Measurement to Error Stability: a Notion of Partial Detectability for Nonlinear Systems

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Abstract

In previous work the notion of input to state stability (ISS) has been generalized to systems with outputs, yielding a number of useful concepts. When considering a system whose output is to be kept small (i.e. an error output), the notion of input to output stability (IOS) arises. Alternatively, when considering a system whose output is meant to provide information about the state (i.e. a measurement output), one arrives at the detectability notion of output to state stability (OSS). Combining these concepts, one may consider a system with *two* outputs, an error and a measurement. This leads naturally to a notion of *partial detectability* we call measurement to error stability (MES). This property characterizes systems in which the error signal is detectable through the measurement signal. This paper provides a partial Lyapunov characterization of the MES property. A closely related property of *stability in three measures* (SIT) is introduced, which characterizes systems for which the error decays whenever it dominates the measurement. The SIT property is shown to imply MES, and the two are shown to be equivalent under an additional boundedness assumption. A nonsmooth Lyapunov characterization of the SIT property is provided, which yields the partial characterization of MES. The analysis is carried out on systems described by differential inclusions – implicitly incorporating a disturbance input with compact value-set.

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1 Introduction

The notion of *input to state stability* (ISS), introduced in [19], provides a theoretical framework in which to formulate questions of robustness with respect to inputs (seen as disturbances) acting on a system. An ISS system is, roughly, one which has a "finite nonlinear gain" with respect to inputs and whose transient behavior can be bounded in terms of the size of the initial state and inputs; the precise definition is in terms of $\mathcal{K}$-function gains. ISS systems have been treated by a number of authors (e.g. [9, 10, 11, 12, 14, 16, 21, 27]).

In light of the duality between input/state and state/output behaviour which is common in control theory, it is natural to ask whether an ISS-like notion of output to state stability can be formulated. This concept, called OSS, is the subject of [13, 22, 23]. The definition given is precisely the same as that of ISS with outputs in the place of inputs. In the case of linear systems this property is equivalent to *detectability.* (When applied to nonlinear systems, OSS is more properly described as zero-detectability).

The paper [23] contains a discussion of various definitions of detectability for nonlinear systems which have appeared in the literature. Several of these definitions are given in terms of the existence of a Lyapunov or "storage" function for the system. The main result of [22] is the fact that the OSS property is equivalent to the existence of an appropriate Lyapunov function. These papers also contain a discussion of a generalized notion in which both inputs and outputs are considered (input-output to state stability, or IOSS). This property is addressed more completely in [13] where a Lyapunov characterization is provided and the construction of nonlinear observers is discussed.

This work addresses a generalization of the OSS property to a notion of "partial detectability". When discussing systems with outputs, the output signal typically plays one of two roles. A common situation is when the outputs are considered as *measurements*. Here, one supposes that knowledge of the whole state is not available, but rather that only partial knowledge of the state can be used. (Most commonly the output map is a projection, which corresponds simply to the ability to measure some, but not all, of the components of the state. More generally, one may only have access to some function of the state variables – e.g. the sum of two components – and so we allow for more general output mappings in the theory). This is the role of the output in OSS, and in the theory of detectability and observers in general.

A second role for outputs occurs when the goal of the control design is not to regulate the behaviour of the entire state, but rather only to regulate the output signal. The theory of *output regulation* addresses precisely this situation (see e.g. [8]). In the case of systems with no inputs, the problem of *stability* of a subset of the state variables (i.e. stability of an output signal which is a projection) has been addressed in the ordinary differential equations literature under the name "partial stability" [28]. Within the ISS framework, the notion of stability of the output signal has been described by *input to output stability* (IOS) [5, 24, 25].

Consider now the case in which both the above situations occur. That is, there are *two* output signals, one which is measured, and the other which must be regulated. A special case of this situation has been addressed in the output regulation theory, under the name "error feedback". This theory formulates the question of regulating an output of the system (the error) with knowledge of that output only. The more general case is when there are two distinct channels playing these two roles. In this paper we generalize the notion of OSS to this situation by introducing the concept of *measurement to error stability* (MES), which can be viewed as a notion of *partial detectability* through the measurement channel.

In this paper we will present a partial Lyapunov characterization of the MES property. This will be accomplished by first comparing the MES property to a notion of output stability relative to a set. This notion, which will be called *stability in three measures* (SIT) (cf. [15]) will be characterized by the existence of a lower semicontinuous Lyapunov function. It will be shown that the SIT property implies MES, and that the converse holds under an additional boundedness assumption.

All stability notions discussed in this paper are defined "robustly" with respect to disturbances. Disturbances are incorporated implicitly into the model by describing the dynamics of the system by a differential inclusion.

2 Basic Definitions and Notations

We consider the differential inclusion

$$\dot{x}(t) \in F(x(t)) \tag{1}$$

with two output maps

$$y(t) = h(x(t)), \quad w(t) = g(x(t)),$$

and a map $\omega : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. We take the state $x \in \mathbb{R}^n$. We assume that the set-valued map F from $\mathbb{R}^n$ to subsets of $\mathbb{R}^n$ is locally Lipschitz (precise definitions to follow) with nonempty compact values. In addition, we assume that the differential inclusion (1) is forward complete. We assume that the output maps $h : \mathbb{R}^n \to \mathbb{R}^{p_y}$ and $g : \mathbb{R}^n \to \mathbb{R}^{p_w}$ are locally Lipschitz. The map ω is assumed to be continuous and proper; it will be used as a measurement of the magnitude of the state vector. We will denote $|\cdot|_\omega := \omega(\cdot)$. The use of $|\cdot|_\omega$ allows a framework which includes the Euclidean norm, distance to a compact set, and more general measures of the magnitude of the state.

The Euclidean norm in a space $\mathbb{R}^k$ is denoted simply by $|\cdot|$. If z is a function defined on a real interval containing $[0,t]$, $\|z\|_{[0,t]}$ is the sup norm of the restriction of z to $[0,t]$, that is $\|z\|_{[0,t]} = \operatorname{ess\,sup} \{|z(t)| : t \in [0,t]\}$. For each $p \in \mathbb{R}^n$ and $r \geq 0$ let $B(p,r) := \{x \in \mathbb{R}^n : |x-p| \leq r\}$, the ball of radius r centered at p. Let $\mathcal{B}$ denote the unit ball $B(0,1)$.

To formulate the statement that a nonsmooth function decreases in an appropriate manner, we will make use of the notion of the *viscosity subgradient* (cf. [1]).

Definition 2.1 A vector $\zeta \in \mathbb{R}^n$ is a *viscosity subgradient* of the function $V : \mathbb{R}^n \to \mathbb{R}$ at $\xi \in \mathbb{R}^n$ if there exists a function $g : \mathbb{R}^n \to \mathbb{R}$ satisfying $\lim_{h\to 0} \frac{g(h)}{|h|} = 0$ and a neighbourhood $\mathcal{O} \subset \mathbb{R}^n$ of the origin so that

$$V(\xi + h) - V(\xi) - \zeta \cdot h \geq g(h)$$

for all $h \in \mathcal{O}$.

The (possibly empty) set of viscosity subgradients of V at ξ is called the *viscosity subdifferential* and is denoted $\partial_D V(\xi)$. We remark that if V is differentiable at ξ, then $\partial_D V(\xi) = \{\nabla V(\xi)\}$.

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{K}$ (denoted $\gamma \in \mathcal{K}$) if it is continuous, positive definite, and strictly increasing; and is of class $\mathcal{K}_\infty$ if in addition it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{KL}$ if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \geq 0$, $\beta(s,t)$ decreases to zero as $t \to \infty$.

2.1 Differential Inclusions

We review some standard concepts from set-valued analysis (See e.g. [1, 2, 3]). The following statements apply to a map F from $\mathbb{R}^n$ to subsets of $\mathbb{R}^n$.

Definition 2.2 Let $0 < T \leq \infty$. A function $x : [0,T) \to \mathbb{R}^n$ is said to be a *solution of the differential inclusion* (1) if it is absolutely continuous and satisfies

$$\dot{x}(t) \in F(x(t)),$$

for almost every $t \in [0,T)$. A function $x : [0,T) \to \mathbb{R}^n$ is called a *maximal solution of the differential inclusion* (1) if it does not have an extension which is a solution. That is, either $T = \infty$ or there does not exist a solution $\widehat{x} : [0, T_+) \to \mathbb{R}^n$ with $T_+ > T$ so that $\widehat{x}(t) = x(t)$ for all $t \in [0,T)$.

Definition 2.3 The differential inclusion (1) is said to be *forward complete* on $\mathbb{R}^n$ if every maximal solution is defined for all $t \geq 0$.

For each $C \subseteq \mathbb{R}^n$ we let $\mathbf{S}(C)$ denote the set of maximal solutions of (1) satisfying $x(0) \in C$ equipped with the topology of uniform convergence on compact intervals. If C is a singleton $\{\xi\}$ we will use the shorthand $\mathbf{S}(\xi)$. We set $\mathbf{S} := \mathbf{S}(\mathbb{R}^n)$, the set of all maximal solutions. Given a trajectory $x(\cdot) \in \mathbf{S}(\xi)$ for some $\xi \in \mathbb{R}^n$, we denote

$$y(t) = h(x(t)) \qquad w(t) = g(x(t)),$$

for all $t \geq 0$.

Definition 2.4 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$. The set-valued map F is said to be *locally Lipschitz* on $\mathcal{O}$ if, for each $\xi \in \mathcal{O}$, there exists a neighbourhood $U \subset \mathcal{O}$ of ξ and an $L > 0$ so that for any η, ζ in U,

$$F(\eta) \subseteq F(\zeta) + L\,|\eta - \zeta|\,\mathcal{B}.$$

3 Stability and Detectability Properties

The following definitions are given for a forward complete system with two output channels as in (1). The outputs y and w are considered as error and measurement signals, respectively.

Our primary motivation is the following notion.

Definition 3.1 We say that the system (1) is *measurement to error stable* (MES) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ so that

$$|y(t)| \leq \max\{\beta(|x(0)|_\omega, t), \gamma(\|w\|_{[0,t]})\}$$

for each $x(\cdot) \in \mathbf{S}$, and all $t \geq 0$.

In the investigation of the MES property, the following notion of relative stability of the error will be useful. This is a notion of output stability which is applicable to systems with a single output y.

Definition 3.2 Given a closed subset D of the state space $\mathbb{R}^n$, we say that the system (1) is *relatively error stable (RES) with respect to* D if there exists $\beta \in \mathcal{KL}$ so that for any solution $x(\cdot) \in \mathbf{S}$, if there exists $t_1 > 0$ so that $x(t) \notin D$ for all $t \in [0, t_1]$, then

$$|y(t)| \leq \beta(|x(0)|_\omega, t) \qquad \forall t \in [0, t_1].$$

A special case of this property occurs for a system with two outputs when the set D is defined by an inequality involving the two output maps, as follows.

Definition 3.3 Let $\rho \in \mathcal{K}$. We say that the system (1) satisfies the *stability in three measures* (SIT) property (with gain ρ) if there exists $\beta \in \mathcal{KL}$ so that for any solution $x(\cdot) \in \mathbf{S}$, if there exists $t_1 > 0$ so that $|y(t)| > \rho(|w(t)|)$ for all $t \in [0, t_1]$, then

$$|y(t)| \leq \beta(|x(0)|_\omega, t) \qquad \forall t \in [0, t_1].$$

It is immediate that SIT is equivalent to relative error stability with respect to the set $D := \{\xi \in \mathbb{R}^n : |h(\xi)| \leq \rho(|g(\xi)|)\}$.
The following relative stability property will also be considered.

Definition 3.4 We say the system (1) satisfies the *relative measurement to error bounded property* (RMEB) if there exist $\mathcal{K}$ functions ρ_1, σ_1, and σ_2 so that for any solution $x(\cdot) \in \mathbf{S}$, if there exists $t_1 > 0$ so that $|y(t)| > \rho_1(|w(t)|)$ for all $t \in [0, t_1]$, then for all $t \in [0, t_1]$,

$$|y(t)| \leq \max\{\sigma_1(|h(x(0))|), \sigma_2(\|w\|_{[0,t]})\}. \qquad (2)$$

Remark 3.5 The RMEB property is equivalent to the following seemingly stronger property: there exist $\mathcal{K}$ functions ρ_2 and σ so that for any solution $x(\cdot) \in \mathbf{S}$, if there exists $t_1 > 0$ so that $|y(t)| > \rho_2(|w(t)|)$ for all $t \in [0, t_1]$, then

$$|y(t)| \leq \sigma(|h(x(0))|) \qquad \forall t \in [0, t_1].$$

The equivalence can be shown by setting $\rho_2(r) = \max\{\rho_1(r), \sigma_2(r)\}$ and $\sigma(r) = \sigma_1(r)$. □

In the next section we provide a Lyapunov characterization for the relative error stability property.

4 Lyapunov Functions

We give definitions of the appropriate Lyapunov functions.

Definition 4.1 Given an open set $E \subseteq \mathbb{R}^n$, we say that a lower semicontinuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a *lower semicontinuous RES-Lyapunov function* for system (1) on E if

- there exist α_1, $\alpha_2 \in \mathcal{K}_\infty$ so that for all $\xi \in E$,

$$\alpha_1(|h(\xi)|) \leq V(\xi) \leq \alpha_2(|\xi|_\omega), \qquad (3)$$

- there exists $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ continuous positive definite so that for each $\xi \in E$,

$$\zeta \cdot v \leq -\alpha_3(V(\xi)) \qquad (4)$$

for all $\zeta \in \partial_D V(\xi)$ and all $v \in F(\xi)$.

We say that V is a *lower semicontinuous exponential decay RES-Lyapunov function* for system (1) on E if in addition (4) holds with $\alpha_3(r) = r$.

We specialize the above definitions for the notion of stability in three measures as follows.

Definition 4.2 Let $\rho \in \mathcal{K}$. We say that a lower semicontinuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a *lower semicontinuous SIT-Lyapunov function* for system (1) with gain ρ if

- there exist α_1, $\alpha_2 \in \mathcal{K}_\infty$ so that for each ξ so that $|h(\xi)| > \rho(|g(\xi)|)$, it follows that

$$\alpha_1(|h(\xi)|) \leq V(\xi) \leq \alpha_2(|\xi|_\omega),$$

- there exists $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ continuous positive definite so that for each ξ so that $|h(\xi)| > \rho(|g(\xi)|)$,

$$\zeta \cdot v \leq -\alpha_3(V(\xi)) \qquad (5)$$

for all $\zeta \in \partial_D V(\xi)$ and all $v \in F(\xi)$.

We say that V is a *lower semicontinuous exponential decay SIT-Lyapunov function* for system (1) with gain ρ if in addition (5) holds with $\alpha_3(r) = r$.

We next remark that the decrease statements (4) and (5) can be written equivalently in an integral formulation. Using a standard result (a minor extension of Theorem 4.6.3 in [1], see e.g. [18] for details) the decrease statements (4) and (5) above can be written equivalently as (after possibly replacing α_3 by a locally Lipschitz function dominated by the original α_3)

$$V(x(t)) - V(x(0)) \leq -\int_0^t \alpha_3(V(x(s)))\, ds, \qquad (6)$$

for all $x(\cdot) \in \mathbf{S}$ which remain in the appropriate set on the interval $[0, t]$.
The Lyapunov characterizations are as follows.

Theorem 1 *Let a system of the form (1) and a closed set $D \subset \mathbb{R}^n$ be given. Let $E = \mathbb{R}^n \backslash D$. The following are equivalent.*

1. *The system is relatively error stable with respect to D.*
2. *The system admits a lower semicontinuous RES-Lyapunov function on E.*
3. *The system admits a lower semicontinuous exponential decay RES-Lyapunov function on E.*

The implication (3) $\Rightarrow$ (2) is immediate. In the interests of space, proofs of the other implications are omitted.

Corollary 4.3 *Let a system of the form (1) and a function $\rho \in \mathcal{K}$ be given. The following are equivalent.*

1. *The system satisfies the SIT property with gain ρ.*
2. *The system admits a lower semicontinuous SIT-Lyapunov function with gain ρ.*
3. *The system admits a lower semicontinuous exponential decay SIT-Lyapunov function with gain ρ.*

$\square$

The corollary follows immediately by setting $D = \{\xi \in \mathbb{R}^n : |h(\xi)| \leq \rho(|g(\xi)|)\}$.

5 Relationships between Notions

Having given a characterization of the SIT property, we now indicate how this notion is related to measurement to error stability. Proofs are omitted due to space requirements.

Lemma 5.1 If the system (1) satisfies the MES property, then it satisfies the SIT property.

Lemma 5.2 If the system (1) satisfies the SIT property and the RMEB property, then it satisfies the MES property.

It is easy to give an example to show that the converse of Lemma 5.1 does not hold in general. The following partial characterization of MES is an immediate consequence of Corollary 4.3 and the two preceding lemmas.

Corollary 5.3 *If the system (1) satisfies MES, then it admits a lower semicontinuous exponential decay SIT-Lyapunov function. If the system satisfies the RMEB property and admits a lower semicontinuous SIT-Lyapunov function, then it satisfies MES.* $\square$

6 Discussion

As previously mentioned, the MES property (or more precisely IMES – partial detectability under explicit inputs) is a natural combination of the notions of IOS and IOSS. As such, one would hope that a Lyapunov characterization of the IMES property would include as special cases the existing characterizations for IOS and IOSS (derived in [25] and [13], respectively). The work presented here is a first step toward such a single unifying result.

Several extensions to this result will be needed to complete this program. Firstly, an explicit input can be included by modelling the system as a forced differential inclusion. Secondly, a complete Lyapunov characterization is needed, with no recourse to an additional boundedness assumption. Finally, one would hope to prove that the stability property implies the existence of a smooth Lyapunov function, rather than the discontinuous case described here. When and if these problems are addressed, there will be a single characterization which would encompass the Lyapunov results on ISS, IOS and IOSS.

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