TIME-OPTIMAL CONTROL OF MANIPULATORS

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ABSTRACT

This paper studies time-optimal control questions for a certain class of nonlinear systems. This class includes a large number of mechanical systems, in particular, rigid robotic manipulators with torque constraints. As nonlinear systems, these systems have many properties that are false for generic systems of the same dimensions.

1. Introduction.

A recent paper by the authors ([SS]) established a number of results about the time optimal control problem for the two link rigid rotational manipulator model described in Paul's book ([PA], equations 6.16 and 6.20). It became evident later that many of the results could in fact be derived just from the general form of the equations of a manipulator, and in fact that a great deal of these results apply to (rigid) manipulators with more than two links as well as to a rather general class of nonlinear systems. This class of systems, which we shall call *mechanical systems with full control*, is characterized by the fact that, as with many Lagrangian formulations of mechanical systems, the evolution equations arise from a nonsingular set of second order differential equations (an Euler-Lagrange equation, typically) for a set of variables ("positions"), and are such that derivatives of these variables ("velocities") appear only quadratically. (Alternatively, a Hamiltonian formulation is also possible.) This note will establish a number of basic optimal control results for mechanical systems, and then apply them to a two-link manipulator model. We view this work as only a (small) first step towards the understanding of this class of systems.

Of course, we do not mean to imply that our "mechanical systems" encompass all possible models of control systems in mechanics. For instance, certain types of frictional effects cannot be included in such models.

Mechanical systems -in the sense of this paper- constitute a very restrictive class of systems when viewed in the context of general nonlinear systems. In particular, the Lie algebra of vector fields associated to such a system must satisfy a large number of nongeneric relations. Since the structure of this Lie algebra characterizes most interesting optimal control properties, one can expect, and indeed one finds, many properties of the time optimal problem for these systems which are false in general for nonlinear systems of the same order. For instance, it is almost trivial to establish that if all controls except possibly one are singular along an extremal, then the remaining control cannot be singular, and if fact must be bang-bang.

Most of the results that we obtain are about the singular structure of the optimal control problem, rather than about optimal controls themselves. The study of the singular structure of the problem is of great interest in itself, for the following reason. One of the main techniques used in practical robotic control consists in dividing the design effort into two stages: (1) find an open-loop control which achieves the desired state transfer, and (2) linearize along the resulting trajectory, and use a linear controller to regulate deviations from this trajectory. The essential point is that this last step will typically depend on controllability of the obtained linearization (as a time-varying linear system), and *a trajectory is singular precisely when this linearization is uncontrollable.* Thus, our characterizations of singular trajectories should help in determining if a trajectory suggested by step (1) is suitable for step (2).

The literature in (numerical) optimal control of manipulators is rather extensive; see for instance the papers [RA], [SD], and [SH], as well as the references there and other papers in the conference volume in which they appear. As far as we are aware, a systematic study of singularities as the one started here has not been attempted in previous work. We intend to direct further research both to theoretical topics and to the understanding of what implications our results have for the algorithms given in the literature. For instance, they may help in the "pruning" of possibilities in dynamic programing numerical methods.

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The only other theoretical work on this problem that we are aware of is that in [AL]. That paper is devoted mainly to the proof of existence of optimal controls (basically, one needs to establish that there are no finite escape times), but the last section proves that for a two link manipulator both controls cannot be simultaneously singular. Note that the result proved here, mentioned earlier, will imply a much stronger fact, namely, that one of the controls has only finitely many switchings. (The proof given in [AL] does not rule out a phenomenon like "Fuller's problem", which occurs in many systems, and in which optimal controls switch infinitely often in a bounded interval, nor for that matter even more pathological behaviors, like Cantor sets of discontinuities for the switching function.)

We shall not repeat most of the material in [SS], which should be consulted for further and more detailed discussions. The model in that reference was, as mentioned earlier, that in Paul's textbook. We have since noticed that there seems to be an error in the Lagrangian derivations in the book, which results in an extra term which cannot be disregarded if both links are moving at high speed. This means that a few of the statements in [SS], while true for the system studied in [PA], are not necessarily true for a two-link manipulator, -specifically, lemmas 4.5 and 5.4, and formulas 4.2-4.5,- though they can be in most cases modified in trivial ways. The theorems in [SS] are still valid (with signs interchanged in 6.1), using the corrected formulas, and in fact will be established in more generality in this note.

2. Mechanical systems with full control.

All vectors will be column vectors, but for printing purposes we shall often display them as rows.

A *(finite dimensional) mechanical system (with full control)* will be, for the purposes of this paper, a system defined by equations (omitting the time arguments for simplicity):

$$u = M(\theta)\ddot{\theta} + N(\theta,\dot{\theta}) ,$$

where θ is a vector (of positions) in $\Re^n$, u is a vector (of controls) in $\Re^n$, and where M is an $n\times n$ matrix of functions of θ, symmetric positive definite for each $\theta\in\Re^n$, and N is an n-vector of functions of θ and $\dot{\theta}$ with the property that, as functions of $\dot{\theta}$, each of its entries is quadratic, i.e. is a polynomial of degree at most 2. (In the robotics literature, N is usually displayed as a sum of two terms, N+Q, the first homogeneous of degree 2 in $\dot{\theta}$ and the second independent of $\dot{\theta}$; we are allowing also for linear terms in the velocities.) The entries u_i are bounded in magnitude:

$$L_i \leq u_i \leq M_i \ , \ i=1,\cdots,n,$$

where the $L_i < M_i$ are given constants. As a function of t, each u_i is measurable essentially bounded. We assume that all the functions of θ and $\dot{\theta}$ that appear are real-analytic (in most applications, functions belong to finitely generated algebras spanned by trigonometric functions and polynomials).

This model includes mechanical manipulators with rigid rotational links as well as many other systems of interest. (The "full control" qualifier refers to the fact that every degree of freedom can be independently controlled; certain lumped models for flexible arms, as well as models that include actuator dynamics, result in very similar equations but without this latter property.) We take the positions θ as belonging to $\Re^n$ rather than to a subset -or even a manifold like S^1, as is natural for some robotics problems-for notational simplicity; in any case, all of the results obtained depend on local methods.

The state space model associated to the above is given by equations

$$\dot{x} = f(x) + G(x)u \ , \tag{2.1}$$

where x is a vector in $\Re^{2n}$; denoting the first n entries of x as θ and last n entries as $\dot{\theta}$, f(x) is the 2n-vector

$$\begin{pmatrix} \dot{\theta} \\ -M(\theta)^{-1}N(\theta,\dot{\theta}) \end{pmatrix}$$

(thus as functions of $\dot{\theta}$ the last n coordinates of f are polynomials of degree at most 2) and

$$G(x) = \begin{pmatrix} 0 \\ L(\theta) \end{pmatrix},$$

with $L(\theta)$ being the inverse of $M(\theta)$ -hence also symmetric positive definite for each x.

Such systems are "linearizable under feedback", in the sense that the the transformation

$$u = N(\theta,\dot{\theta}) + M(\theta)v \ ,$$

where v is a new control, results in a set of decoupled double integrators. This transformation is typically used in control ("computed torque method", etc.); however in optimal control it does not seem to be useful, since the torque constraints get transformed into state-dependent constraints for the linear problem. A discussion is given in [SS] showing that full-control mechanical systems may have very different behavior than double integrators, in terms of certain degeneracies that may appear in optimal problems.

For each i=1,...,n, we shall let g_i denote the ith column of G; its entries are all functions of the first n coordinates θ of x, and are nonzero for every x (positive definiteness of L).

2.1. Basic Lie theoretic properties.

We shall identify functions $\Re^{2n}\to\Re^{2n}$ with vector fields on $\Re^{2n}$, and apply differential geometric notation. In particular, $ad_a(b) = [a,b]$ denotes the Lie bracket of the vector fields a and b, i.e. b'a-a'b, where prime indicates Jacobian. When we refer to the "ith coordinate" of a vector field, we mean coordinates with respect to above identification.

We denote by $\mathcal{G}$ the module over $C^\omega(\Re^{2n})$ generated by the vector fields $g_1,\cdots,g_n$, and by $\mathcal{L}$ the Lie algebra of vector fields generated by $\{f,g_1,\cdots,g_n\}$, , the vector fields appearing in the model (2.1); we also use the shorthand notation:

$$X = X_1X_2\cdots X_k := [X_1,[X_2,[\cdots,[X_{k-1},X_k]\cdots]]] \qquad (2.2)$$

for iterated Lie compositions of vector fields. Note that $X^kY = ad_X^kY$ with this notation. By an *iterated bracket* we shall mean an expression as above for which each X_i is one of f, g_1, ..., g_n. (We also say, abusing terminology, that the vector field X is an iterated bracket.) Given one such $X = X_1X_2\cdots X_k$, the *index* $\iota(X)$ of X is defined as the difference n_f-n_g, where n_f is the number of times that the vector field f appears in the expression, and n_g is the total number of g_i's, counting multiplicities.

With these notations, $fg_i = [f,g_i]$, i=1,⋯,n. Because of the form of f and of the vector fields g_i, it follows that the matrix $(fg_1,\cdots,fg_n)$ has the form

$$-\binom{L(\theta)}{*},$$

from which it follows that the set of vector fields

$$\{g_1,\cdots,g_n,fg_1,\cdots,fg_n\} \text{ constitutes a frame,} \qquad (2.3)$$

i.e. is everywhere linearly independent, and hence generates (as a module over analytic functions) the set of all analytic vector fields on the state-space $\Re^{2n}$.

Let Z_i denote the set of all functions on R^{2n} which, as functions of θ_1, ⋯, θ_n, are polynomials of degree at most i. (With the convention that $Z_i = \{0\}$ if i<0.)

Let S_i be the set of vector fields X with the property that the first n coordinates of X are in Z_i, and the last n coordinates are in Z_{i+1}. (So, for instance, vectors in S_{-1} have their first n coordinates zero and last n coordinates dependent only on θ, and hence belong to $\mathcal{G}$.) If X is in S_i and Y is in S_j, then [X,Y] belongs to S_{i+j}. The sets S_i provide a filtration of $\mathcal{L}$. In fact, since f is in S_1 and all g_i are in S_{-1}, by induction on the formation of iterated brackets it follows that:

Lemma 2.1: For any iterated bracket X, $X \in S_{\iota(X)}$∎

In particular, for all i,j,

$$[g_i,g_j] = 0 \qquad (2.4)$$
$$g_ifg_j \in \mathcal{G}.$$

3. Singular extremals.

An *extremal* on an interval I = [0,T] is given by functions (x,λ,u) on I which satisfy (2.1) together with

$$\dot\lambda = D[f(x)+G(x)u]\lambda \qquad (3.1)$$

(where D[⋯] denotes transpose Jacobian), $\lambda \neq 0$ everywhere, u is measurable essentially bounded, and, for each i=1,⋯,n and for almost every t where the *i-th switching function*

$$\phi_i(t) := \langle\lambda(t),g_i(x((t))\rangle$$

is nonzero,

$$u_i(t) = \begin{cases} M_i & if\ \phi_i(t)>0 \\ L_i & if\ \phi_i(t)<0 \end{cases} .$$

An extremal is *admissible* if $L_i \le u_i(t) \le M_i$ for each i and almost all t. A *u_i-singular extremal* is one for which ϕ_i is identically zero; a singular extremal is one that is u_i-singular for some i.

From the maximum principle, it follows that, if $\{(x(t),u(t)),\ t\in[0,T]\}$ is a time-optimal trajectory then there exists a λ such that (x,λ,u) is an extremal. If the set of zeroes of ϕ_i is *finite* along such an extremal, then necessarily u_i is a *bang-bang control*, i.e., it equals a.e. a function that is piecewise constant, with values equal to either L_i or M_i in each of finitely many subintervals.

We fix an extremal and an i=1,⋯,n. Consider the switching function ϕ_i; its derivative is

$$\phi_i' = \langle\lambda,[f,g_i]\rangle + \sum_{j=1}^n u_j\langle\lambda,[g_j,g_i]\rangle \qquad (3.2)$$
$$= \langle\lambda,[f,g_i]\rangle.$$

In particular, the derivative exists everywhere, and is itself absolutely continuous. (Note that we are abusing notation, in that the precise meaning of the above is as follows: $\phi_i'(t) = \langle\lambda(t),[f,g_i](x(t))\rangle$. We shall often omit the arguments t, or x(t), for notational simplicity.)

Let J_i [resp., J_i'] be the set of limit points of zeroes of ϕ_i [resp., of ϕ_i']. Since ϕ_i is continuously differentiable, J_i is contained in J_i'. Thus, at points in J_i the equations

$$\langle\lambda(t),g_i(x(t))\rangle = \langle\lambda(t),[f,g_i](x(t))\rangle = 0 \qquad (3.3)$$

hold. If t is a limit point of all ϕ_i, then these equations hold for all i, contradicting the facts that λ is always nonzero and that (2.3) holds. We have then established:

Lemma 3.1: $J_1 \cap J_2 \cap \cdots \cap J_n$ is empty.∎

Thus, if n-1 of these sets equal all of I, the remaining one, say J_i, must be empty, i.e. the set of zeroes of ϕ_i is finite. This says that not all controls may be simultaneously singular, and more precisely:

Corollary 3.2: If the extremal (x,λ,u) is u_j-singular for all j≠i, then u_i is bang-bang.∎

This result suggests the study of extremals that are singular for n-1 of the controls. We now consider this question in some more detail, and then specialize to the case n=2. There are of course many other cases of interest (e.g., n-2 controls are singular and the rest are bang-bang), but they will not be studied in this note.

3.1. A generic situation.

By the second formula in (2.4), there exist analytic functions $\{\alpha_{ijk},\ i,j,k = 1,\cdots,n\}$ such that, for each i,j = 1,⋯,n:

$$g_ifg_j = \sum_{k=1}^n \alpha_{ijk}g_k .$$

These coefficients can be computed explicitely, as follows. Let v be any vector field whose first n coordinates vanish identically, and let w be the vector function obtained from the last n coordinates. Then, v can be expressed as $\sum_{k=1}^{n} \alpha_k g_k$, where α_k is the k-th entry of the n-vector $M(\theta)w$. When $v = g_i f g_j$, which is in $\mathcal{S}_{-1}$, the obtained coefficients are in fact functions of θ alone.

Assume an extremal (x,λ,u) has been fixed. Taking a further derivative in (3.2) results for each i in:

$$\phi_i'' = \langle\lambda, ffg_i\rangle + \sum_{k=1}^{n} \beta_{ik}\phi_k \quad (3.4)$$

where for each i,k:

$$\beta_{ik}(t) = \sum_{j=1}^{n} \alpha_{ijk}(x(t))u_j(t) .$$

This formula holds under no assumptions of singularity whatsoever. Assume now, however, that the extremal in question is u_i-singular for all $i \neq k$, for some given k. This means that ϕ_i vanishes identically for $i \neq k$, and hence (3.4) reduces to:

$$\phi_i'' = \langle\lambda, ffg_i\rangle + \beta_{ik}\phi_k . \quad (3.5)$$

Thus, for every $i \neq k$, the equations in (3.3) hold at all t, as well as the almost everywhere vanishing of ϕ_i'' for such i. Assume that, at some t_o in the interval of definition of the extremal, $\phi_k(t_o) = 0$. In that case,

$$\langle\lambda(t_o), ffg_i(x(t_o))\rangle = 0$$

for all $i \neq k$. (The precise argument is as follows: the coefficient $\beta_{ik}(t)$ is essentially bounded, and $\phi_k \to 0$ as t approaches t_o. Thus the last term in (3.5) approaches 0 except at most along a set of measure zero. Since the first term is continuous, it must be zero at t_o.) Consider, for the given k, the set of vector fields

$$\{g_i, i=1,\cdots,n\} \cup \{fg_i, i=1,\cdots,n, i \neq k\} \cup \{ffg_i, i=1,\cdots,n, i \neq k\} ,$$

and let S_k be the set of states x at which these span the entire (2n-dimensional) tangent space. This set S_k is an open set, and since the vector fields are all analytic it is in fact open dense, provided only that it be nonempty. Since there are 3n-2 vectors in $\Re^{2n}$, one may expect that it is indeed nonempty (assuming $n \geq 2$), and this does happen in the 2-link manipulator example discussed later. The above arguments establish, by contradiction, the following fact:

Theorem 3.1: If (x,λ,u) is an u_i-singular extremal for all $i \neq k$ and $x(t)$ remains in S_k for all t, then u_k is constant (equal to L_k or M_k).∎

(Of course, as with all statements in optimal control, "constant" here means equal to a constant almost everywhere.) Later, we shall see how one may sometimes determine whether u_k is equal to L_k or M_k, based on higher order conditions.

For each $k=1,\cdots,n$, let A_k be the $(n-1)\times(n-1)$ matrix (α_{ijk}), where i and j each take the values $1,\cdots,k-1,k+1,\cdots,n$. (That is, delete the k-th row and column of (α_{ijk}), seen for fixed k as an $n\times n$ matrix.) Let

$$\Delta_k := \det(A_k) .$$

This is again an analytic function of x. Finally, let

$$R_k := S_k \cap \{x \mid \Delta_k \neq 0\} .$$

This set is either empty or open dense. If an extremal as in the previous theorem is such that $x(t)$ in fact remains in R_k, then not only is u_k constant, but we may determine the remaining controls. Fix one such extremal, and assume $u \equiv c$ constant (one of the above two values). The set of simultaneous equations $\{\phi_i''=0, i \neq k\}$, is by (3.5) equivalent to the following matrix equation:

$$A_k(x(t))\omega_k(t) = \psi_k(t) , \quad (3.6)$$

where ω_k is the column (n-1)-vector $(u_1,\cdots,\hat{u}_k,\cdots,u_n)'$ (we use the "^" to indicate a missing element), and where

$$\psi = (\psi_{k1},\cdots,\hat{\psi}_{kk},\cdots,\psi_{kn})'$$

and for each $i \neq k$,

$$\psi_{ki} := \langle\lambda(t), ffg_i(x(t))\rangle/\phi_k(t) - \alpha_{ikk}c .$$

(Recall that ϕ_k is always nonzero along this type of extremal.) If $x(t)$ remains in R_k, then we can solve (3.6) for ω_k as an analytic function of $\lambda(t)$ and $x(t)$. We may substitute the obtained expressions for $u_i(t)$, $i \neq k$, as well as $u_k \equiv c$, into the system equation (2.1) and the adjoint equation (3.1). These two together become a system of 2n ordinary differential equations with analytic right-hand side (there are no controls u left), and the solutions, that is both $\lambda(t)$ and $x(t)$, are analytic functions of time. Since ω_k was expressed as an analytic function of them, we also conclude:

Theorem 3.2: If (x,λ,u) is an u_i-singular extremal for all $i \neq k$ and $x(t)$ remains in R_k for all t, then all controls u_i are analytic as functions of time. (And can in fact be computed in the above way.)∎

Note that the conditions $\langle\lambda, g_i\rangle \equiv \langle\lambda, fg_i\rangle \equiv 0$ for $i \neq k$ result in 2n-2 independent constraints, by (2.3). So the costate λ depends globally on only two parameters, and it is possible to give thus a 2-dimensional equation for these parameters. We discuss this in more detail below when treating the 2-link manipulator case.

3.2. A degenerate case.

It may happen that Δ_k is identically zero, so that R_k is empty and the above theorem doesn't provide any information. On sets where the rank of A_k is constant, it is possible to provide some results, using pseudoinverses instead of inverses. In particular, assume that there is a row, say the i-th, of some A_k, which is identically zero. For this particular i, then, (3.5) says that the equation

$$\phi_i'' = \langle\lambda, ffg_i\rangle \equiv 0$$

must also hold. There are now 2n-1 conditions for λ, and if independent these determine λ up to a constant multiple (and hence essentially uniquely as a multiplier). If the rank of A_k is constantly n-2, one also can determine by an argument as above the controls u_j, $j \neq i,k$ as feedback functions of the state alone. However, also the condition $\phi_i''' \equiv 0$ must then hold, and this may result in yet another equation for λ, inconsistent with the previous 2n-1. This case appears in the two-link manipulator discussed next.

4. A manipulator example.

We computed explicitely with the 2-link (n=2) model given in [SH], with the numerical parameters provided in their Figure 1. With the use of MACSYMA, we deduced the following facts:

$$\alpha_{ij1} \equiv 0 \quad \text{for all } i,j \text{ (so } A_1 \equiv 0), \tag{4.1}$$
$$\Delta_2 = A_2 = \alpha_{112} = \gamma(\theta_2)\sin 2\theta_2,$$
$$S_2 = \{x \mid \beta(\theta_2)(\dot\theta_1 + \dot\theta_2)\sin\theta_2 = 0\},$$

where γ and β are functions which are nonzero, (in fact, always negative,) for all θ_2. Note also that

$$g_2 f f g_2 = f g_2 f g_2 = L_f(\alpha_{222}) g_2 + \alpha_{222} f g_2 \ . \tag{4.2}$$

4.1. Second control singular.

We first consider the case corresponding to k=1 in the discussion in the previous section. Thus fix any extremal which is u_2-singular. We calculate the third derivative of ϕ_2, and obtain:

$$\phi_2''' = \langle \lambda, fffg_2 \rangle + u_1 \langle \lambda, g_1 ffg_2 \rangle + u_2 \langle \lambda, g_2 ffg_2 \rangle \ .$$

By (4.2), the last term is a linear combination of ϕ_2 and ϕ_2'', and hence vanishes by singularity. The control u_1 must be constant. Let B be the set in which the vectors

$$\{g_2, fg_2, ffg_2, fffg_2 + c g_1 ffg_2\}$$

are linearly independent, for $c = L_i$ and $c = M_i$. This is open, and a calculation shows that it is nonempty; thus:

Theorem 4.1: There are no v-singular extremals for which x(t) intersects the open dense set B.∎

States with $\dot\theta = 0$ are especially interesting. The intersection of B with the set of such states is still nonempty (hence, open dense in that subset).

4.2. First control singular.

Consider now the case k=2. From the calculations in (4.1) it follows that there is an easy geometric characterization of R_2:

$$R_2 = \{x \mid \theta_2 \neq k\pi/2 \text{ and } \dot\theta_1 + \dot\theta_2 \neq 0\} \ .$$

There, u_2 is constant=c (=M_2 or L_2), and u_1 is analytic, as discussed earlier. We now provide some details of the way u_1 is computed.

Note that g_1 has the form $(0,0,\mu,\nu)'$, where -by positive definiteness of M- μ is everywhere nonzero (positive). Correspondingly, fg_1 has the form $(-\mu,-\nu,0,0)'$. The vectors

$$a := (-\nu/\mu, 1, 0, 0)' \quad \text{and}$$
$$b := (0, 0, -\nu/\mu, 1)'$$

are orthogonal to fg_1 and g_1 respectively. Since λ is also orthogonal to these vectors along a u_1-singular trajectory, it follows that λ is a combination of the two independent vectors a,b, i.e.:

$$\lambda(t) = \lambda_2(t) a(x(t)) + \lambda_4(t) b(x(t)) \ .$$

Because of the fact that the last 2 entries of a vanish, a is also ortogonal to g_1 and g_2 along this trajectory. It follows from the definition of S_2 that $\langle a, ffg_1 \rangle$ can never vanish. Further,

$$\phi_2 = \langle \lambda, g_2 \rangle = \lambda_4 \langle b, g_2 \rangle \ ,$$

so λ_4 cannot vanish at any point of the interval (otherwise, this would give $\phi_2 = 0$, a contradiction). It follows that

$$q(t) := \lambda_2(t)/\lambda_4(t)$$

is well-defined. We compute the derivative of q using the adjoint equations for λ; this results in a Riccati differential equation

$$q(t) = q^2(t) + \psi(x(t)) q(t) + \chi(x(t), u_1(t)) \ , \tag{4.3}$$

where $\chi(x, u_1)$ and $\psi(x)$ are explicitely computed functions, the former linear in u_1 (and dependent on the constant value c of the control u_2). If x(t) remains in R_2, then we may as before solve $\phi_1'' = 0$ for u_1, there resulting the control law

$$u_1(t) = r(x(t)) q(t) + s(x(t)) \ , \tag{4.4}$$

where s(x) is easily computed (and depends on c) and where

$$r(x) := -\langle a, ffg_1 \rangle \tag{4.5}$$

is always nonzero as remarked earlier. If we substitute the control law (4.4) into (4.3), we get a similar equation but with the function χ now independent of u. Alternatively, we may solve for q in (4.4), and substitute into (4.3) in order to obtain a similar differential equation for u.

The construction can be reversed, in the following sense. Given any x_o in R_2, solving (4.3) for any given initial q(0) results, via the rule (4.4), in a singular extremal (for either fixed value of u_2), defined at least for small time. Moreover, *since* r(x) *is never zero*, we may always find q(0) so that $L_1 < u_1(0) < M_1$, and hence so that the extremal is admissible (for small enough time). We have then recovered theorem 5.1 of [SS], for the present system (and with a somewhat simpler proof, based on considerably less computations):

Theorem 4.2: Assume that (x,p,u) is a u_1-singular extremal such that x(t) is in the open set R_2 for all t in I = [0,T]. Then there is a solution q(t) of the Riccati equation (4.3) on I such that the control law (4.4) holds, while u_2 equals one of the constant values $c = L_2$ or M_2.

Conversely, for each $x_o \in R_2$, each $c = L_2$ or M_2, and each real q_o, there is a u_1-singular extremal (x,p,u), and a solution of equation (4.3), both defined on an interval I which contains 0 in its interior, such that $x(0)=x_o$, $q(0)=q_o$, and equation (4.4) holds. Moreover, there is for each x_o in R_2 a nonempty open interval $Q(x_o)\subseteq\Re$ with the following property: If $q_0 \in Q$ then the singular extremal so constructed, for either of the two values of c, is *admissible.*■

One can apply a higher order test for optimality in order to determine the exact value of c for the above extremals, just as done in [SS]. Various authors (see e.g. [KR], [HE], [MO], and references in these papers,) have found stronger constraints than those implied by the maximum principle. The simplest of these generalizes the classical *Legendre-Clebsch* condition from variational calculus. We apply these conditions to the single-control system that results when u_2 is set identically equal to c. The necessary condition is then that, along the singular extremal,

$$<\lambda, g_1 f g_1> \geq 0 \ . \tag{4.6}$$

Note that, by definition of α_{112},

$$<\lambda, g_1 f g_1> = \alpha_{112} <\lambda, g_2> = \alpha_{112}\phi_2 \ .$$

Since α_{112} here equals Δ_2, we know that it is never zero along an extremal for which x(t) is in R_2. Thus, since ϕ_2 never vanishes either, it follows that the inequality (4.6) *is strict*, and hence:

$$\text{sign of } \phi_2 = \text{sign of } \Delta_2 \ .$$

Equivalently, we obtain the following precise characterization of the value of the constant control:

$$u_2(t) = \begin{cases} M_i \ if \ sin 2\theta_2 < 0 \\ L_i \ if \ sin 2\theta_2 > 0 \end{cases} .$$

5. References.

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