picture is as follows:



(Actually, this sketch is not quite correct: one needs to make a slight adjustment in order to obtain also continuity and differentiability at the origin; the actual coordinate change is $z = \gamma(V(x))\theta(x')$, so $W(z) = \gamma(|z|)$, for a suitable γ.)

3 Integral-Input to State Stability

The "$L^2 \to L^\infty$" operator gain property led to iISS:

$$\underline{\alpha}\,(|x(t)|) \;\leq\; \beta(|x^0|,t) \;+\; \int_0^t \gamma(|u(s)|)\,ds\,.$$

There is a dissipation characterization here as well.

A smooth, proper, and positive definite $V : \mathbb{R}^n \to \mathbb{R}$ is an *iISS-Lyapunov function* for $\dot{x} = f(x,u)$ if for some *positive definite* continuous α and $\gamma \in \mathcal{K}_\infty$

$$\nabla V(x)\,f(x,u) \;\leq\; -\alpha(|x|) + \gamma(|u|) \quad \forall\, x \in \mathbb{R}^n,\, u \in \mathbb{R}^m$$

— observe that we are not requiring now $\alpha \in \mathcal{K}_\infty$. (Intuitively: even for constant u one may have $\dot{V} > 0$, but $\gamma(|u|) \in \mathcal{L}^1$ means that $\dot{V}$ is "often" negative.)

A recent result from [5] is this:

Theorem. *A system is iISS if and only if it admits an iISS-Lyapunov function.*

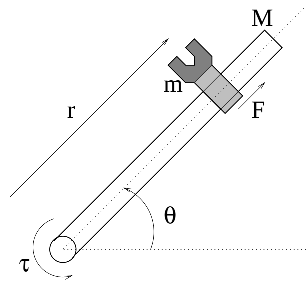
Since any $\mathcal{K}_\infty$ function is positive definite, every ISS system is also iISS, but the converse is false. For example, a bilinear system

$$\dot{x} = (A + \sum_{i=1}^{m} u_i A_i)x + Bu$$

is iISS if and only if A is a Hurwitz matrix, but in general it is not ISS — e.g., if $B = 0$ and $A + \sum_{i=1}^{m} u_i^0 A_i$ is not Hurwitz for some u^0. As another example, take $\dot{x} = -\tan^{-1} x + u$. This is not ISS, since bounded inputs may produce unbounded trajectories; but it is iISS, since $V(x) = x\,\tan^{-1} x$ is an iISS-Lyapunov function.

3.1 An Application of iISS Theory

Let us illustrate the iISS results through an application which, as a matter of fact, was the one that originally motivated much of the work in [5]. Consider a rigid manipulator with two controls:



The arm is modeled as a segment with mass M and length L, and the hand as a point with mass m. Denoting by r the position and by θ the angle of the arm, the resulting equations are:

$$(mr^2 + ML^2/3)\,\ddot{\theta} + 2mr\dot{r}\dot{\theta} = \tau\,, \quad m\ddot{r} - mr\dot{\theta}^2 = F$$

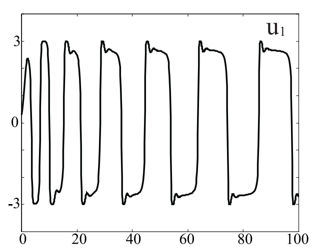
where F and τ are the external force and torque. In a typical passivity-based tracking design one takes

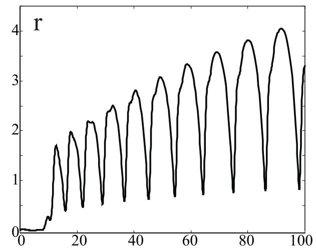
$$\begin{aligned} \tau &:= -k_{d_1}\dot{\theta} - k_{p_1}(\theta - \theta_d) \\ F &:= -k_{d_2}\dot{r} - k_{p_2}(r - r_d) \end{aligned}$$

where r_d and θ_d are the desired signals and the gains $(k_{d_1}, \ldots)$ are > 0. For constant reference θ_d, r_d, there is tracking: $\theta \to \theta_d$, $\dot{\theta} \to 0$, and analogously for r.

But, what about time-varying θ_d, r_d? Can these destabilize the system? **Yes:** there are bounded inputs which produce “nonlinear resonance” — so the system can’t be ISS (not even bounded-input bounded-state).

The figures that follow show the “r” component of the state of a certain solution which corresponds to the shown input (see [5] for details on how this input and trajectory were calculated).





On the other hand, many inputs are not destabilizing — how does one formulate qualitatively this fact? One way is by showing that the system is iISS. The closed-loop system is 4-dimensional, with states $(q, \dot{q})$, $q = (\theta, r)$ and $u = (k_{p_1}\theta_d, k_{p_2}r_d)$:

$$\begin{aligned}(mr^2 + ML^2/3)\ddot{\theta} + 2mr\dot{r}\dot{\theta} &= u_1 - k_{d_1}\dot{\theta} - k_{p_1}\theta \\ m\ddot{r} - mr\dot{\theta}^2 &= u_2 - k_{d_2}\dot{r} - k_{p_2}r\end{aligned}$$

To prove iISS, we consider the mechanical energy V and note the following passivity-type estimate:

$$\frac{d}{dt}V(q(t), \dot{q}(t)) \;\leq\; -c_1|\dot{q}(t)|^2 + c_2|u(t)|^2$$

for sufficiently small $c_1 > 0$ and large $c_2 > 0$.

In general, we say that a system is *h-dissipative* with respect to an output function $y = h(x)$ (continuous and with $h(0) = 0$) if for some $\mathcal{C}^\infty$ positive definite, proper $V : \mathbb{R}^n \to \mathbb{R}$, and for some γ, α as above

$$\nabla V(x)\, f(x, u) \;\leq\; -\alpha(h(x)) + \gamma(|u|) \quad \forall\, x \in \mathbb{R}^n,\, u \in \mathbb{R}^m$$

and *weakly h-detectable* if, for all trajectories, $y(t) = h(x(t)) \equiv 0$ implies that $x(t) \to 0$ as $t \to \infty$.

This is proved in [5]:

Theorem. *A system is iISS if and only if it is weakly h-detectable and h-dissipative for some output h.*

With output $\dot{q}$, our example is weakly zero-detectable and dissipative, since $u \equiv 0$ and $\dot{q} \equiv 0$ imply $q \equiv 0$. Thus it is iISS, as claimed.

3.2 Mixed Notions

Changes of variables transformed "finite L^2 gain" to an "integral to integral" property, which turns out to be equivalent to ISS. Finite gain as operators between L^p and L^q spaces, with $p \neq q$ both finite, lead instead to this type of "weak integral to integral" estimate:

$$\int_0^t \underline{\alpha}(|x(s)|)\, ds \;\leq\; \kappa(|x^0|) \,+\, \alpha\left(\int_0^t \gamma(|u(s)|)\, ds\right)$$

for appropriate $\mathcal{K}_\infty$ functions (note the additional "α"). See [6] for more discussion on how this estimate is reached, as well as this result:

Theorem. *A system satisfies a weak integral to integral estimate if and only if it is iISS.*

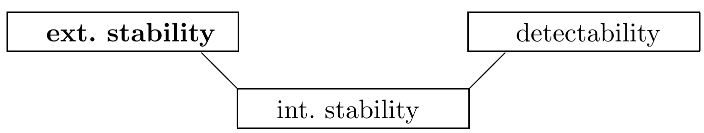
Another interesting variant results by studying *mixed* integral/supremum estimates:

$$\underline{\alpha}(|x(t)| \;\leq\; \beta(|x^0|, t) \,+\, \int_0^t \gamma_1(|u(s)|)\, ds \,+\, \gamma_2(\|u\|_\infty)$$

for suitable $\beta \in \mathcal{KL}$ and $\underline{\alpha}, \gamma_i \in \mathcal{K}_\infty$. This result is also from [6]:

Theorem. *The system $\dot{x} = f(x,u)$ satisfies a mixed estimate if and only if it is iISS.*

4 Input/Output Stability



The second component of the fundamental triad is *input to output* stability (IOS) for systems with outputs $\dot{x} = f(x,u)$, $y = h(x)$:

$$|y(t)| \;\leq\; \beta(|x^0|, t) \,+\, \sup_{s\in[0,t]} \gamma(|u(s)|)$$

for all solutions, assuming completeness (for some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$). This is closely related to "partial stability" (if h is a projection, so y is a subset of variables), and "stability with respect to two measures".

A dissipation (Lyapunov-) type characterization of this property is as follows. An *IOS-Lyapunov function* is a smooth $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ so that, for some $\alpha_i \in \mathcal{K}_\infty$, for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$:

$$\alpha_1(|h(x)|) \;\leq\; V(x) \;\leq\; \alpha_2(|x|)$$

and

$$V(x) > \alpha_3(|u|) \;\Rightarrow\; \nabla V(x)\, f(x,u) < 0\,.$$

For systems that are bounded-input bounded-state stable, we have (see [77]):

Theorem. *A system $\dot{x} = f(x,u)$, $y = h(x)$ is IOS if and only if it admits an IOS-Lyapunov function.*

4.1 Motivation: Regulator Theory

One may re-interpret this result as the existence of a new output map $\widetilde{y} = \alpha_1^{-1}(V(x))$ which dominates the original output ($y \leq \widetilde{y}$) and which is monotonically decreasing (no overshoot) as long as inputs are small. This is, in fact, one generalization of a central argument used in regulator theory (Francis equations).

Let us sketch how IOS is motivated by regulator theory. (See the paper [76] for some more details.) In a regulator system, for each exogenous signal $d(\cdot)$ (a disturbance to be rejected, a signal to be tracked), the output $y(\cdot)$ ("error") must decay to zero as $t \to \infty$. One assumes that the exogenous signal is generated by an "exosystem" described by differential equations.

For example, for linear systems (a nonlinear version is also well-known, cf. [23]) one studies the closed-loop system

$$\dot{z} = Az + Pw\,, \quad \dot{w} = Sw\,, \quad y = Cz + Qw$$

seen as a system $\dot{x} = f(x)$, $y = h(x)$, where the extended state x consists of z and w, and the z-subsystem incorporates both the state of the system being regulated (the plant) and the state of the controller, and the equation $\dot{w} = Sw$ describes the exosystem. (Later, we introduce inputs into the model.)

For example, a second order system $\ddot{y} - y = u + w$ under the action of all possible constant disturbances w leads to the conventional proportional-integral-derivative (PID) controller given by a feedback law $u(t) = c_1 q(t) + c_2 y(t) + c_3 v(t)$, for appropriate gains c_1, c_2, c_3, where $q = \int y$ and $v = \dot{y}$. Let us take $c_1 = -1$, $c_2 = c_3 = -2$. Viewing disturbances as produced by the exosystem $\dot{w} = 0$, the complete system is

$$\dot{q} = y,\; \dot{y} = v,\; \dot{v} = -q - y - 2v + w,\; \dot{w} = 0$$

with output y, $z = \mathrm{col}\,(q, y, v)$.

The routine way to verify the regulation objective is: one assumes that A is Hurwitz (after feedback) and that there is some matrix Π solving Francis' equations:

$$\Pi S = A\Pi + P\,, \quad 0 = C\Pi + Q\,.$$

Consider the new variable $\hat{y} := z - \Pi w$. The first identity for Π allows decoupling $\hat{y}$ from w, leading to $\dot{\hat{y}} = A\hat{y}$. Since A is a Hurwitz matrix, one concludes that $\hat{y}(t) \to 0$ for all initial conditions. As the second identity for Π gives that $y(t) = C\hat{y}(t)$, one has the desired conclusion that $y(t) \to 0$. The key fact is that the new output $\hat{y}$ dominates the old ($|y| \leq c\,|\hat{y}|$) and (for some $\sigma \in \mathcal{K}_\infty$)

$$|\hat{y}(t)| \leq \sigma(|\hat{y}(0)|)\,, \quad \forall\, t \geq 0$$

— i.e., the overshoot for this (also stable) output depends only on its initial condition. Note that a zero initial value $\hat{y}(0)$ implies $\hat{y} \equiv 0$ (initial state of the internal model and exosignal match), but this is false for the regulated variable.

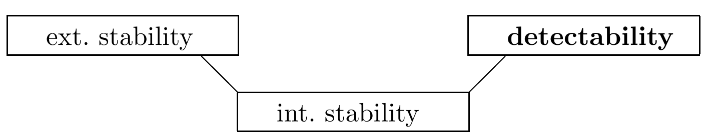
For example, in the PID regulator, $\hat{y}$ replaces q by $q - w$ (internal model $-$ disturbance); but with e.g. $x(0) = y(0) = v(0) = 0$ and $w(0) = 1$, $y(t) = \frac{1}{2}t^2e^{-t}$ overshoots (even if $y(0) = 0$).

By a further modification (introduce a Lyapunov function for the $\hat{y}$ subsystem), we also have that $\hat{y}(t)$ can be defined so that it decreases monotonically. The significance of this interpretation is that, instead of *output zeroing submanifolds*, one considers two functions to be compared in amplitude, one corresponding to zero error, the other to a new and well-behaved output map. This “comparison in amplitude” (CIA) principle is a general theorem for nonlinear systems, via the results in [76] and [77].

The usual formulation, motivated by linear theory, includes no external inputs. Inputs allow studying the effect on the feedback system of exosignals not exactly represented by the exosystem model. The IOS property amounts to asking small steady-state error if the exosignal is “close” to the model.

The paper [76] should be consulted for a “catalog” of variants of the IOS notion, and its companion paper [77] for the corresponding Lyapunov characterizations.

5 Zero-Detectability: IOSS

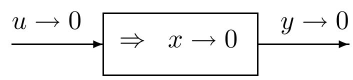


The third component of the fundamental triad, zero-detectability is typically defined by asking “$u \equiv 0$ and $y \equiv 0 \Rightarrow x(t) \to 0$ as $t \to \infty$” — this is too weak a property for nonlinear systems: it is not “well-posed” (what happens if $u, y \approx 0$?).

More natural is *input/output to state* stability (IOSS):

$$|x(t)| \;\leq\; \beta(|x^0|,t) \;+\; \sup_{s\in[0,t]} \gamma(|u(s)|) \;+\; \sup_{s\in[0,t]} \gamma(|y(s)|)$$

along all solutions (for some $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$) which results from the linear detectability estimate under coordinate changes. It implies, in particular, $x \to 0$ if both $u, y \to 0$ as $t \to \infty$.



The terminology IOSS is self-explanatory: formally, there is “stability from the i/o data to the state”.

5.1 Dissipation Characterization of IOSS

A smooth, proper, and positive definite $V : \mathbb{R}^n \to \mathbb{R}$ is an *IOSS-Lyapunov function* if, for some $\alpha_i \in \mathcal{K}_\infty$,

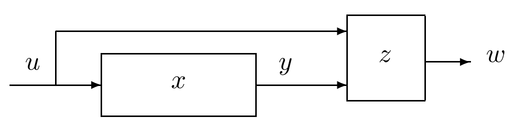
$$\nabla V(x)\, f(x,u) \;\leq\; -\alpha_1(|x|) + \alpha_2(|u|) + \alpha_3(|y|)$$

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$.

This is from [35] and [36]:

Theorem. *A system $\dot{x} = f(x,u)$, $y = h(x)$ is IOSS if and only if it admits an IOSS-Lyapunov function.*

As a corollary, IOSS is equivalent to the existence of a *norm-estimator*: driven by the i/o data generated by the original system, it estimates an upper bound on the internal state.



This is defined as a system $\dot{z} = g(z,u,y)$, $w = \ell(z)$, whose inputs are the i/o pairs of the original system, which is ISS with respect to u, y as inputs (so that there is robustness to signal errors), and, for some $\rho \in \mathcal{K}$ and $\beta \in \mathcal{KL}$,

$$|x(t)| \;\leq\; \beta(|x^0|+|z^0|,t) \;+\; \rho(|w(t)|) \quad \forall\, t \geq 0$$

for all initial states x^0 and z^0. (See the paper [36] for the precise definition.)

6 Comments

There are many foundational directions still being explored. Let me summarize just a few of them:

- Observers require a notion of *incremental* IOSS, not merely distinguishing from $x = 0$. This is a very appealing problem. See [75] for some preliminary remarks.
- An ISS-like "globally minimum phase" property — meaning that "the zero-dynamics are ISS" can be defined with no recourse to normal forms. (See [41] for a preliminary version.)
- A common generalization of IOSS and IOS is, for "regulated" and "measured" outputs w, y:

  $$|w(t)| \;\leq\; \beta(|x^0|,t) \,+\, \sup_{s\in[0,t]} \gamma(|u(s)|) \,+\, \sup_{s\in[0,t]} \gamma(|y(s)|)$$

  along along all solutions, for appropriate comparison functions; moreover, one may allow the overshoot to depend on yet another fixed function of x^0, such as a distance to a set. Characterizations are now being worked out, cf. [21]. One might call this property "input/measurement to ouput stability" (IMOS).

I have focused on basic theoretical constructs, instead of on applications, in this brief survey. The next section provides references to more work related to ISS-related theory and applications.

7 Additional References

Textbooks and research monographs which make use of ISS and related concepts include [13,24,37,38,32,60].

After the definition in [63] and the basic characterizations in [69], the main results on ISS are given in [73]. See also [8,78] for early uses of asymptotic gain notions. "Practical" ISS is equivalent to ISS with respect to compact attractors, see [71].

Several authors have pointed out that *time-varying* system versions of ISS are central to the analysis of asymptotic tracking problems, see e.g. [87]. In [10], one can find further results on Lyapunov characterizations of the ISS property for time-varying (and in particular periodic) systems, as well as a small-gain theorem based on these ideas.

Perhaps the most interesting set of open problems concerns the construction of feedback laws that provide ISS stability with respect to observation errors. Actuator errors are far better understood (cf. [63]), but save for the case of special structures studied in [13], the one-dimensional case (see e.g. [11])

and the counterexample [12], little is known of this fundamental question. Recent work analyzing the effect of small observation errors (see [68]) might provide good pointers to useful directions of research (indeed, see [40] for some preliminary remarks in that direction). For special classes of systems, even output feedback ISS with respect to observation errors is possible, cf. [52].

Both ISS and iISS properties have been featured in the analysis of the performance of switching controllers, cf. [17] and [18].

Coprime factorizations are the basis of the parameterization of controllers in the Youla approach. As a matter of fact, as the paper's title indicates, their study was the original motivation for the introduction of the notion of ISS in [63]. Some further work can be found in [64], see also [14], but much remains to be done.

There are now results on averaging for ISS systems, see [54], as well as on singular perturbations, see [7].

Discrete-time ISS systems are studied in [31] and in [29]; the latter paper provides Lyapunov-like sufficient conditions and an ISS small-gain theorem, and more complete characterizations and extensions of many standard ISS results for continuous time systems are given in [30].

Discrete-time iISS systems are the subject of [2], who proves the very surprising result that, in the discrete-time case, iISS is actually no different than global asymptotic stability of the unforced system (this is very far from true in the continuous-time case, of course). In this context, of interest are also the relationships between the ISS property for a continuous-time system and its sampled versions. The result in [80] shows that ISS is recovered under sufficiently fast sampling; see also the technical estimates in [53].

Stochastic ISS properties are treated in [86].

A very interesting area regards the combination of clf and ISS like-ideas, namely providing necessary and sufficient conditions, in terms of appropriate clf-like properties, for the existence of feedback laws (or more generally, dynamic feedback) such that the system $\dot{x} = f(x, d, u)$ becomes ISS (or iISS, etc) with respect to d, once that $u = k(x)$ is substituted. Notice that for systems with disturbances typically $f(0, d, 0)$ need not vanish (example: additive disturbances for linear systems), so this problem is qualitatively different from the robust-clf problem since uniform stabilization is not possible. There has been substantial work by many authors in this area; let us single out among them the work [81], which deals primarily with systems of the form $\dot{x} = f(x, d) + g(x)u$ (affine in control, and control vector fields are independent of disturbances) and with assigning precise upper bounds to the "nonlinear gain" obtained in terms of d, and [9], which, for the class of systems that can be put in output-feedback form (controller canonical form with an added stochastic output injection term), produces, via appropriate clf's, stochastic ISS behavior ("NSS" = noise to state stability, meaning that solutions converge in probability to a residual set whose radius is proportional to bounds on covariances).

In connection with our example from tracking design for a robot, we mention here that the paper [50] proposed the reformulation of tracking problems by means of the notion of input to state stability. The goal was to strengthen the robustness properties of tracking designs, and the notion of ISS was instrumental in the precise characterization of performance. Incidentally, the same example was used, for a different purpose — namely, to illustrate a different nonlinear tracking design which produces ISS, as opposed to merely iISS, behavior — in the paper [1].

Neural-net control techniques using ISS are mentioned in [59].

A problem of decentralized robust output-feedback control with disturbance attenuation for a class of large-scale dynamic systems, achieving ISS and iISS properties, is studied in [28].

Incremental ISS is the notion that estimates differences $|x_1(t) - x_2(t)|$ in terms of $\mathcal{KL}$ decay of differences of initial states, and differences of norms of inputs. It provides a way to formulate notions of sensitivity to initial conditions and controls (not local like Lyapunov exponents or as in [46], but of a more global character, see [3]); in particular when there are no inputs one obtains "incremental GAS", which can be completely characterized in Lyapunov terms using the result in [45], since it coincides with stability with respect to the diagonal of the system consisting of two parallel copies of the same system. This area is of interest, among other reasons, because of the possibility of its use in information transmission by synchronization of diffusively coupled dynamical systems ([56]) in which the stability of the diagonal is indeed the behavior of interest.

Small-gain theorems for ISS and IOS notions originated with [27]; a purely operator version (cf. [20]) of the IOS small-gain theorem holds as well. There are ISS-small gain theorems for certain infinite dimensional classes of systems such as delay systems, see [79].

The notion of IOSS is called "detectability" in [64] (where it is phrased in input/output, as opposed to state space, terms, and applied to questions of parameterization of controllers) and was called "strong unboundedness observability" in [27]. IOSS and its incremental variant are very closely related to the OSS-type detectability notions pursued in [34]; see also the emphasis on ISS guarantees for observers in [49]. The use of ISS-like formalism for studying observers, and hence implicitly the IOSS property, has also appeared several times in other authors' work, such as the papers [19,47,55].

It is worth pointing out that several authors had independently suggested that one should *define* "detectability" in dissipation terms. For example, in [48], Equation 15, one finds detectability defined by the requirement that there should exist a differentiable storage function V satisfying our dissipation inequality but with the special choice $\alpha_3(r) := r^2$ (there were no inputs in the class of systems considered there). A variation of this is to weaken the dissipation inequality, to require merely

$$x \neq 0 \Rightarrow \nabla V(x)\, f(x,u) < \alpha_3(|y|)$$

(again, with no inputs), as done for instance in the definition of detectability given in [51]. Observe that this represents a slight weakening of our property, in so far as there is no "margin" of stability $-\alpha_1(|x|)$.

Norm-estimators are motivated by developments appeared in [26] and [57].

The notion studied in [62] is very close to the combination of IOSS and IOS being pursued in [21].

Partial asymptotic stability for differential equations is a particular case of output stability (IOS when there are no inputs) in our sense; see [90] for a survey of the area, as well as the book [58], which contains a converse theorem for a restricted type of output stability. (We thank Anton Shiriaev for bringing this latter reference to our attention.)

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