

The ISS Philosophy

as a Unifying Framework for

for Stability-Like Behavior

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What is “ISS”?



Sorry, not ISS ...



Indonesian Skeptic Society
International Superstar Soccer
Imprinted Sportswear Shows
Intelligence Studies Section
Internet Security Systems
Internet Support Service
Institute of Social Studies
International Summer School
Information Systems Support
Istituto Superiore di Sanita'
Industry Solutions & Services
Institute of Social Science
International Shared Services
International Sound Symposium

...

Scope: ISSomics

“input to state stability” *paradigm* for stability
among coauthors in the work discussed here are:

■ *Yuan Wang*

■ *David Angeli*

● Misha Krichman

● Andy Teel

● Lars Grüne

● Fabian Wirth

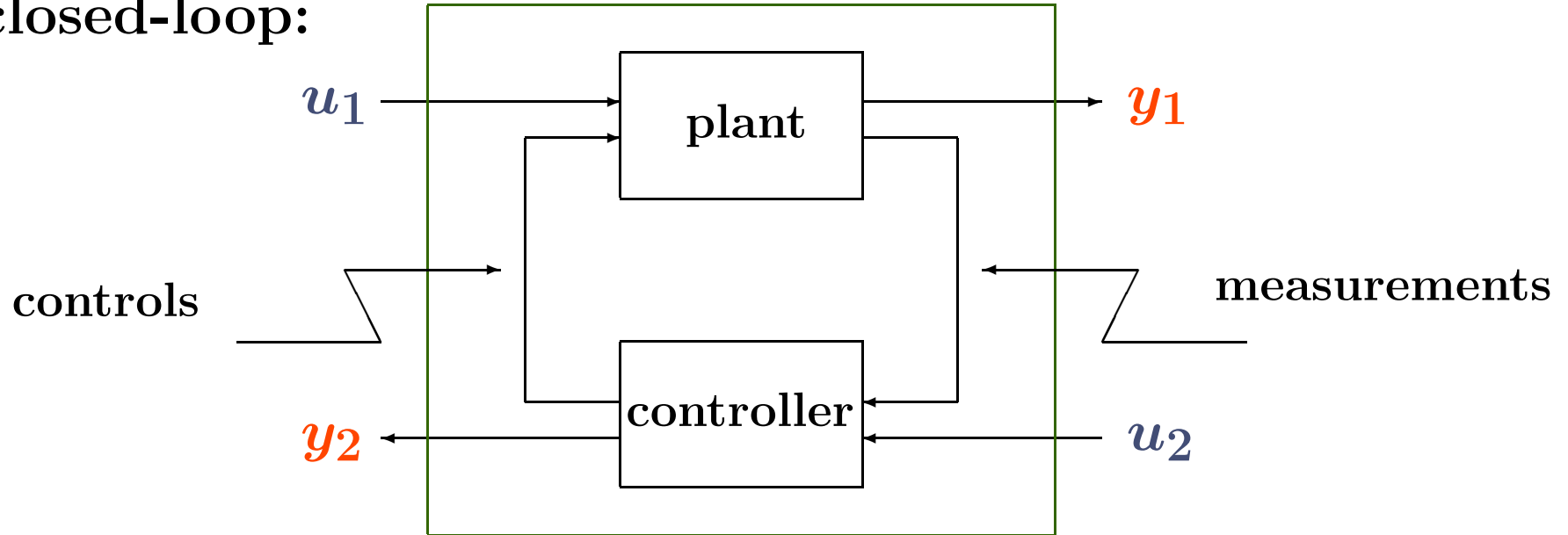
● also work by Teel-Jiang-Praly

no applications here (see papers, books) ... *theory* talk!

I/O Stability

Overall theme: study “stability” of $u \longrightarrow \boxed{\text{system}} \longrightarrow y$

e.g. closed-loop:



$u = (u_1, u_2)$ = errors, disturbances, tracking signals, ...

$y = (y_1, y_2)$ = distance to desired states, tracking error, ...

formalization of “stability” of $u(\cdot) \mapsto y(\cdot)$?

Stability = ?

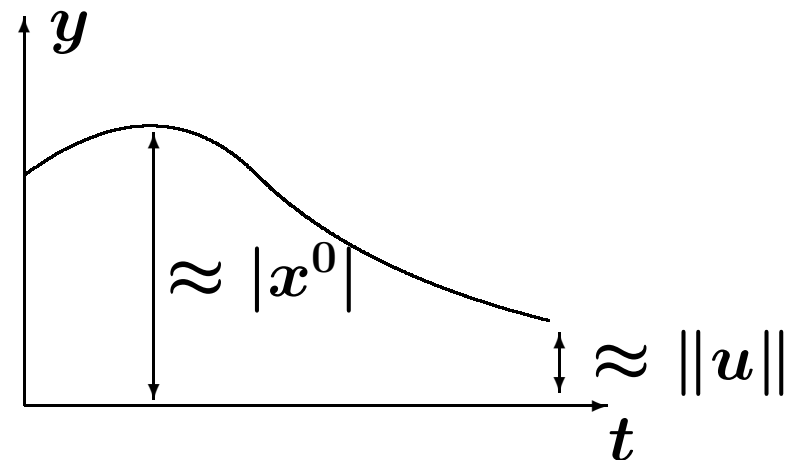
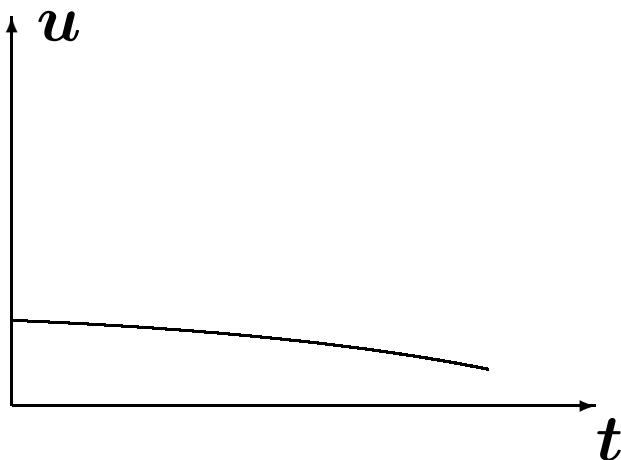
two desirable (and complementary) features of stability:

■ *asymptotically, u small $\Rightarrow y$ small, e.g.:*

- $\rightarrow 0$ when $t \rightarrow +\infty$
- bounded
- $\varepsilon - \delta$ definition

■ *transient (overshoot) depends on $x(0)$*

- fading memory



Classical Approaches

■ *Lyapunov Stability*

- well-developed general theory
- no systematic study forcing/disturbances effects

■ *Operator Boundedness of $u(\cdot) \mapsto y(\cdot)$*

- Zames, Sandberg, Willems, Safanov, ...
- robustness metrics, small-gain theorems, ...
- hard to deal with arbitrary nonlinearities
- not most natural for internal stability

goal: combine best of both approaches

Mostly Today: Input to State

ideas easiest to explain in input/state ($y = x$) case

but “philosophy” and many results extend:

- “magnitude” of signal \rightsquigarrow error $|y(t) - y_{\text{desired}}(t)|$
- or distance to a set \mathcal{A} : $|y(t)|_{\mathcal{A}} = \text{dist}(y(t), \mathcal{A})$
— e.g. $\mathcal{A} =$ periodic orbit, ask $y(t) \rightarrow \mathcal{A}$ as $t \rightarrow \infty$
- robust/adaptive most important in applications
- external stability (IOS = input to output)
- detectability (IOSS = input/output to state)
- incremental ISS (compare incremental gain)
- relative stability (regulation of w based on y)
- notions of minimum-phase

$\dot{x} = f(x, 0)$ Stable Not Enough

consider *linear system*

$$\dot{x} = Ax + Bu$$

suppose A Hurwitz matrix: all eigens have $\text{Re } \lambda < 0$,

i.e. $x(t) \rightarrow 0 \forall$ solutions with $u = 0$

then $x(t) \rightarrow 0$ whenever $u(t) \rightarrow 0$

but “ $u \rightarrow 0 \Rightarrow x \rightarrow 0$ ” is false in general, e.g.:

$$\dot{x} = -x + (x^2 + 1)u \quad \text{with input } u(t) = \frac{1}{\sqrt{2t + 2}}$$

admits unbounded solution $x(t) = \sqrt{2t + 2}$

(even though $x(t) \rightarrow 0 \forall$ solutions with $u = 0$)

even worse: $u \equiv 1 \Rightarrow$ explosion (!) (no “BIBS”)

Linear Case, for Motivation

$$\dot{x} = Ax + bu \rightsquigarrow \boxed{|x(t)| \leq \beta(t) |x(0)| + \gamma \|u\|_\infty} \quad (\beta \rightarrow 0, \gamma < \infty):$$

$$|x(t)| \leq \underbrace{\|e^{tA}\|}_{\beta(t)} |x(0)| + \underbrace{\left(\|B\| \int_0^\infty \|e^{sA}\| ds \right)}_{\gamma} \|u\|_\infty$$

superposition of transient and asymptotic effects:

t large: $x(t)$ bounded by $\gamma \|u\|_\infty$
independently of initial conditions

t small: effect of initial states may dominate

what is a reasonable nonlinear version of this estimate?
linear functions of $|x^0|$ and $\|u\|_\infty \rightsquigarrow$ nonlinear ones

Input-to-State Stability

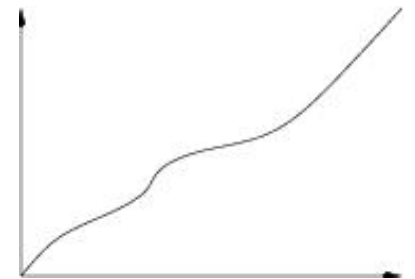
guiding principle:

notions of stability should be invariant under (nonlinear) changes of variables

\therefore estimate “ $|x(t)| \leq \beta(t) |x(0)| + \gamma \|u\|_\infty$ ” transforms into:

$$|x(t)| \leq \beta(|x^0|, t) + \gamma (\|u\|_\infty)$$

Definition: a system is *input to state stable (ISS)* if such an estimate holds, for some $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$, for all inputs and solutions



\mathcal{K}_∞ : $\gamma(0) = 0$, \mathcal{C}^0 str. incr., unbounded

\mathcal{KL} : $\beta(\cdot, t) \in \mathcal{K}_\infty$ for all t , and $\beta(r, t) \searrow 0$ as $t \rightarrow \infty$

Discussion

may equivalently ($\neq \beta, \gamma$) use “max” instead of “+”

$$|x(t)| \leq \max \{ \beta(|x^0|, t), \gamma(\|u\|_\infty) \}$$

\Rightarrow bounded-input/bounded-state, “ $u \rightarrow 0 \Rightarrow x \rightarrow 0$ ”, etc

two central characteristics of the ISS philosophy are:

■ *nonlinear gains*

■ *not asking about exact forms of gains*

i.e. *qualitative* questions of *existence*

— a “topological” vs. a “metric” point of view

analogy: “is gain $< \infty$?” “is operator bounded?”

Natural Notion?

a mathematical concept is “natural” (\Rightarrow useful!)
if it has many *equivalent* characterizations

ISS turns out to be equivalent to:

- \exists *proper robustness margins*
- *dissipativity (in “input – state” form)*
- *separation: GAS + asymptotic gain*
- *energy-like stability*

and other properties

much of talk will concentrate on explaining these

Fine Print Stuff

unless otherwise stated, all results hold for arbitrary (finite dimensional) systems

$$\dot{x}(t) = f(x(t), u(t)) \quad (x(t) \in \mathbb{R}^n)$$

inputs are Lebesgue-measurable locally (essentially) bounded

$$u(\cdot) : [0, \infty) \rightarrow \mathbb{R}^m$$

the map $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is locally Lipschitz, $f(0, 0) = 0$

when outputs appear, output map is continuous

when feedbacks appear, they are loc. Lipschitz, $k(0) = 0$

partial results available (but not discussed here)

for time-varying, some infinite-dimensional system classes, discrete-time, etc

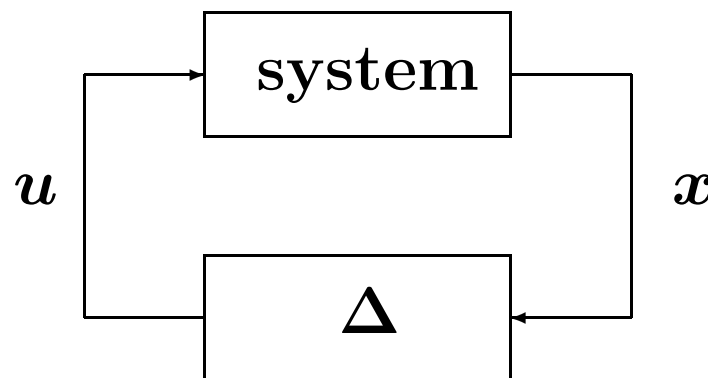
ISS \equiv Robust Stability

$$|x(t)| \leq \max \{ \beta(|x^0|, t), \gamma(\|u\|_\infty) \}$$

ISS $\iff \exists$ “robustness margin” $\rho \in \mathcal{K}_\infty$, i.e.:

$$\dot{x} = f(x, \Delta(t, x))$$

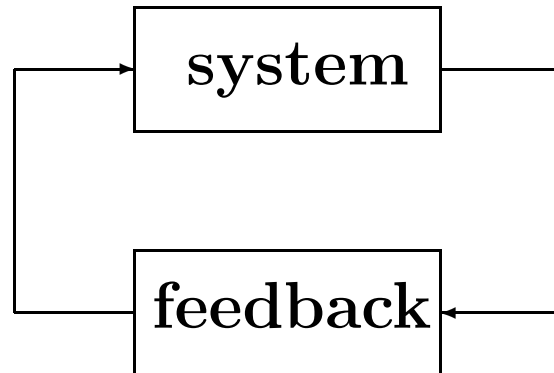
has $x = 0$ uniformly globally asymptotically stable
 \forall time-varying feedback laws Δ s.t. $|\Delta(t, x)| \leq \rho(|x|)$



intuition: for u “smaller than x ” the β term dominates

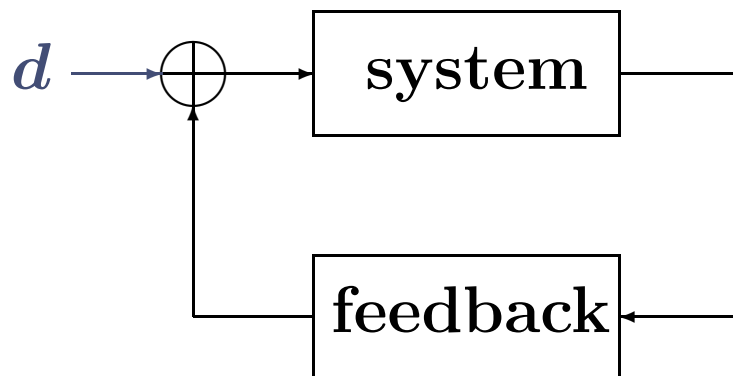
Feedback Redesign

suppose $\dot{x} = f(x, u)$ stabilized under feedback $u = k(x)$:



$$\dot{x} = f(x, k(x))$$

but what is effect of actuator disturbances $d(\cdot)$?



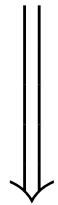
$$\dot{x} = f(x, k(x) + d)$$

noise may destabilize (not ISS) !

Example

may happen under feedback linearization design:

$$\dot{x} = x + (x^2 + 1)u$$



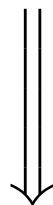
$$u := \frac{-2x}{x^2 + 1} + d$$

$$\dot{x} = -x + (x^2 + 1)d$$

$$\text{not ISS: } u(t) = \frac{1}{\sqrt{2t+2}} \not\rightarrow x(t) \rightarrow 0$$

but if, instead:

$$\dot{x} = x + (x^2 + 1)u$$



$$u := \frac{-2x}{x^2 + 1} \boxed{-x} + d$$

$$\dot{x} = -2x + \boxed{x^3} + (x^2 + 1)d$$

this is still stable when $u \equiv 0$

but in addition is now ISS: $-x^3$ dominates $c(x^2 + 1)$

General Theorem

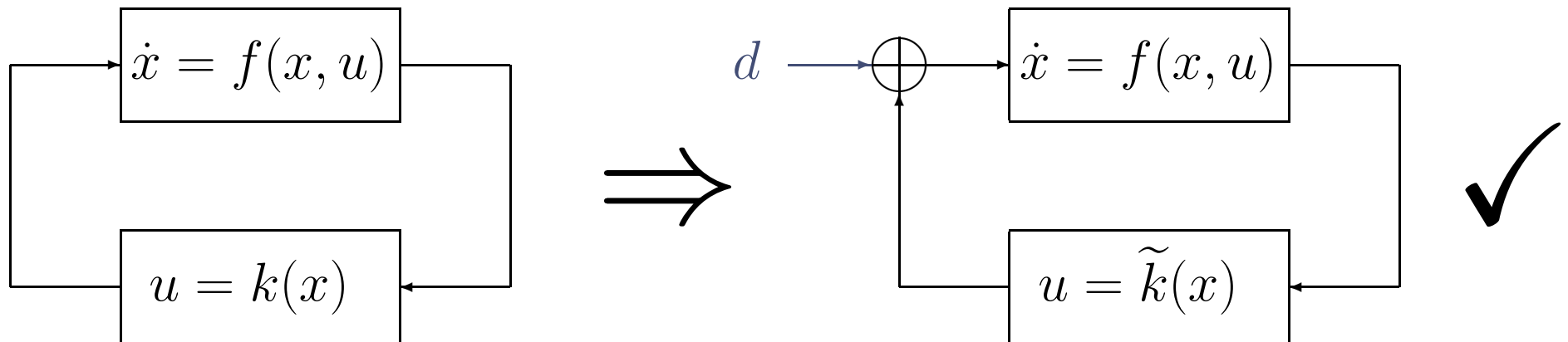
$$\dot{x} = f(x, u) = g_0(x) + \sum_{i=1}^m u_i g_i(x) \quad (g_0(0) = 0)$$

suppose \exists feedback $u = k(x)$ s.t.

$\dot{x} = f(x, k(x))$ has $x = 0$ as GAS equilibrium

then \exists feedback $u = \tilde{k}(x)$ s.t.

$\dot{x} = f(x, \tilde{k}(x) + d)$ is ISS with input $d(\cdot)$



e.g. corollary: feedback linearizable \Rightarrow ISS-stabilizable
(not completely obvious - one of original motivations)

ISS Superposition Principle

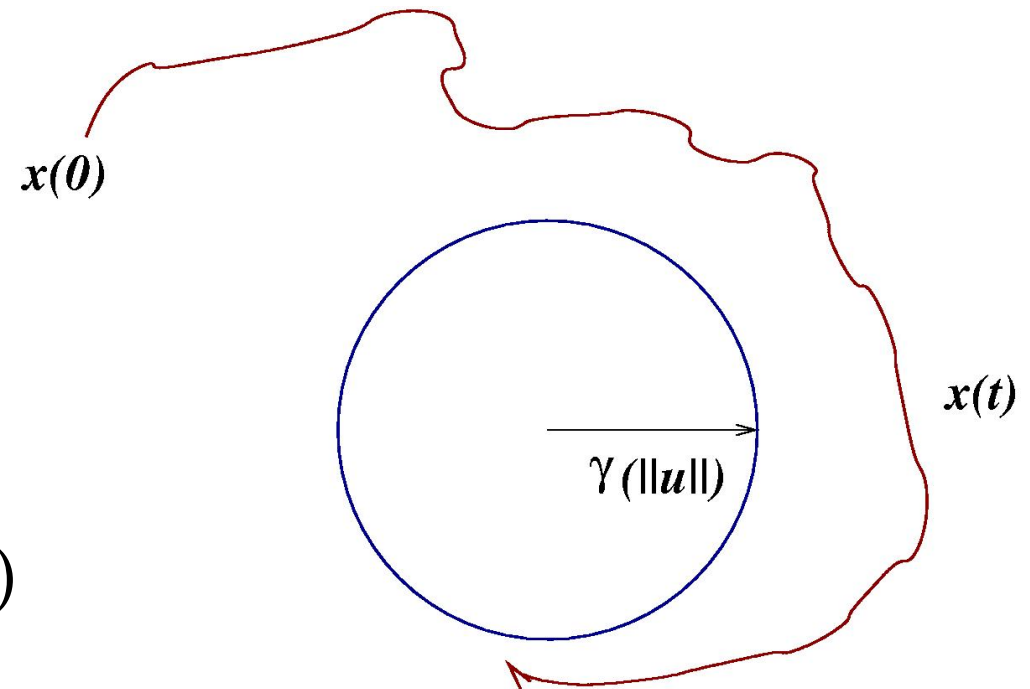
$$|x(t)| \leq \max \{ \beta(|x^0|, t), \gamma(\|u\|_\infty) \}$$

$$\Rightarrow |x(t)| \leq \beta(|x^0|, t)$$

when $u \equiv 0$ (GAS) &

\exists asymptotic gain $\gamma \in \mathcal{K}_\infty$

$$\overline{\lim}_{t \rightarrow +\infty} |x(t, x^0, u)| \leq \gamma(\|u\|_\infty)$$



Theorem: ISS $\iff \exists$ asympt gain & unforced sys GAS

nontrivial because of non-uniformity on controls; no (even weak) compactness

Dissipation Characterization

$V : \mathbb{R}^n \rightarrow \mathbb{R}$ is an *ISS-Lyapunov function* for $\dot{x} = f(x, u)$ if smooth, proper, positive definite, and $\exists \gamma, \alpha \in \mathcal{K}_\infty$ s.t.:

$$\dot{V}(x, u) = \nabla V(x) f(x, u) \leq -\alpha(|x|) + \gamma(|u|) \quad \forall x, u$$

i.e., dissipation inequality

$$V(x(t_2)) - V(x(t_1)) \leq \int_{t_1}^{t_2} w(u(s), x(s)) ds$$

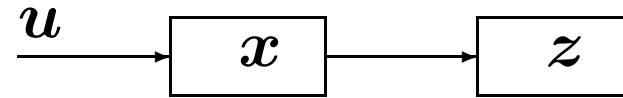
holds along all trajectories of the system,
with “supply” function $w(u, x) = \gamma(|u|) - \alpha(|x|)$

Theorem: ISS $\iff \exists$ ISS-Lyapunov function

generalizes converse Lyapunov theo: $u = 0 \rightsquigarrow$ Lyap function

ISS and Interconnections

$$\begin{aligned} \dot{z} &= f(z, x) & \text{ISS (} x \text{ input)} \\ \dot{x} &= g(x, u) & \text{ISS (} u \text{ input)} \end{aligned}$$



$$\begin{aligned} \dot{V}_1(z, x) &\leq \theta(|x|) - \alpha(|z|) & \text{matching ISS-Lyapunov} \\ \dot{V}_2(x, u) &\leq \tilde{\theta}(|u|) - 2\theta(|x|) & \text{functions for subsystems} \end{aligned}$$

$\rightsquigarrow W(x, z) := V_1(z) + V_2(x)$ is ISS-Lyap for cascade:

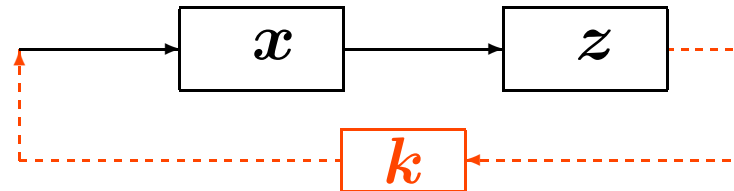
$$\dot{W}(x, z) \leq \tilde{\theta}(|u|) - \theta(|x|) - \alpha(|z|) \quad \checkmark$$

\Rightarrow **cascade of ISS is ISS** (for $u = 0$: GAS & ISS \Rightarrow GAS)

generalization:

small-gain theorem

$u = k(z)$, k small

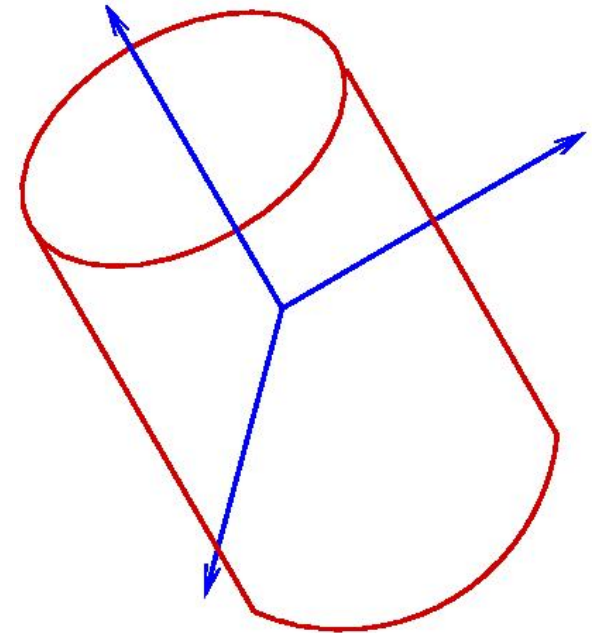


$$\tilde{\theta}(|u|) \leq (1-\varepsilon)\alpha(|z|) \rightsquigarrow \dot{W}(x, z) \leq -\theta(|x|) - \varepsilon\alpha(|z|) < 0 \quad \checkmark$$

E.g. Cascades

academic “application” just for illustration of ideas
angular momentum stabilization of rigid body
controls = two torques acting along principal axes

$\omega = (\omega_1, \omega_2, \omega_3)$ angular velocity
body-attached frame wrt inertial coords
 $I = \text{diag}(I_1, I_2, I_3)$ principal
moments of inertia



$$I\dot{\omega} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} I\omega + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} v$$

assume $I_2 \neq I_3$

Change State & Input Coords

$$(I_2 - I_3)x_1 = I_1\omega_1, x_2 = \omega_2, x_3 = \omega_3, I_2u_1 = (I_3 - I_1)\omega_1\omega_3 + v_1, I_3u_2 = (I_1 - I_2)\omega_1\omega_2 + v_2$$

$$\dot{x}_1 = x_2x_3$$

$$\dot{x}_2 = u_1$$

$$\dot{x}_3 = u_2$$

this feedback globally stabilizes when $v_1 = v_2 \equiv 0$:

$$u_1 = -x_1 - x_2 - x_2x_3 + v_1$$

$$u_2 = -x_3 + x_1^2 + 2x_1x_2x_3 + v_2$$

because, with $z_2 := x_1 + x_2, z_3 := x_3 - x_1^2$:

$$\dot{x}_1 = -x_1^3 + \alpha(x_1, z_2, z_3)$$

$$\dot{z}_2 = -z_2 + v_1$$

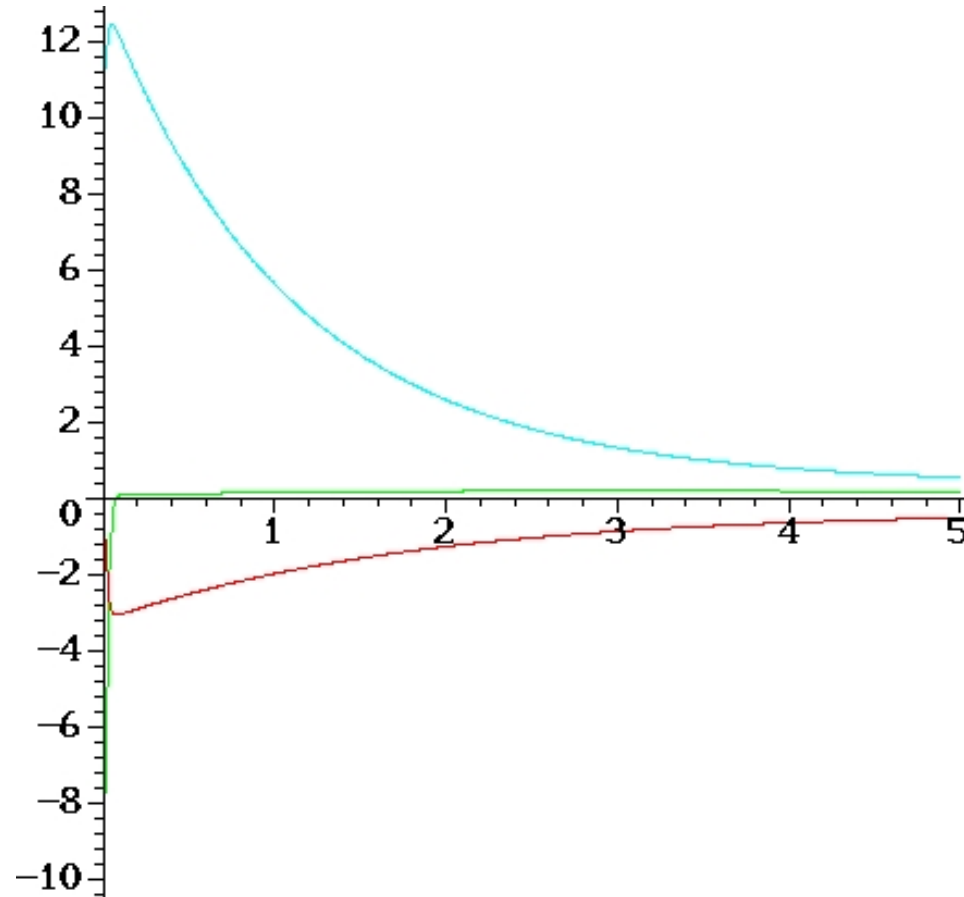
$$\dot{z}_3 = -z_3 + v_2$$

x_1 -subsys ISS: $\deg_{x_1} \alpha \leq 2$ so cubic dominates for $x_1 \gg 1$

thus cascade ISS; GAS when $v_1 = v_2 \equiv 0$

(& proved much more: global robustness to actuator error)

Sample Closed-Loop Solution



($x_1(0) = 0$, overshoots; rest start at ± 10)

Nonlinear “ H_∞ ”

ISS motivation coord changes from “ $L^\infty \rightarrow L^\infty$ ” estimates

what about $L^2 \rightarrow L^2$ (“ H_∞ ”)?

$$\int_0^t |x(s)|^2 ds \leq c|x^0|^2 + c \int_0^t |u(s)|^2 ds \quad \forall t \geq 0$$

leads to (for appropriate comparison functions):

$$\int_0^t \gamma_1(|x(s)|) ds \leq \kappa(|x^0|) + \int_0^t \gamma_2(|u(s)|) ds \quad \forall t \geq 0$$

Theorem: ISS $\iff \exists$ an \int / \int estimate

Neat Little Theorem

Theorem: if $n \neq 4, 5$, ISS \Rightarrow under coordinate changes

$$\int_0^t |x(s)|^2 ds \leq |x(0)|^2 + \int_0^t |u(s)|^2 ds \quad (\text{finite } H_\infty \text{ gain})$$

(“coord change” = homeomorphism & \mathcal{C}^1 except at zero)

similarly: global *exponential* stability \equiv asymptotic stab
— emphasizes that ISS is to lin gains as Lyap is to eigens
(no CM obstruction: coord change not necessarily \mathcal{C}^∞ at 0)

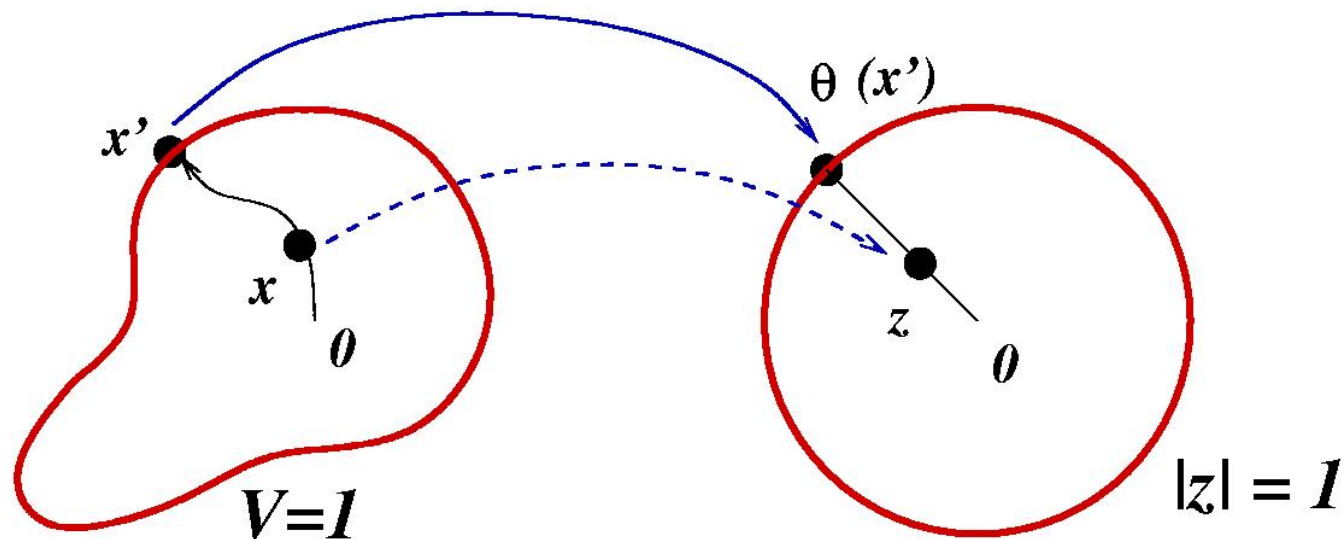
cases $n = 4, 5$ are still open

key steps: introduce an ISS-Lyapunov function; prove that every Lyapunov function is quadratic under coord change (!)

uses Poincaré, more specifically Smale/Milnor h -cobordism

Proof Sketch: V 's Quadratic

levels $S := \{V(x) = 1\}$ homotopically equivalent to \mathbb{S}^{n-1} :
 $S \simeq S \times \mathbb{R}$ (\mathbb{R} contractible) & $S \times \mathbb{R} \approx \mathbb{R}^n \setminus \mathbf{0} \simeq \mathbb{S}^{n-1}$ (flow)
 $\Rightarrow \{V(x) = 1\}$ diffeomorphic to \mathbb{S}^{n-1} , provided $n \neq 4, 5$
(h -cobordism; Poincaré would give homeomorphism if $n \neq 4$)



go to level set via an appropriate normalized gradient flow
and then use diffeomorphism $\theta : \{V = 1\} \simeq \{|z| = 1\}$

must adjust s.t. smooth away from 0 and \mathcal{C}^0 at 0; must apply robustness characterization to \rightsquigarrow UGAS problem
for $n \geq 6$ sublevel set is compact, connected smooth manifold with a simply connected boundary \rightsquigarrow diffeo to disk

A Different Notion: iISS

most concepts equivalent to ISS ... a new one which is *not* obtained as a nonlinear version of “finite $L^2 \rightarrow L^\infty$ gain”

$$\gamma_1(|x(t)|) \leq \beta(|x^0|, t) + \int_0^t \gamma_2(|u(s)|) ds$$

iISS = *integral input* to (sup-norm) state stability

$V : \mathbb{R}^n \rightarrow \mathbb{R}$ is an *iISS-Lyapunov function* for $\dot{x} = f(x, u)$ if smooth, proper, pos def, and $\exists \gamma \in \mathcal{K}_\infty$ & α pos def s.t.:

$$\dot{V}(x, u) = \nabla V(x) f(x, u) \leq -\alpha(|x|) + \gamma(|u|) \quad \forall x, u$$

observe: not asking $\alpha \in \mathcal{K}_\infty$: even for constant u may have $\dot{V} > 0$, but $\gamma(|u|) \in \mathcal{L}^1$ means \dot{V} is “often” negative

Theorem: iISS $\iff \exists$ iISS-Lyapunov function

ISS \implies iISS, but converse false, e.g. bilinear $\dot{x} = (A + \sum_{i=1}^m u_i A_i)x + Bu$ is iISS if A Hurwitz, but not ISS, e.g. $\dot{x} = -x + ux, u \equiv 2, \therefore$ not BIBS

iISS and Dissipation

particularly useful if dissipation only wrt some variables

say system *h-dissipative* wrt output function $y = h(x)$ if

$$\nabla V(x) f(x, u) \leq -\alpha(h(x)) + \gamma(|u|) \quad (\exists V, \alpha, \gamma)$$

and *h-detectable* if $y(t) = h(x(t)) \equiv 0 \Rightarrow x(t) \rightarrow 0$ as $t \rightarrow \infty$

Theorem: iISS $\iff \exists h$ s.t. *h-detectable* and *h-dissipative*

e.g.'s with tracking control, iISS wrt time-varying signals in larger class than designed for; $V =$ energy, $y =$ velocities

Input/Output Stability

input to output stability (IOS)

for systems with outputs $\dot{x} = f(x, u)$, $y = h(x)$:



$$|y(t)| \leq \max \{ \beta(|x^0|, t), \gamma(\|u\|) \}$$

observe: overshoot bounded by $|x^0|$, as in regulator theory

closely related to “partial stability” (subset of variables)

\exists necessary and sufficient dissipation characterization, too

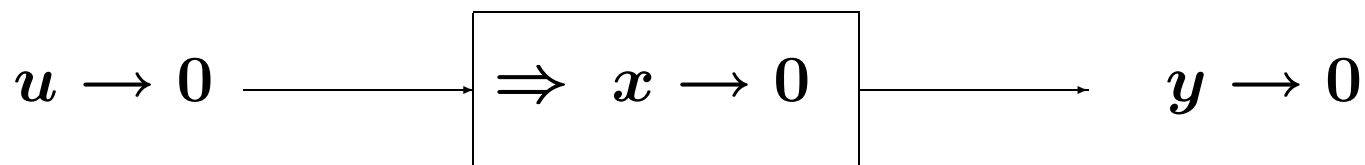
Zero-Detectability: IOSS

zero-detectability typically “ $u \equiv 0 \ \& \ y \equiv 0 \Rightarrow x(t) \rightarrow 0$ ”
— this is a bit weak for nonlinear systems:
not “well-posed” (what happens if $u, y \approx 0$?)

input/output to state stability (IOSS):

$$|x(t)| \leq \max \{ \beta(|x^0|, t), \gamma_1(\|u\|), \gamma_2(\|y\|) \}$$

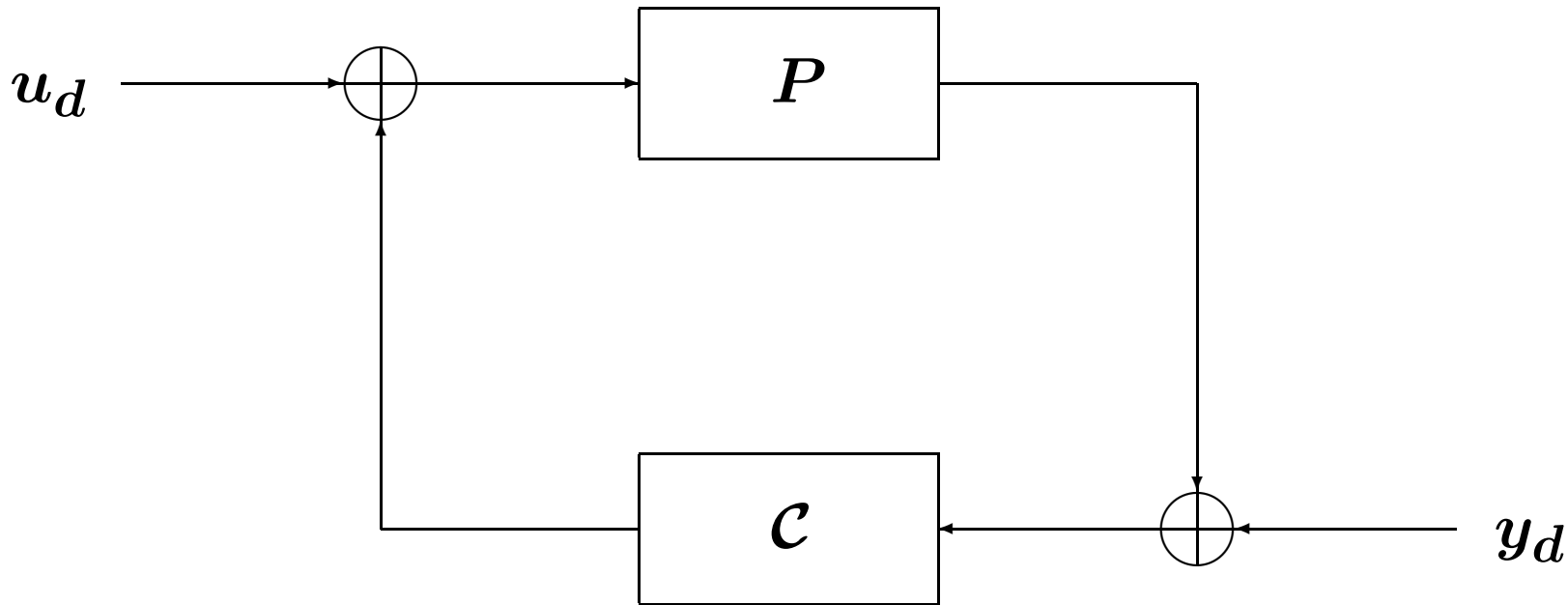
(terminology: stability from the i/o data to the state)
(more precisely: norms restricted to $[0, t]$)



Theorem: IOSS \iff IOSS-Lyapunov function

$$\nabla V(x) f(x, u) \leq -\alpha_1(|x|) + \alpha_2(|u|) + \alpha_3(|y|)$$

Output Stabilization \Rightarrow IOSS



$\exists \mathcal{C}$ ISS stabilizing w.r.t. external “disturbances” u_d and y_d

\Rightarrow original system is IOSS

assuming $y_e \equiv 0 \Rightarrow u_e \equiv 0$

for any initial state ξ and control u , pick $u_d := u$ and $y_d := -y_{\xi,u}$

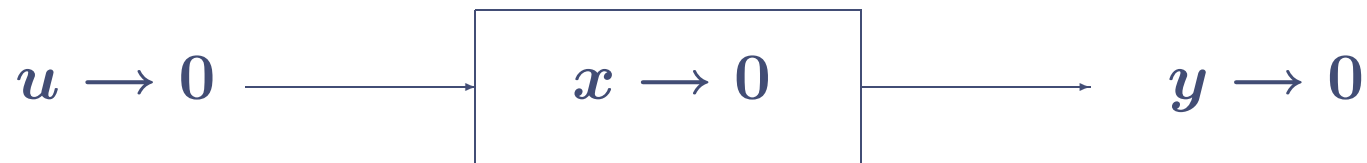
Relation Between Concepts

basically tautological, *given the definitions*:

$$\boxed{\text{IOS \& IOSS} \iff \text{ISS}}$$

i.e., intuitively:

external stability & detectability \iff internal stability



if internally stable, then $u \rightarrow 0 \Rightarrow x \rightarrow 0$,
so in particular this happens when $y(t) \rightarrow 0$ (detectability)
and it always holds that $y(t) \rightarrow 0$ (i/o stability)

conversely, if $u \rightarrow 0$ then $y \rightarrow 0$ (by external stability),
and this then implies $x \rightarrow 0$ (by detectability)

An Example of “ISSomics”

e.g. of systematic use in output feedback stabilization
(Arcak and Kokotovic, Automatica Dec. 2001)

suppose system has equations:

$$\dot{x} = f(x, z, u)$$

$$\dot{z} = g(x, z)$$

and only an output $y = h(x)$ available for stabilization

view z -subsystem as unknown (robust design)

design state-feedback law $u = k(x)$ and

observer, producing estimate \hat{x} such that:

- error $e = x - \hat{x}$ is ISS with respect to z
- $\dot{x} = f(x, z, k(\hat{x})) = F(x, z, e)$ is ISS with respect to e, z
- $\dot{z} = g(x, z)$ is ISS with respect to x

\rightsquigarrow (w/small-gain condition) stability of entire system

A&K e.g.: Jet Engine Stall

(single-mode approximation of) Moore-Greitzer PDE

$$\dot{\phi} = -\psi + (3/2)\phi + 1/2 - (1/2)(\phi + 1)^3 - 3(\phi + 1)R$$

$$\dot{\psi} = \frac{1}{\beta^2}(\phi + 1 - u)$$

$$\dot{R} = \sigma R(-2\phi - \phi^2 - R) \quad (R \geq 0)$$

ϕ = mass flow relative to setpoint

ψ = pressure rise relative to setpoint

R = magnitude of first stall mode

view R as “ z ” (unmodeled); measurement is ψ (pressure)

complete GAS design based on above & new observer design
– veritable tour de force based on circle criterion/LMI's

Incremental (“ Δ ”) ISS

$$|x_1(t) - x_2(t)| \leq \max\{\beta(|x_1^0 - x_2^0|, t), \gamma_1(\|u_1 - u_2\|)\}$$

again, cascades are well-behaved:

$$\begin{array}{ll} \dot{x} = f(x, y, u) & \Delta\text{-ISS wrt } (y, u) \\ \dot{y} = g(y, u) & \Delta\text{-ISS wrt } u \end{array}$$

\Rightarrow overall system is Δ -ISS with respect to input u

some applications:

- master-slave synchronization
(chaotic systems, secure communications)
- observers

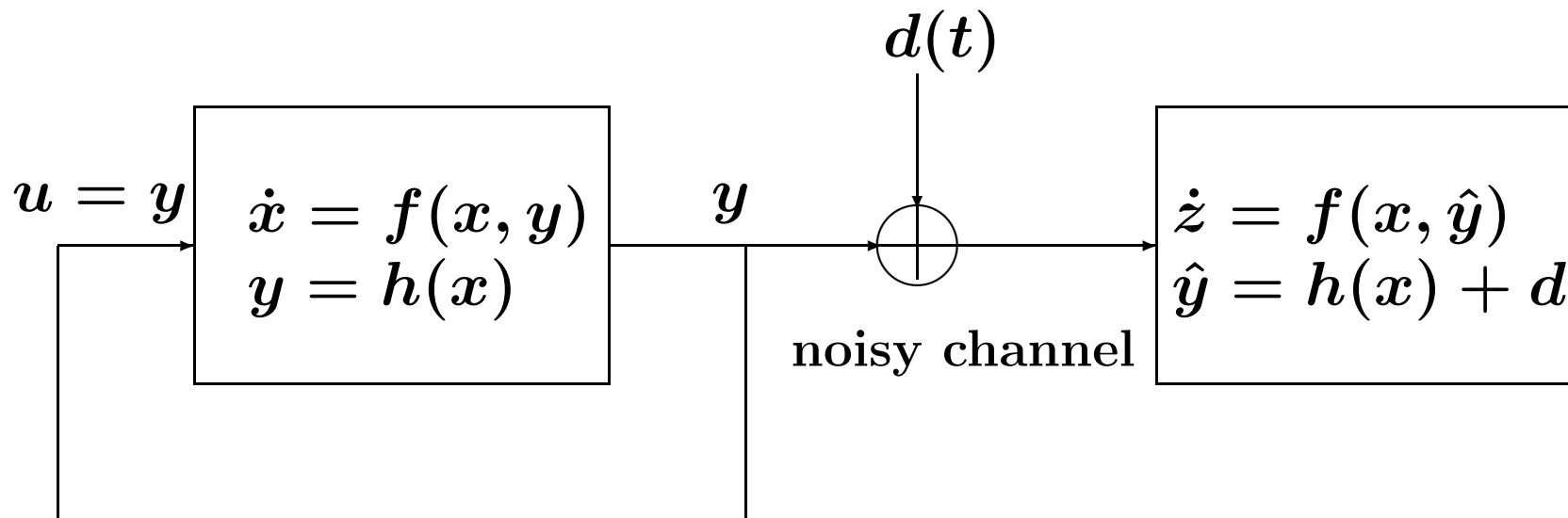
example from Angeli, TAC’02 of former:

Synchronization

suppose $\dot{x} = f(x, u)$ is Δ -ISS

consider any feedback law $u = h(x)$

and driven copy of system & possible transmission noise:



then states synchronize:

$$|x(t) - z(t)| \leq \max\{\beta(|x^0 - z^0|, t), \|d\|\}$$

(“master-slave”, not symmetric oscillator coupling)

numerical example next

Lorentz Attractor

$$\dot{x}_1 = -\beta x_1 + \text{sat}(x_2)\text{sat}(x_3)$$

$$\dot{x}_2 = \sigma(x_3 - x_2)$$

$$\dot{x}_3 = -x_3 + u$$

$$y = \rho x_2 - x_1 x_2$$

where $\beta = 8/3$, $\sigma = 10$, $\rho = 28$

(saturation here for technical reasons - does not affect application)

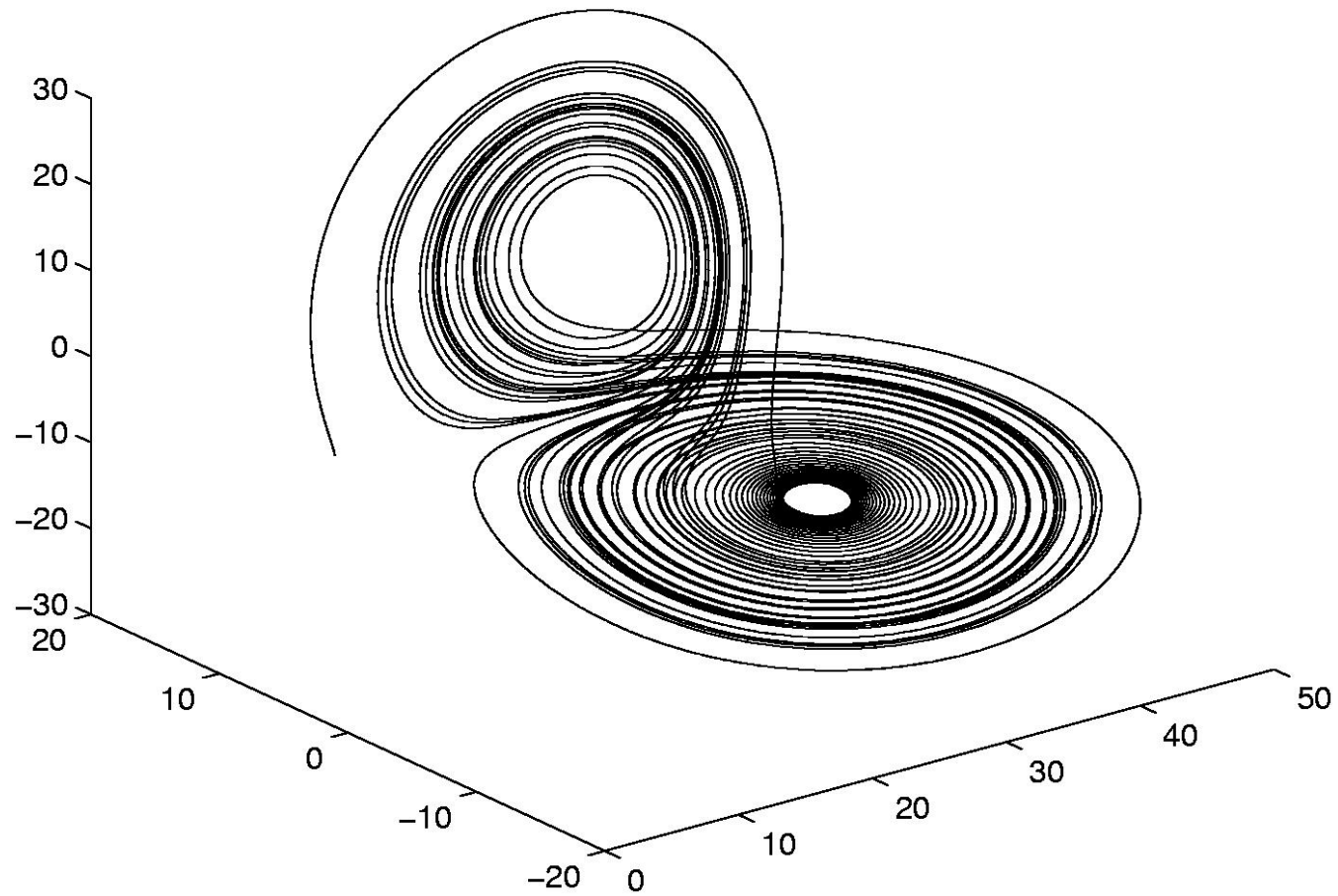
can view system as *cascade* of Δ -ISS:

x_1 subsystem linear & stable; Lipschitz function of input and x_2, x_3 subsystem linear stable

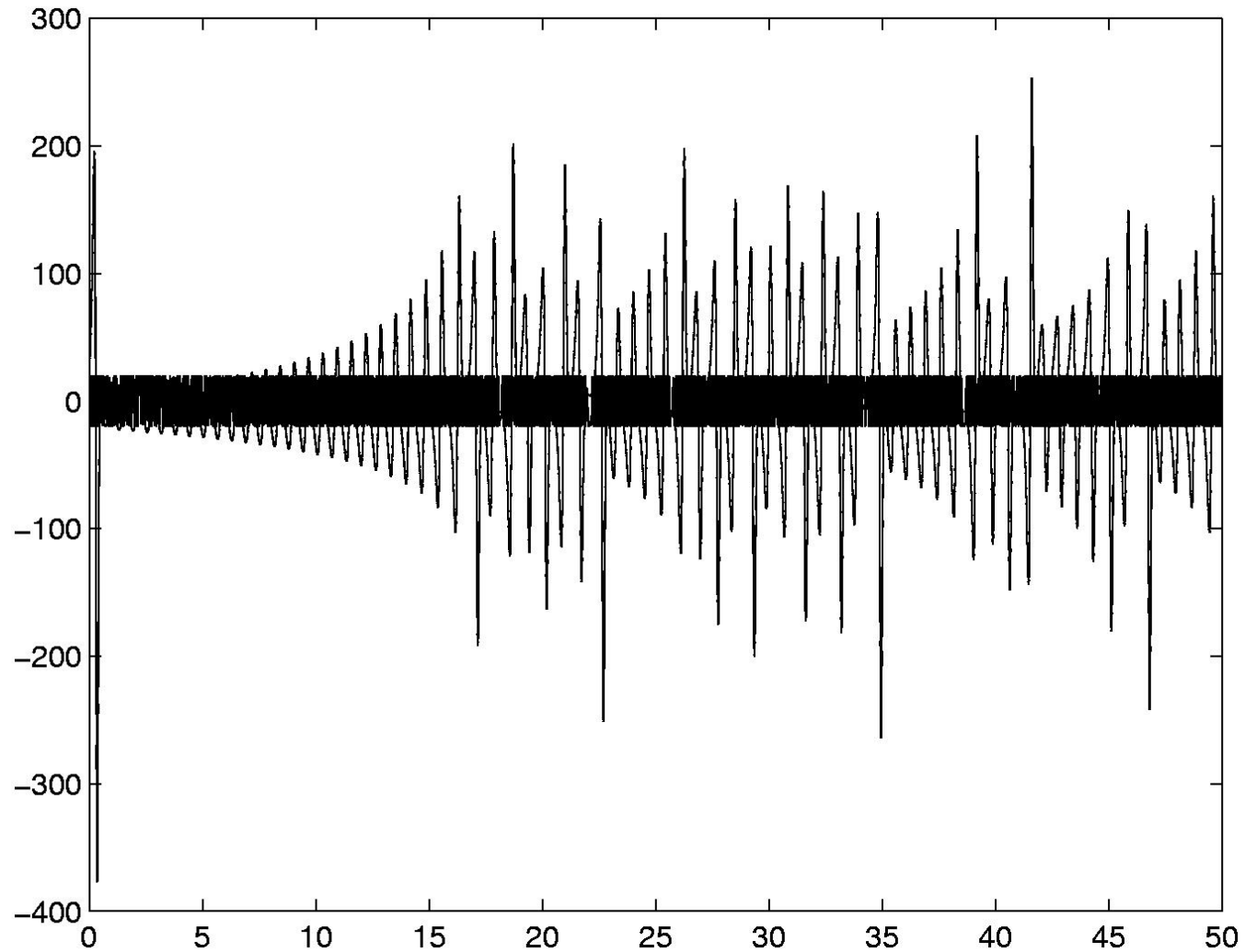
\Rightarrow system is Δ -ISS, and theory applies

simulations:

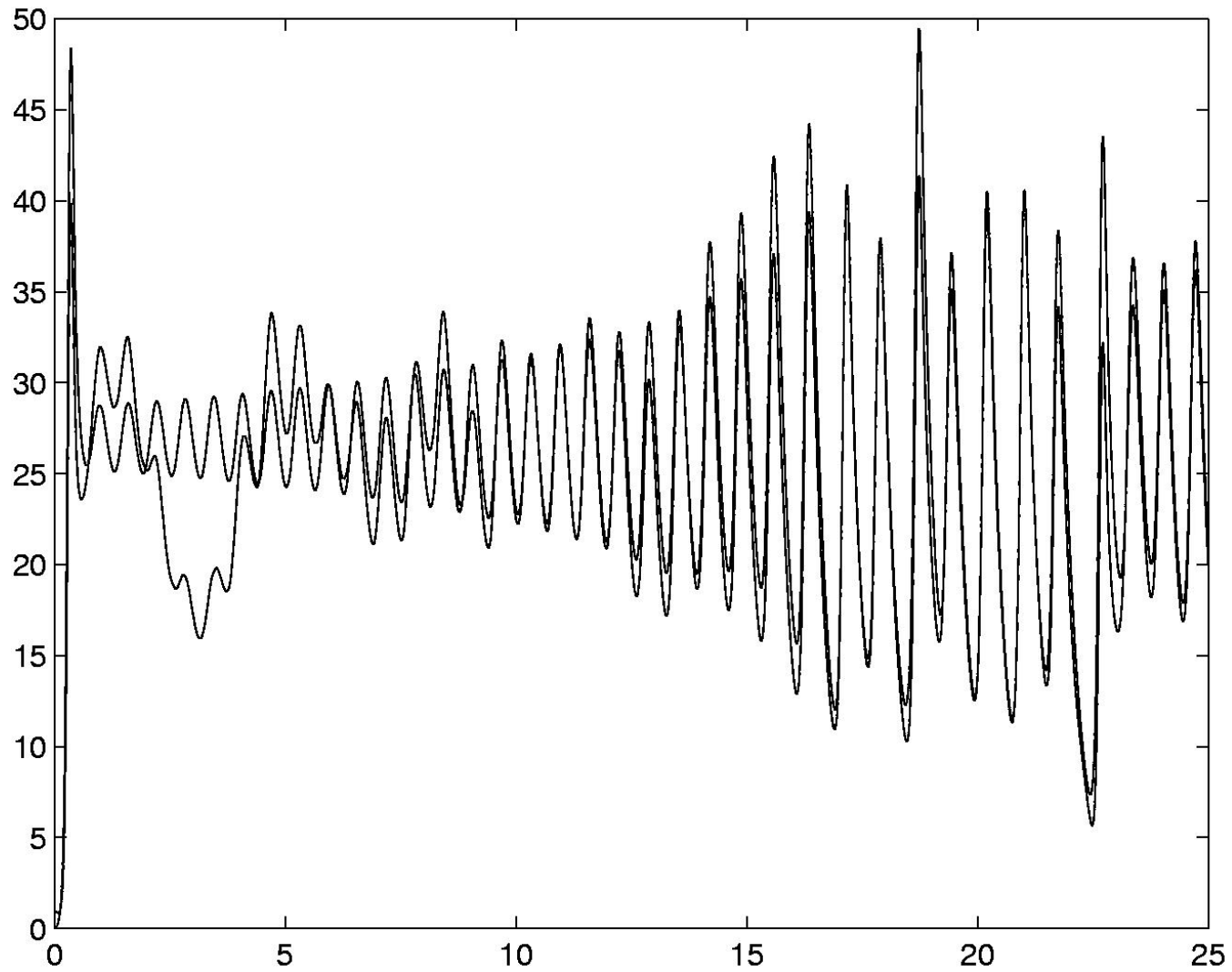
Trajectory of System



Signal and Noise

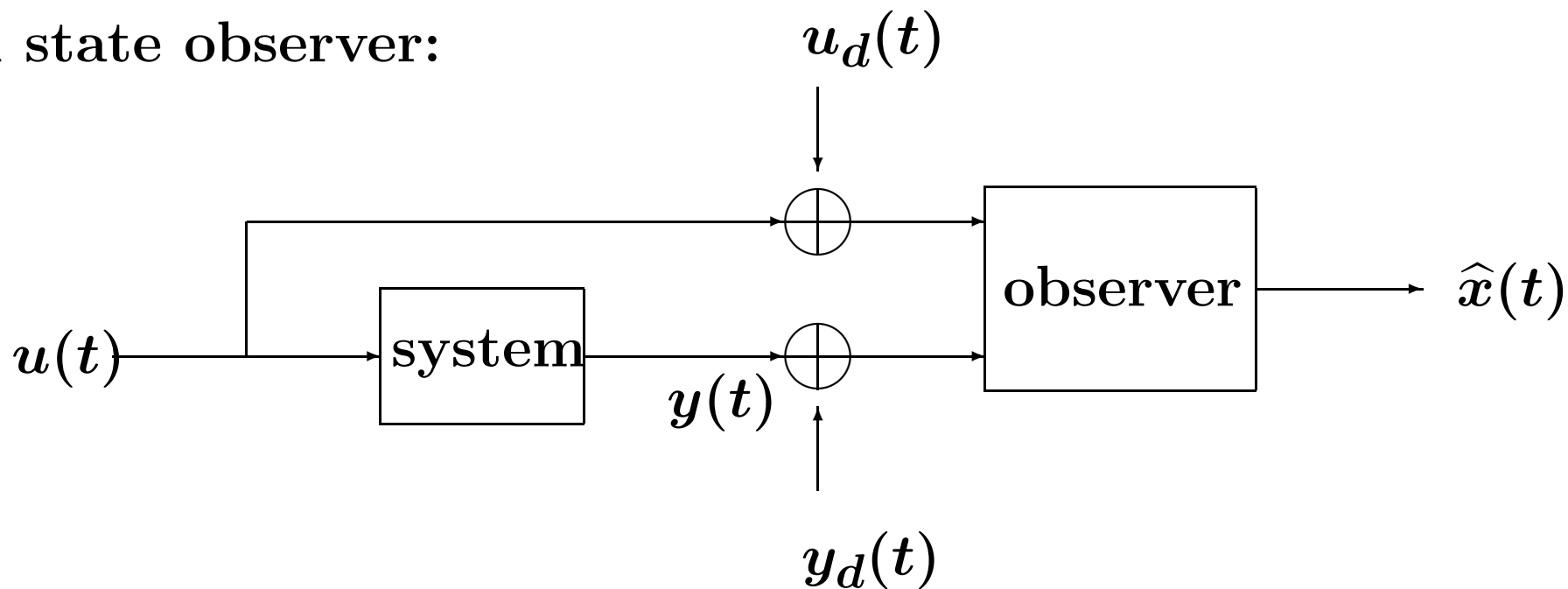


x_1 VS z_1



Remark: Observers

full state observer:



$$|x(t) - \hat{x}(t)| \leq \max \left\{ \beta(|x^0 - \hat{x}^0|, t), \gamma_1(\|u_d\|), \gamma_2(\|y_d\|) \right\}$$

implies *incremental* (“ Δ ”) *IOSS*:

$$|x_1(t) - x_2(t)| \leq \max \left\{ \beta(|x_1^0 - x_2^0|, t), \gamma_1(\|u_1 - u_2\|), \gamma_2(\|y_1 - y_2\|) \right\}$$

(more precisely: norms restricted to $[0, t]$)

Comments

there are many foundational directions still being explored
– just a few of them:

- ISS wrt measurement error: $\dot{x} = f(x, k(x + u))$
– this would allow estimators in feedback loop
- development of basic theorems on incremental IOSS
- a common generalization of IOSS and IOS is,
for “regulated” and “measured” outputs w, y :
$$|w(t)| \leq \beta(|x^0|, t) + \sup_{s \in [0, t]} \gamma(|u(s)|) + \sup_{s \in [0, t]} \gamma(|y(s)|)$$

theory needs major effort
- infinite-dimensional theory open
- ...

focused on core theoretical constructs; for more theory see
textbooks by Isidori, Kokotovic et. al., Khalil, ...

Summary

- linear stability, coord changes; $\dot{x} = f(x, 0)$ stable not enough
- ISS \equiv Robust Stability
- “robustifying” feedback laws
- superposition principle
- dissipation characterization
- interconnections, cascades, sgt
- “nonlinear H_∞ ” \equiv ISS; $n \neq 4, 5$ little theorem
- integral ISS, dissipation
- input/output stability
- zero-detectability: IOSS; output stabilization \Rightarrow IOSS
- external stability & detectability \iff internal stability
- engine stall example
- incremental (“ Δ ”) ISS; synchronization; full state observer

Details & Background:

<http://www.math.rutgers.edu/~sontag>

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