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Control-Lyapunov functions

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1 Motivation and history of the problems

The main objective of control is to modify the behavior of a dynamical system, typically with the purpose of regulating certain variables or of tracking desired signals. Often, either *stability* of the closed-loop system is an explicit requirement, or else the problem can be recast in a form that involves stabilization (e.g., of an error signal). For linear systems, the associated problems can now be treated fairly satisfactorily, but in the nonlinear case the area is still far from being settled. Both of the late 1980s reports [9] and [18], with dealt with challenges and future directions for research in control theory, identified the problem of stabilization of finite-dimensional deterministic systems as one of the most important open problems in nonlinear control. We discuss some questions in this area.

Specifically, this chapter deals with systems of the following general form:

$$\dot{x}(t) \;=\; f(x(t), u(t))\,. \tag{40.1}$$

The states $x(t)$ take values in Euclidean space $\mathbb{R}^n$ and the controls $u(t)$ take values in $\mathbb{R}^m$. The map $f(x,u)$ is continuous, and is locally Lipschitz in x, uniformly for u in compacts. In addition, $f(0,0) = 0$, that is to say, the zero state is an equilibrium when no inputs are applied. By a *control* we mean a measurable function $u : [0, +\infty) \to \mathbb{R}^m$ which is locally essentially bounded (meaning that, for each $T > 0$ there is some compact subset $K \subseteq \mathbb{R}^m$ so that $u(t) \in K$ for a.a. $t \in [0,T]$). In general, we use the notation $x(t; x_0, u)$ to denote the solution of (40.1) at time $t \geq 0$, with initial condition x_0 and control u. The expression $x(t; x_0, u)$ is defined on some maximal interval $[0, t_{\max}(x_0, u))$.

A common approach for the stabilization of (40.1) to $x = 0$ relies on the use of abstract "energy" or "cost" functions which can be made to infinitesimally decrease in directions corresponding to possible controls. Let us review some basic definitions. A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be *positive* (definite) if

$$V(x) > 0 \quad \forall\, x \neq 0,\; V(0) = 0, \tag{40.2}$$

and it is *proper* if the sublevel set $\{x | V(x) \leq a\}$ is compact, for each $a > 0$. The function V is said to be *infinitesimally decreasing* if there exists a continuous positive function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that, for each compact set $K \subset \mathbb{R}^n$, there exists a compact set $U \subset \mathbb{R}^m$ so that

$$\min_{u \in U} \langle \nabla V(x), f(x,u) \rangle \;\leq\; -W(x) \quad \forall\, x \in K\,. \tag{40.3}$$

Note that this implies, in particular, the Hamilton-Jacobi inequality

$$\sup_{x \in \mathbb{R}^n} \; \inf_{u \in \mathbb{R}^m} \; \langle \nabla V(x), f(x,u) \rangle + W(x) \;\leq\; 0\,. \tag{40.4}$$

Definition. A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a *differentiable control Lyapunov function* (CLF) if it is positive, proper, and infinitesimally decreasing.

From a numerical point of view, given the objective of approaching the state $x = 0$, the use of control-Lyapunov functions reduces the search for stabilizing inputs to the iterative solution of a static nonlinear programming problem: when at state ξ, find u such that $\min_{u \in U} \langle \nabla V(x), f(x,u) \rangle \;\leq\; -W(x)$. This paradigm underlies the optimal control approach of Bellman, "artificial intelligence" techniques based on position evaluations in games and "critics" in learning programs, and several "neural-network" approaches to control.

Mathematically, the main implication of the existence of a CLF is null-asymptotic controllability. This means that for each initial state ξ there is some control function $u(\cdot)$ which steers the state ξ asymptotically to the origin, while not producing large excursions. More precisely:

Definition. The system (40.1) is *(globally) null-asymptotically controllable* if:

1. **(attractiveness)** for each $x_0 \in \mathbb{R}^n$ there exists some control u such that the trajectory $x(t) = x(t; x_0, u)$ is defined for all $t \geq 0$ and $x(t) \to 0$ as $t \to +\infty$;

2. **(Lyapunov stability)** for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x_0 \in \mathbb{R}^n$ with $|x_0| < \delta$ there is a control u as in 1. such that $\|x(t)\| < \varepsilon$ for all $t \geq 0$;

3. **(bounded controls)** there are a neighborhood X of 0 in $\mathbb{R}^n$, and a compact subset U of $\mathbb{R}^m$ such that, if the initial state x_0 in 2. satisfies also $x_0 \in X$, then the control in 2. can be chosen with $u(t) \in U$ for almost all t.

This is a natural generalization to control systems of the concept of uniform asymptotic stability of solutions of differential equations.

2 Past work and new questions

A natural question is: can one use a known CLF in order to effectively design control laws that achieve stabilization? For systems affine in controls, Artstein's Theorem says that the existence of a differentiable CLF implies (and is also implied by) the existence of a continuous feedback stabilizer, cf. [1], meaning a $k : \mathbb{R}^n \to \mathbb{R}^m$ with the property that the closed-loop system $\dot{x} = f(x, k(x))$ has the origin $x = 0$ as a globally asymptotically stable equilibrium, and k is continuous away from the origin. CLF-based feedback designs provide one of the main current approaches to nonlinear control, and are discussed in detail in several recent textbooks, including [11, 10, 16, 20]. More precisely, one can find *universal formulas* for the stabilizer, which depend analytically on the directional derivatives of the CLF V. For example, for systems with one input ($m = 1$) $\dot{x} = f(x, u) = g_0(x) + ug_1(x)$, one has that

$$k(x) \;:=\; -\,\frac{a(x) + \sqrt{a(x)^2 + b(x)^4}}{b(x)} \qquad (0 \text{ if } b = 0)$$

stabilizes the system, where we are denoting $a(x) := \nabla V(x)\, g_0(x)$ and $b(x) := \nabla V(x)\, g_1(x)$. (This expression is analytic on $a(x), b(x)$ when $x \neq 0$; cf. [23, 20].)

Often, controls are restricted to lie in constrained sets $\mathbb{U} \subseteq \mathbb{R}^m$, for instance due to actuator saturation effects. As Artstein's Theorem holds for arbitrary convex input-value sets $\mathbb{U}$, it would be desirable for design purposes to have universal formulas that lead to controls satisfying the same constraints, under the assumption that the CLF satisfies condition (40.3) with controls in $\mathbb{U}$. Examples of such formulas are given in ([14]), for open balls in $\mathbb{R}^n$.

Problem. *Find universal formulas for* CLF *stabilization, for general (convex) control-value sets* $\mathbb{U}$.

The above considerations motivate an obvious fundamental theoretical question: is the existence of a differentiable CLF *equivalent* to null-asymptotic controllability?

When there are no controls, the equivalence between asymptotic controllability and the existence of CLF's amounts to the equivalence between (global) asymptotic stability of an equilibrium and the existence of classical Lyapunov functions, and in that case the answer to the question is positive, and even an infinitely differentiable CLF always exists, as shown in the fundamental contributions of Massera and Kurzweil [15, 12]. When there are controls, but the system is linear, again the answer is positive, and there is a quadratic CLF, as discussed in linear systems textbooks, e.g. [20]. In general, however, the answer to the question is negative. There do exist systems which are null-asymptotically controllable yet for which there is no possible differentiable CLF. There are various ways to prove this negative result. One is as a corollary of Artstein's Theorem, since in general continuous stabilizers are known not to exist for asymptotically controllable systems, cf. [17, 24, 21, 3, 2, 20]. See also [19, 7, 13] for many further recent results regarding the connection between continuous stabilizability and the existence of differentiable CLF's.

Faced with the negative answer, a very attractive relaxation of the question involves dropping the requirement that V be differentiable, and reinterpreting the differential inequality (40.3) in a weak sense. There are many candidates for this interpretation: viscosity solutions, generalized differentials, proximal subgradients, and others. These different interpretations, for the purpose of this note, are equivalent, as discussed in [5]. We pick one of them, proximal subgradients, for concreteness. Specifically, we say that a *continuous function* V is *infinitesimally decreasing* if the same definition as earlier holds, except that we substitute (40.3) by:

$$\min_{u\in U} \langle \zeta, f(x,u)\rangle \;\leq\; -W(x) \quad \forall\, x \in K \;\; \forall\, \zeta \in \partial_{\mathrm{P}} V(x)\,, \tag{40.5}$$

where $\partial_{\mathrm{P}} V(x)$ is the proximal subdifferential of the function V at the point x. We say that a continuous $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a (continuous) CLF if it is positive, proper, and infinitesimally decreasing in this sense.

Recall that a vector $\zeta \in \mathbb{R}^n$ is a proximal subgradient of V at x if there exists some $\sigma > 0$ such that, for all y in some neighborhood of x,

$$V(y) \;\geq\; V(x) + \langle \zeta, y - x\rangle - \sigma \left\|y - x\right\|^2 .$$

That is, ζ is the gradient of a supporting quadratic function at x to the graph of V. The set of proximal subgradients of V at x (which may be empty) is $\partial_{\mathrm{P}} V(x)$. The use of proximal subgradients as substitutes for the gradient for a nondifferentiable function was originally proposed in nonsmooth analysis for the study of optimization problems, see [4]. It is also possible to use continuous CLF's as a basis for feedback design; see [6]. The following positive result holds:

Theorem. A system is globally null-asymptotically controllable if and only if there exists a continuous CLF for it.

This was proved in [22]. More precisely, it was stated there in terms of Dini derivatives; the translation into the far more elegant and powerful language of proximal subgradients was later remarked in [6].

Thus, there is always a continuous V, but not always a differentiable V. This leads us to state:

Problem. *What is the best degree of regularity that can be assured for a* CLF*, for any globally null-asymptotically controllable system?*

Specifically, one may ask if one can always find a CLF that is piecewise differentiable on a locally finite (away from the origin) partition of $\mathbb{R}^n$, so that universal formulas for feedback stabilization can be employed locally.

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