A CHARACTERIZATION OF ASYMPTOTIC CONTROLLABILITY

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ABSTRACT

A Lyapunov-like function is used to characterize the asymptotic controllability of nonlinear control systems.

1. INTRODUCTION

The systems to be considered are described by controlled differential equations of the type

(1.1) $$(dx/dt)(t) = f(x(t),u(t)),$$

with states $x(t)$ in an Euclidean space $\mathbb{R}^n$ and control values $u(t)$ in a metric space U. The map f is assumed locally Liptchitz, and suitable Caratheodory-type conditions are assumed to hold, insuring existence and uniqueness of solutions for small times. Admissible ("ordinary") control functions u(.) are all measurable and locally essentially bounded functions defined on real intervals. Let W be the set of probability measures on U, topologized using the weak topology. The space of relaxed controls is the space of measurable maps from real intervals into W, topologized as usual in optimal control theory (see, e.g., Warga [1972]).

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The system (1.1) is asymptotically controllable in case the following properties hold:

(1.2) for each state x there exists an (ordinary) control for which the corresponding solution $\phi(t,x,u)$ is defined for all t, and converges to zero;

(1.3) if $\varepsilon > 0$ is given, there is a $\delta > 0$ such that, for any x of norm less than δ, there is a u as in (1.2) with the ensuing trajectory having $\|x(t)\| < \varepsilon$ for all t;

(1.4) there is a neighborhood L of the zero state, and a compact subset K of U such that, if $x(0)$ is in L, there exists an input as in (1.3) with values in K a.e..

For interpretations of these definitions, and for proofs of the results to be stated, see Sontag [1981].

A Lyapunov-like function for the system (1.1) is a real function V defined on the state set R^n such that:

(1.5) V is continuous;

(1.6) $V(x) > 0$ for $x \neq 0$, $V(0) = 0$;

(1.7) $\{x \mid V(x) < a\}$ is bounded for all a;

(1.8) for each $x \neq 0$ there is a relaxed control u with $V'(x,u) < 0$;

(1.9) there is a neighborhood L of the zero state and a compact subset K of U such that, for states x in L, the control in (1.8) can be chosen with values in K (more precisely, the measures $u(t)$ are supported in K a.e.).

The meaning of the derivative along the chosen trajectory is:

$$(1.10) \qquad V'(x,u) := \liminf_{t \to 0^+} [(1/t)(V(\phi(t,x,u)) - V(x))]$$

2. THE MAIN RESULT

Theorem 2.1. The system (1.1) is asymptotically controllable if and only if there is a Lyapunov-like function for it.

We sketch the idea of the proof. The sufficiency is the easier (and rather standard) part. One considers the infimum of the set of values V(a) of states reachable from a given state x. This infimum has to be zero; otherwise property (1.8) allows further diminishing of V(a) (relaxed controls can be approximated arbitrarily close by ordinary ones). But then states converge to zero under suitably chosen controls.

For the necessary part, one starts by constructing a function N(x) on states with the property that, along suitably chosen trajectories, the integral of N(ϕ(t,x, u)) is finite and further, this integral is small for small initial states x. The "suitably chosen trajectories" do not reach far from the origin for small x(0), and in general are required to reach neighborhoods of the origin in times uniformly bounded on compacts. The functional N is also constructed in such a way that it has very large values away from the origin. Once this function N is constructed, a cost function R(x,w) is associated to each state x and relaxed control w; this is just the integral of N along the corresponding trajectory ϕ(t,x,w) plus a term measuring input magnitudes. (If the corresponding solution has a finite escape time, just let R(x,w) be infinite.) The functional V is now the Bellman function for R, i.e.,

$$V(x) := \inf \{R(x,w) \mid w \text{ relaxed control}\}. \qquad (2.2)$$

This function is finite at all x, and satisfies all the required properties. In particular, continuity follows from the integral being small on trajectories, if close to the origin, and the (easier to establish) lower semicontinuity of V. The derivative condition is easy to establish once that it is shown that the inf in (2.2) is always achieved. This in turn follows from the fact that, for bounded states, there is a bound on input magnitudes that need be considered (recall the term included in R), and sequential compactness of the set of relaxed controls on compact control sets. (Trajectories are well defined for the minimizing w, because the function N is large far from the origin, implying bounded trajectories when using w.)

The above gives the general ideas of the proof; for details see the reference given earlier. The proof reduces in the classical case (U = one point) to a usual inverse-Lyapunov argument, but is complicated here by the necessary choice of controls.

REFERENCES

[1] E. D. Sontag, *A Lyapunov-like characterization of asymptotic controllability*, to appear, 1981.

[2] J. Warga, *Optimal control of differential and functional equations*, Academic Press, N.Y., 1972.

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