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Nonlinear Feedback Stabilization Revisited

Eduardo D. Sontag*

1 Introduction

Methods of feedback design are undergoing an exceptionally rich period of progress and maturation, fueled to a great extent by the discovery of new conceptual notions as well as by the systematic application of certain ideas such as that of control-Lyapunov functions (clf's). This paper, which can be seen as an updated version of [22], discusses the problem of state stabilization, the understanding of which is a fundamental prerequisite to the solution of control problems such as tracking, disturbance rejection, output feedback, or adaptive and robust design.

It is known that, in general, in order to control nonlinear systems one must use switching (discontinuous) mechanisms of various types. Of course, time-optimal solutions for even linear systems often involve such discontinuities, see for instance [23], Chapter 10. But, whereas for linear systems most control problems often admit also (perhaps suboptimal) continuous solutions, when dealing with arbitrary systems discontinuities are unavoidable, even when no optimality objectives are imposed. As in [22], we begin therefore by discussing the necessity of such discontinuities, and explain the characterization of regular stabilizability in terms of differentiable clf's.

Among other results which only became available after [22] was written, we will mention the use of differentiable clf's as a tool in the characterization of robustness with respect to small observation noise. Another major way in which this paper extends the material from [22] is in its treatment of *non*smooth clf's and, associated to them, techniques of discontinuous stabilization. This leads us into the subject of precisely defining what we mean by "solution" of a discontinuous differential equation, and makes contact with the literature on differential games as well as nonsmooth analysis.

There is often a tradeoff between generality and clarity of exposition. In this paper, we have opted for clarity, not necessarily presenting results in their most general formulations. The citations should be consulted for generalizations as well as for the omitted technical details. (The web site: **http://www.math.rutgers.edu/~sontag** contains several of the papers referenced.) However, we have included technical proofs of a few minor extensions of results, which were not available in the literature in the form needed for this exposition.

One subject which was covered in the lecture, but which we cannot include here because of space limitations, is that dealing with the study of

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the effect of "large" disturbances on the behavior of feedback systems. This study leads one to the very active area of input to state stability (ISS) and related notions (output to state stability as a model of detectability, input to output stability for the study of regulation problems, and so forth), and constitutes, in the author's view, the most exciting area of current work in nonlinear control. The web site referenced above may be consulted for several recent papers as well as expository articles on that subject.

2 Preliminaries

In this paper, we consider exclusively continuous-time systems evolving in finite-dimensional spaces $\mathbb{R}^n$, and we suppose that controls take values in $\mathcal{U} = \mathbb{R}^m$. A control or input is any measurable locally essentially bounded map $u(\cdot) : [0, \infty) \to \mathcal{U} = \mathbb{R}^m$. In general, we use the notation $|x|$ for Euclidean norms, and use $\|u\|$, or $\|u\|_\infty$ for emphasis, to indicate the essential supremum of a function $u(\cdot)$. For basic terminology and facts about control systems, we rely upon [23]. Given a map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ which is locally Lipschitz and satisfies $f(0,0) = 0$, we consider the system

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

and, when f does not depend on u (for instance, when we substitute, later, a feedback law $u = k(x)$, leading to $\dot{x} = f(x, k(x))$), we have a system with no inputs

$$\dot{x}(t) = f(x(t)). \tag{2}$$

All definitions for such systems are implicitly applied as well to systems with inputs (1) by setting $u \equiv 0$; for instance, we define the global asymptotic stability (GAS) property for (2), but we say that (1) is GAS if $\dot{x} = f(x, 0)$ is. The maximal solution $x(\cdot)$ of (1), corresponding to a given initial state $x(0) = x^0$, and to a given control u, is defined on some maximal interval $[0, t_{\max}(x^0, u))$ and is denoted by $x(t, x^0, u)$. For systems with no inputs (2) we write just $x(t, x^0)$.

2.1 Stability and Asymptotic Controllability

The use of "comparison functions" has become widespread in stability analysis, as this formalism allows elegant formulations of most concepts. We recall the relevant definitions here. The class of $\mathcal{K}_\infty$ functions consists of all $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which are continuous, strictly increasing, unbounded, and satisfy $\alpha(0) = 0$, cf. Figure 1. The class of $\mathcal{KL}$ functions consists of those $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with the properties that (1) $\beta(\cdot, t) \in \mathcal{K}_\infty$ for all t, and (2) $\beta(r, t)$ decreases to zero as $t \to \infty$. (It is worth remarking, cf. [24], that for each $\beta \in \mathcal{KL}$, there exist two functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that $\beta(r, t) \leq \alpha_2\left(\alpha_1(r) e^{-t}\right)$ for all s, t. Thus, as every function of the latter form

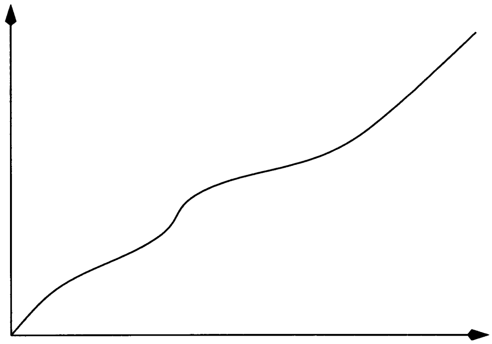

Figure 1: A function of class $\mathcal{K}_\infty$

is in $\mathcal{KL}$, and since $\mathcal{KL}$ functions are only used in order to express upper bounds, in a sense there is no need to introduce the class $\mathcal{KL}$.) We will also use $\mathcal{N}$ to denote the set of all nondecreasing functions $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

Expressed in this language, the property of *global asymptotic stability (GAS)* for a system with no inputs (2) becomes:

$$(\exists \beta \in \mathcal{KL}) \quad |x(t, x^0)| \leq \beta(|x^0|, t) \quad \forall x^0, \forall t \geq 0.$$

It is an easy exercise to show that this definition is equivalent to the usual "$\varepsilon - \delta$" definition; for one implication, simply observe that

$$|x(t, x^0)| \leq \beta(|x^0|, 0)$$

provides the stability (or "small overshoot") property, while

$$|x(t, x^0)| \leq \beta(|x^0|, t) \underset{t \to \infty}{\longrightarrow} 0$$

gives attractivity.

More generally, we define what it means for the system with inputs (1) to be (open loop, globally) *asymptotically controllable (AC)*. The definition amounts to requiring that for each initial state x^0 there exists some control $u = u_{x^0}(\cdot)$ defined on $[0, \infty)$, such that the corresponding solution $x(t, x^0, u)$ is defined for all $t \geq 0$, and converges to zero as $t \to \infty$, with "small" overshoot. Moreover, we wish to rule out the possibility that $u(t)$ becomes unbounded for x near zero. The precise formulation is as follows.

$$(\exists \beta \in \mathcal{KL})\,(\exists \sigma \in \mathcal{N}) \quad \forall x^0 \in \mathbb{R}^n \;\; \exists u(\cdot),\; \|u\|_\infty \leq \sigma(|x^0|),$$

$$|x(t, x^0, u)| \leq \beta(|x^0|, t) \quad \forall t \geq 0.$$

Finally, we say that $k : \mathbb{R}^n \to \mathcal{U}$ is a *feedback stabilizer* for the system with inputs (1) if k is locally bounded (that is, k is bounded on each bounded subset of $\mathbb{R}$), $k(0) = 0$, and the closed-loop system

$$\dot{x} = f(x, k(x)) \tag{3}$$

is GAS, i.e. there is some $\beta \in \mathcal{KL}$ so that $|x(t)| \leq \beta(|x(0)|, t)$ for all solutions and all $t \geq 0$. Obviously, if there exists a feedback stabilizer for (1), then (1) is also AC (just use $u(t) := k(x(t, x^o))$ as u_{x^o}). A most natural question is to ask if the converse holds as well, namely: *is every asymptotically controllable system also feedback stabilizable?* We will see that the answer is "yes", provided that we allow discontinuous feedbacks k, which in turn leads to the technical problem of defining precisely what one means by a "solution" of an initial value problem for (1) when $f(x, k(x))$ is not continuous (since, in that case, the standard theorems on existence do not apply). This is a question that is very much central to the rest of the paper.

2.2 Regularity of Feedback

As already mentioned, one of the central issues with which we will be concerned is that of dealing with possibly discontinuous feedback laws k. Before addressing that subject, however, we will study what can be done with continuous feedback.

It turns out that requirements away from 0, say asking whether k is continuous or smooth, are not very critical; it is often the case that one can "smooth out" a continuous feedback (or, even, make it real-analytic, via Grauert's Theorem) away from the origin. So, in order to avoid unnecessary complications in exposition due to nonuniqueness, let us call a feedback k *regular* if it is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$. For such k, solutions of initial value problems $\dot{x} = f(x, k(x))$, $x(0) = x^o$, are well defined (at least for small time intervals $[0, \varepsilon)$) and, provided k is a stabilizing feedback, are unique (cf. [23], Exercise 5.9.9).

On the other hand, behavior at the origin cannot be "smoothed out" and, at zero, the precise degree of smoothness plays a central role in the theory. For instance, consider the system

$$\dot{x} = x + u^3 .$$

The continuous (and, in fact, smooth away from zero) feedback $u = k(x) := -\sqrt[3]{2x}$ globally stabilizes the system (the closed-loop system becomes $\dot{x} = -x$). However, there is no possible stabilizing feedback which is differentiable at the origin, since $u = k(x) = O(x)$ implies that

$$\dot{x} = x + O(x^3)$$

about $x = 0$, which means that the solution starting at any positive and small point moves to the right, instead of towards the origin. (A general result, assuming that A has no purely imaginary eigenvalues, cf. [23], Section 5.8, is that if –and only if– $\dot{x} = Ax + Bu + o(x, u)$ can be locally asymptotically stabilized using a feedback which is differentiable at the origin, the linearization $\dot{x} = Ax + Bu$ must be itself stabilizable. In the example that we gave, this linearization is just $\dot{x} = x$, which is not stabilizable.)

3 Nonexistence of Regular Feedback

We now turn to the question of existence of regular feedback stabilizers. We first study a comparatively trivial case, namely systems with one state variable and one input. After that, we turn to multidimensional systems.

3.1 The Special Case $n = m = 1$

There are algebraic obstructions to the stabilization of $\dot{x} = f(x,u)$ if the input u appears nonlinearly in f. Ignoring the requirement that there be a $\sigma \in \mathcal{N}$ so that controls can be picked with $\|u\| \leq \sigma(|x^0|)$, asymptotic controllability is, for $n = m = 1$, equivalent to:

$$(\forall\, x \neq 0)\, (\exists\, u) \;\; xf(x,u) < 0 \tag{4}$$

(this is proved in [25]; it is fairly obvious, but some care must be taken to deal with the fact that one is allowing arbitrary measurable controls; the argument proceeds by first approximating such controls by piecewise constant ones). Let us introduce the following set:

$$\mathcal{O} \;:=\; \{(x,u) \mid xf(x,u) < 0\}\,,$$

and let $\pi : (x,u) \mapsto x$ be the projection into the first coordinate in $\mathbb{R}$. Then, (4) is equivalent to:

$$\pi\mathcal{O} \;=\; \mathbb{R} \setminus \{0\}\,.$$

(One can easily include the requirement "$\|u\| \leq \sigma(|x^0|)$" by asking that for each interval $[-K,K] \subset \mathbb{R}$ there must be some compact set $C_K \subset \mathbb{R}^2$ so that $[-K,K] \subseteq \pi(C_K)$. For simplicity, we ignore this technicality.) In these terms, a stabilizing feedback is nothing else than a locally bounded map $k : \mathbb{R} \to \mathbb{R}$ such that $k(0) = 0$ and so that k is a section of π on $\mathbb{R} \setminus \{0\}$:

$$(x, k(x)) \in \mathcal{O} \;\; \forall\, x \neq 0\,.$$

For a regular feedback, we ask that k be locally Lipschitz on $\mathbb{R} \setminus \{0\}$.

Clearly, there is no reason for Lipschitz, or for that matter, just continuous, sections of π to exist. As an illustration, take the system

$$\dot{x} \;=\; x\left[(u-1)^2 - (x-1)\right]\left[(u+1)^2 + (x-2)\right]\,.$$

Let

$$\mathcal{O}_1 = \{(u+1)^2 < (2-x)\} \quad \text{and} \quad \mathcal{O}_2 = \{(u-1)^2 < (x-1)\}$$

(see Figure 2). Here, $\mathcal{O}$ has three connected components, namely $\mathcal{O}_2$ and $\mathcal{O}_1$ intersected with $x > 0$ and $x < 0$. It is clear that, even though $\pi\mathcal{O} = \mathbb{R}$, there is no continuous curve (graph of $u = k(x)$) which is always in $\mathcal{O}$

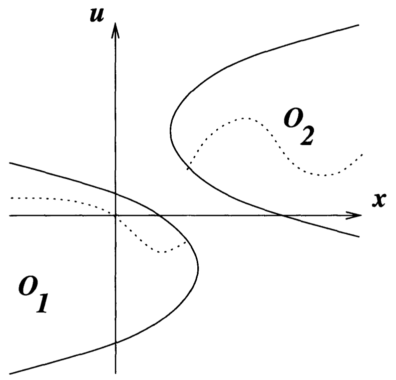

Figure 2: Two regions

and projects onto $\mathbb{R} \setminus \{0\}$. On the other hand, there exist many possible feedback stabilizers provided that we allow one discontinuity (e.g., dotted curve in figure). Although our interest here is primarily in time-invariant feedback $u = k(x)$, it is worth pointing out that it is often possible to overcome obstructions to regular feedback stabilization by means of the use of *time-varying* feedback $u = k(t, x)$. A general result in that direction was proved in [25], which established that every one-dimensional system can be stabilized using $k(t, x)$ continuous (it is not difficult to see that the proof can be adapted to obtain $k(t, x)$ periodic in t, for each x). More recently, a very different construction of time-varying feedbacks was accomplished by Coron, cf. [6], who established a general result valid in any dimension, but restricted to systems with no drift, and by Coron and Rosier, cf. [7], for systems for which smooth clf's (see later) exist.

3.2 Obstructions

When feedback laws are required to be continuous at the origin, new obstructions arise. The case of systems with $n = m = 1$ is also a good way to introduce this subject. The first observation is that stabilization about the origin (even if just local) means that we must have, near zero:

$$f(x, k(x)) \begin{cases} > 0 & \text{if } x < 0 \\ < 0 & \text{if } x > 0 \\ = 0 & \text{if } x = 0 \,. \end{cases}$$

In fact, all that we need is that $f(x_1, k(x_1)) < 0$ for some $x_1 > 0$ and $f(x_2, k(x_2)) > 0$ for some $x_2 < 0$. This guarantees, via the intermediate-value theorem that, if k is continuous, the projection

$$(-\varepsilon, \varepsilon) \to \mathbb{R}, \quad x \mapsto f(x, k(x))$$

is onto a neighborhood of zero, for each $\varepsilon > 0$, see Figure 3. It follows, in

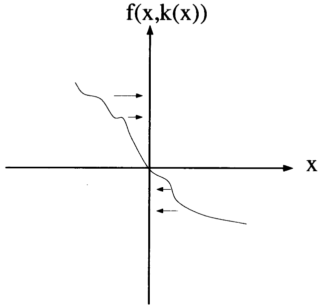

Figure 3: Onto projection, for case $n = m = 1$

particular, that

$$(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \to \mathbb{R}, \quad (x, u) \mapsto f(x, u)$$

also contains a neighborhood of zero, for any $\varepsilon > 0$ (that is, the map $(x, u) \mapsto f(x, u)$ is open at zero). This last property is intrinsic, being stated in terms of the original data $f(x, u)$ and not depending upon the feedback k. Brockett's condition, to be described next, is a far-reaching generalization of this argument; in its proof, degree theory replaces the use of the intermediate value theorem.

Logical Decisions are Often Necessary

If there are *global obstacles* in the state-space (that is, if the state-space is a proper subset of $\mathbb{R}^n$), discontinuities in feedback laws cannot in general be avoided. Even if it is in principle possible to reach the origin, it may not be possible to find a regular feedback stabilizer. Actually, this is fairly obvious, and is illustrated in intuitive terms by Figure 4. We think of the position of the (immobile) cat as the origin. In deciding in which way to move, as a function of its current position, the dog must at some point in the state-space make a discontinuous decision: move to the left or to the right of the obstacle (represented by the shaded rectangle)? Formally, this setup can be modeled as a problem in which the state-space is the complement in $\mathbb{R}^2$ of the obstacle, and the fact that discontinuities are necessary is a particular case of a general fact (a theorem of Milnor's), namely that the domain of attraction of an asymptotically stable vector field must be diffeomorphic to Euclidean space (which the complement of the rectangle is not); see [23] for more on the subject.

The interesting point is that even if, as in this exposition, we assume that states evolve in Euclidean spaces, similar obstructions may arise. These are due not to the topology of the state space, but to "virtual obstacles"

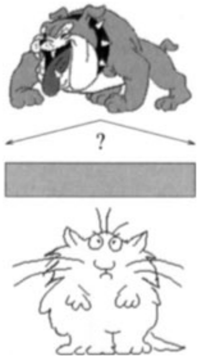

Figure 4: At some point, a discontinuous decision is necessary

implicit in the form of the system equations. These obstacles occur when it is impossible to move *instantaneously* in certain directions, even if it is possible to move *eventually* in every direction ("nonholonomy").

Nonholonomy and Brockett's Theorem

As an illustration, let us consider a model for the "shopping cart" shown in Figure 5 ("knife-edge" or "unicycle" are other names for this example). The state is given by the orientation θ, together with the coordinates x_1, x_2 of the midpoint between the back wheels. The front wheel is a castor, free

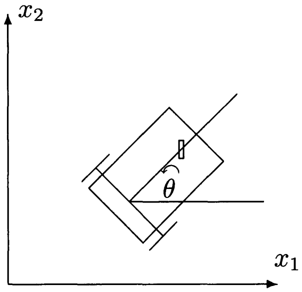

Figure 5: Shopping cart

to rotate. There is a non-slipping constraint on movement: the velocity $(\dot{x}_1, \dot{x}_2)'$ must be parallel to the vector $(\cos\theta, \sin\theta)'$. This leads to the following equations:

$$\begin{aligned}\dot{x}_1 &= u_1 \cos\theta \\ \dot{x}_2 &= u_1 \sin\theta \\ \dot{\theta} &= u_2\end{aligned}$$

where we may view u_1 as a "drive" command and u_2 as a steering control; in practice, one would implement these controls by means of differential forces on the two back corners of the cart. The feedback transformation $z_1 := \theta$, $z_2 := x_1 \cos\theta + x_2 \sin\theta$, $z_3 := x_1 \sin\theta - x_2 \cos\theta$, $v_1 := u_2$, and $v_2 := u_1 - u_2 z_3$ brings the system into the system with equations $\dot{z}_1 = v_1$, $\dot{z}_2 = v_2$, $\dot{z}_3 = z_1 v_2$ known as "Brockett's example" or "nonholonomic integrator" (yet another change can bring the third equation into the form $\dot{z}_3 = z_1 v_2 - z_2 v_1$). We view the system as having state space $\mathbb{R}^3$. Although a physically more accurate state space would be the manifold $\mathbb{R}^2 \times \mathbb{S}^1$, the necessary condition to be given is of a local nature, so the global structure is unimportant.

This system is (obviously) completely controllable (in any case, controllability can be checked using the Lie algebra rank condition, as in e.g. [23], Exercise 4.3.16), and in particular is AC. But we may expect that discontinuities are unavoidable due to the non-slip constraint, which does not allow moving from, for example the position $x_1 = 0$, $\theta = 0$, $x_2 = 1$ in a straight line towards the origin. Indeed, we have:

Theorem A (Brockett [2]) *If there is a stabilizing feedback which is regular and continuous at zero, then the map $(x, u) \mapsto f(x, u)$ is open at zero.*

The test fails here, since no points of the form $(0, \varepsilon, *)$ belong to the image of the map

$$\mathbb{R}^5 \to \mathbb{R}^3 : (x_1, x_2, \theta, u_1, u_2)' \mapsto f(x, u) = (u_1 \cos\theta, u_1 \sin\theta, u_2)'$$

for $\theta \in (-\pi/2, \pi/2)$.

More generally, it is impossible to continuously stabilize any system without drift

$$\dot{x} = u_1 g_1(x) + \ldots + u_m g_m(x) = G(x)u$$

if $m < n$ and $\operatorname{rank}[g_1(0), \ldots, g_m(0)] = m$ (this includes all totally nonholonomic mechanical systems). Indeed, under these conditions, the map $(x, u) \mapsto G(x)u$ cannot contain a neighborhood of zero in its image, when restricted to a small enough neighborhood of zero. Indeed, let us first rearrange the rows of G:

$$G(x) \rightsquigarrow \begin{pmatrix} G_1(x) \\ G_2(x) \end{pmatrix}$$

so that $G_1(x)$ is of size $m \times m$ and is nonsingular for all states x that belong to some neighborhood N of the origin. Then,

$$\begin{pmatrix} 0 \\ a \end{pmatrix} \in \operatorname{Im}[N \times \mathbb{R}^m \to \mathbb{R}^n : (x, u) \mapsto G(x)u] \Rightarrow a = 0$$

(since $G_1(x)u = 0 \Rightarrow u = 0 \Rightarrow G_2(x)u = 0$ too).

If the condition $\operatorname{rank}[g_1(0), \ldots, g_m(0)] = m$ is violated, we cannot conclude a negative result. For instance, the system $\dot{x}_1 = x_1 u$, $\dot{x}_2 = x_2 u$ has $m = 1 < 2 = n$ but it can be stabilized by means of the feedback law $u = -(x_1^2 + x_2^2)$.

Observe that for linear systems, Brockett's condition says that

$$\operatorname{rank}[A, B] = n$$

which is the Hautus controllability condition (see e.g. [23], Lemma 3.3.7) at the zero mode.

Idea of the Proof

One may prove Brockett's condition in several ways. A proof based on degree theory is probably easiest, and proceeds as follows (for details see for instance [23], Section 5.9). The basic fact, due to Krasnosel'ski, is that if the system $\dot{x} = F(x) = f(x, k(x))$ has the origin as an asymptotically stable point and F is regular (since k is), then the degree (index) of F with respect to zero is $(-1)^n$, where n is the system dimension. In particular, the degree is also nonzero with respect to points p near enough 0, which means that the equation $F(x) = p$ can be solved for small p, and hence $f(x, u) = p$ can be solved as well. The proof that the degree is $(-1)^n$ follows by exhibiting a homotopy, namely

$$F_t(x^0) = \frac{1}{t}\left[x\left(\frac{t}{1-t}, x^0\right) - x^0\right],$$

between $F_0 = F$ and $F_1(x) = -x$, and noting that the degree of the latter is obviously $(-1)^n$. An alternative (and Brockett's original) proof uses Lyapunov functions. Asymptotic stability implies the existence of a smooth Lyapunov function V for $\dot{x} = F(x) = f(x, k(x))$, so, on the boundary ∂B of a sublevel set $B = \{x \mid V(x) \leq c\}$ we have that F points towards the interior of B, see Figure 6. So for p small, $F(x) - p$ still points to the interior,

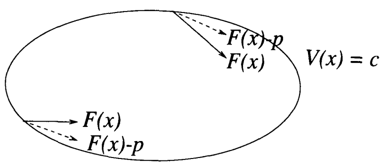

Figure 6: Perturbations of F still point inside B

which means that B is invariant with respect to the perturbed vector field $\dot{x} = F(x) - p$. *Provided that a fixed-point theorem applies to continuous maps* $B \to B$, this implies that $F(x) - p$ must vanish somewhere in B, that is, the equation $F(x) = p$ can be solved. (Because, for each small $h > 0$, the time-h flow ϕ of $F-p$ has a fixed point $x_h \in B$, i.e. $\phi(h, x_h) = x_h$, so picking a convergent subsequence $x_h \to \bar{x}$ gives that $0 = \frac{\phi(h,x_h)-x_h}{h} \to F(\bar{x}) - p$.) A fixed point theorem can indeed be applied, because B is a retract of $\mathbb{R}^n$ (use the flow itself); note that this argument gives a weaker conclusion than the degree condition.

Another Example

There is another nice example of these ideas, which will also be useful later when illustrating Lyapunov techniques (see [1]). It is closely related to the shopping-cart example; in fact, it arises when we control the cart by this procedure: first, we rotate the cart until tangent to, either a circle centered on the x_2-axis and tangent to the x_1-axis (see Figure 7), or the x_1-axis

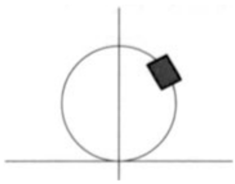

Figure 7: Cart tangent to circle

if we started there; next, we move only with velocities tangential to this circle (steering so as to maintain invariance of the circle). In summary, one obtains a system with state-space $\mathbb{R}^2$, input space $\mathbb{R}$, and equations

$$\dot{x} = g(x)u\,, \text{ where } \quad g(x) = \begin{pmatrix} x_1^2 - x_2^2 \\ 2x_1x_2 \end{pmatrix}.$$

The vector field g and typical orbits of g are shown in Figure 8. In this

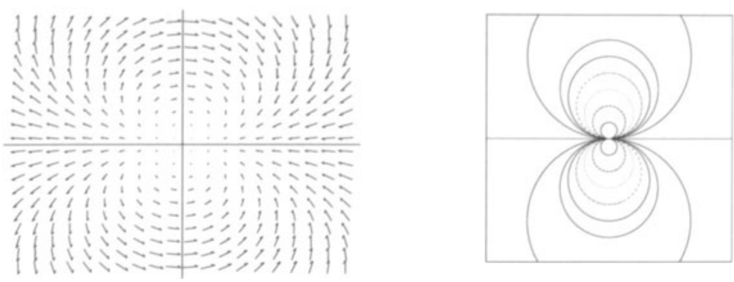

Figure 8: $(x_1^2 - x_2^2)\frac{\partial}{\partial x_1} + 2x_1x_2\frac{\partial}{\partial x_2}$ with typical integral manifolds

system, all motions along the integral curves of g are allowed (clockwise, counterclockwise, or staying at one point). The Brockett condition is satisfied; this is clear if one views the system as a system with complex state space (write $z = x_1 + ix_2$) with equations:

$$\dot{z} = z^2 u .$$

The map $(x, u) \mapsto g(x)u$ is $(z, u) \mapsto z^2 u$, which is clearly onto any neighborhood of zero, even when restricted to any neighborhood of $z{=}u{=}0$ (take square roots). However, the degree condition fails, as for any continuous feedback one would have $k(x) \neq 0$ for all $x \neq 0$ (otherwise, an equilibrium results), so one obtains degree 2 for $f(x, k(x)) = g(x)k(x)$, and not $(-1)^2 = 1$.

Actually, for this example it is easy to see directly that no regular stabilizing feedback can exist. One proof is by noticing that circles would have to be invariant, so motions restricted to them would result in globally asymptotically stable vector fields in manifolds not diffeomorphic to Euclidean space, contradicting Milnor's theorem. Another proof is even easier: take any path $\gamma : [0,1] \to \mathbb{R}^2$ with $\gamma(0) = (1,0)'$ and $\gamma(1) = (-1,0)'$ and so that $\gamma(r) \neq 0$ for all r. Consider the function $\kappa(r) := k(\gamma(r))$, which is continuous if k was assumed continuous. Since $\kappa(0) < 0$ (because, otherwise, trajectories of $\dot{x} = g(x)k(x)$ starting at $(1,0)'$ cannot converge to zero, since the positive axis is invariant and g points towards the right there) and $\kappa(1) > 0$ (analogous argument), it follows that $\kappa(r_0) = 0$ for some $r_0 \in (0,1)$. Therefore, the point $\bar{x} = \gamma(r_0) \neq 0$ is such that $k(\bar{x}) = 0$ and is therefore an equilibrium point, contradicting the fact that $u = k(x)$ is a stabilizer. We shall return to this example when discussing control-Lyapunov functions.

3.3 Control-Lyapunov Functions

The method of control-Lyapunov functions ("clf's") provides a powerful tool for studying stabilization problems, both as a basis of theoretical developments and as a method for actual feedback design.

Before discussing clf's, let us quickly review the classical concept of Lyapunov functions, through a simple example. Consider first a damped spring-mass system $\ddot{y} + \dot{y} + y = 0$, or, in state-space form with $x_1 = y$ and $x_2 = \dot{y}$, $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 - x_2$. One way to verify global asymptotic stability of the equilibrium $x = 0$ is to pick the (Lyapunov) function $V(x_1, x_2) := \frac{3}{2} x_1^2 + x_1 x_2 + x_2^2$, and observe that $\nabla V(x).f(x) = -|x|^2 < 0$ if $x \neq 0$, which means that

$$\frac{dV(x(t))}{dt} = -|x(t)|^2 < 0$$

along all nonzero solutions, and thus the energy-like function V decreases along all trajectories, which, since V is a nondegenerate quadratic form, implies that $x(t)$ decreases, and in fact $x(t) \to 0$. Of course, in this case one could compute solutions explicitly, or simply note that the characteristic equation has all roots with negative real part, but Lyapunov functions are a general technique. (In fact, the classical converse theorems of Massera and Kurzweil show that, whenever a system is GAS, there always exists a smooth Lyapunov function V.)

Now let us modify this example to deal with a control system, and consider a forced (but undamped) harmonic oscillator $\ddot{x} + x = u$, i.e. $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 + u$. The damping feedback $u = -x_2$ stabilizes the system, but let us pretend that we do not know that. If we take the same V as before, now the derivatives along trajectories are, using "$\dot{V}(x,u)$" to denote $\nabla V(x).f(x,u)$ and omitting arguments t in $x(t)$ and $u(t)$:

$$\dot{V}(x,u) = -x_1^2 + x_1 x_2 + x_2^2 - (x_1 + 2x_2)u\,.$$

This expression is affine in u. Thus, if x is a state such that $x_1 + 2x_2 \neq 0$, then we may pick a control value u (which depends on this current state x) such that $\dot{V} < 0$. On the other hand, if $x_1 + 2x_2 = 0$, then the expression reduces to $\dot{V} = -5x_2^2$ (for any u), which is negative unless x_2 (and hence also $x_1 = -2x_2$) vanishes.

In conclusion, for each $x \neq 0$ there is some u so that $\dot{V}(x,u) < 0$. This is, except for some technicalities to be discussed, the characterizing property of control-Lyapunov functions. For any given compact subset B in $\mathbb{R}^n$, we now pick some compact subset $\mathcal{U}_0 \subset \mathcal{U}$ so that

$$\forall x \in B, x \neq 0, \quad \exists\, u \in \mathcal{U}_0 \quad \text{such that} \quad \dot{V}(x,u) < 0\,. \tag{5}$$

In principle, then, we could then stabilize the system, for states in B, by using the steepest descent feedback law:

$$k(x) := \underset{u \in \mathcal{U}_0}{\operatorname{argmin}}\ \nabla V(x) \cdot f(x,u) \tag{6}$$

("argmin" means "pick any u at which the min is attained"; we restricted $\mathcal{U}$ to be assured that $\dot{V}(x,u)$ attains a minimum). Note that the stabilization problem becomes, in these terms, a set of static nonlinear programming problems: minimize a function of u, for each x. Global stabilization is also possible, by appropriately picking $\mathcal{U}_0$ as a function of the norm of x; later we discuss a precise formulation.

Control-Lyapunov functions, if understood non-technically as the basic paradigm "look for a function $V(x)$ with the properties that $V(x) \approx 0$ if and only if $x \approx 0$, and so that for each $x \neq 0$ it is possible to decrease $V(x)$ by some control action," constitute a very general approach to control (sometimes expressed in a dual fashion, as maximization of some measure of success). They appear in such disparate areas as A.I. game-playing programs (position evaluations), energy arguments for dissipative systems, program termination (Floyd/Dijkstra "variant"), and learning control ("critics" implemented by neural-networks). More relevantly to this paper, the idea underlies much of modern feedback control design, as illustrated for instance by the books [8, 12, 15, 14, 23].

Differentiable clf's: Precise Definition

We say that a continuous function

$$V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$$

is *positive definite* if $V(x) = 0$ only if $x = 0$, and it is *proper* (or "weakly coercive") if for each $a \geq 0$ the set $\{x \mid V(x) \leq a\}$ is compact, or, equivalently, $V(x) \to \infty$ as $|x| \to \infty$ (radial unboundedness). A property which is equivalent to properness and positive definiteness together is:

$$(\exists\, \underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty)\ \ \underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|)\ \ \forall x \in \mathbb{R}^n . \tag{7}$$

A *differentiable control-Lyapunov function* (clf) is a differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ which is proper, positive definite, and *infinitesimally decreasing*, meaning that there exists a positive definite continuous function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and there is some $\sigma \in \mathcal{N}$, so that

$$\sup_{x \in \mathbb{R}^n} \ \min_{|u| \leq \sigma(|x|)} \ \nabla V(x) \cdot f(x,u) + W(x) \ \leq \ 0 . \tag{8}$$

This is basically the same as condition (5), with $\mathcal{U}_0 =$ the ball of radius $\sigma(|x|)$ picked as a function of x. The main difference is that, instead of saying "$\nabla V(x) \cdot f(x,u) < 0$ for $x \neq 0$" we write $\nabla V(x) \cdot f(x,u) \leq -W(x)$, where W is negative when $x \neq 0$. The two definitions are equivalent, but the "Hamiltonian" version used here is the correct one for the generalizations to be given, to nonsmooth V.

Theorem B (Artstein [1]) *A control-affine system $\dot{x} = g_0(x) + \sum u_i g_i(x)$ admits a differentiable clf if and only if it admits a regular stabilizing feedback.*

The proof of sufficiency is easy: if there is such a k, then the converse Lyapunov theorem, applied to the closed-loop system $F(x) = f(x, k(x))$, provides a smooth V such that

$$L_F V(x) \ = \ \nabla V(x)\, F(x) \ < \ 0 \quad \forall x \neq 0 .$$

This gives that for all nonzero x there is some u (bounded on bounded sets, because k is locally bounded by definition of feedback) so that $\dot{V}(x,u) < 0$; and one can put this in the form (8).

The necessity is more interesting. The original proof in [1] proceeds by a nonconstructive argument involving partitions of unity, but it is also possible to exhibit explicitly a feedback, written as a function

$$k\left(\nabla V(x)\cdot g_0(x)\,,\,\ldots\,,\,\nabla V(x)\cdot g_m(x)\right)$$

of the directional derivatives of V along the vector fields defining the system (*universal formulas* for stabilization). Taking for simplicity $m=1$, one such formula is:

$$k(x) \;:=\; -\,\frac{a(x)+\sqrt{a(x)^2+b(x)^4}}{b(x)} \qquad (0 \text{ if } b=0)$$

where $a(x) := \nabla V(x)\cdot g_0(x)$ and $b(x) := \nabla V(x)\cdot g_1(x)$. (The expression for k is analytic in a, b when $x \neq 0$, because the clf property means that $a(x)<0$ whenever $b(x)=0$, see [23] for details.)

Thus, the question of existence of regular feedback, for control-affine systems, reduces to the search for differentiable clf's, and this gives rise to a vast literature dealing with the construction of such V's, see [8, 15, 14, 23] and references therein. Many other theoretical issues are also answered by Artstein's theorem. For example, via Kurzweil's converse theorem one has that the existence of k merely continuous on $\mathbb{R}^n\setminus\{0\}$ suffices for the existence of smooth (infinitely differentiable) V, and from here one may in turn find a k which is smooth on $\mathbb{R}^n\setminus\{0\}$. In addition, one may easily characterize the existence of k continuous at zero as well as regular: this is equivalent to the *small control property*: for each $\varepsilon>0$ there is some $\delta>0$ so that $0<|x|<\delta$ implies that $\min_{|u|\leq\varepsilon}\nabla V(x)\cdot f(x,u)<0$ (if this property holds, the universal formula automatically provides such a k). We should note that Artstein provided a result valid for general, not necessarily control-affine systems $\dot{x}=f(x,u)$; however, the obtained "feedback" has values in sets of relaxed controls, and is not a feedback law in the classical sense. Later, we discuss a different generalization.

Differentiable clf's will in general not exist, because of obstructions to regular feedback stabilization. This leads us naturally into the twin subjects of discontinuous feedbacks and non-differentiable clf's.

4 Discontinuous Feedback

The previous results and examples show that, in order to develop a satisfactory general theory of stabilization, one in which one proves the implication "asymptotic controllability implies feedback stabilizability," we must allow discontinuous feedback laws $u=k(x)$. But then, a major technical difficulty arises: solutions of the initial-value problem $\dot{x}=f(x,k(x))$, $x(0)=x^0$, interpreted in the classical sense of differentiable functions or even as (absolutely) continuous solutions of the integral equation $x(t)=$

$x^0 + \int_0^t f(x(s), k(x(s)))\, ds$, do not exist in general. The only general theorems apply to systems $\dot{x} = F(x)$ with continuous F. For example, there is no solution to $\dot{x} = -\operatorname{sign} x$, $x(0) = 0$, where $\operatorname{sign} x = -1$ for $x < 0$ and $\operatorname{sign} x = 1$ for $x \geq 0$. So one cannot even pose the stabilization problem in a mathematically consistent sense.

There is, of course, an extensive literature addressing the question of discontinuous feedback laws for control systems and, more generally, differential equations with discontinuous right-hand sides. One of the best-known candidates for the concept of solution of (3) is that of a *Filippov solution* [9, 10], which is defined as the solution of a certain differential inclusion with a multivalued right-hand side which is built from $f(x, k(x))$. Unfortunately, there is no hope of obtaining the implication "asymptotic controllability implies feedback stabilizability" if one interprets solutions of (3) as Filippov solutions. This is a consequence of results in [20, 7], which established that the existence of a discontinuous stabilizing feedback in the Filippov sense implies the Brockett necessary conditions, and, moreover, for systems affine in controls it also implies the existence of regular feedback (which we know is in general impossible).

A different concept of solution originates with the theory of discontinuous positional control developed by Krasovskii and Subbotin in the context of differential games in [13], and it is the basis of the new approach to discontinuous stabilization proposed in [5], to which we now turn.

4.1 Limits of High-Frequency Sampling

By a *sampling schedule* or *partition* $\pi = \{t_i\}_{i\geq 0}$ of $[0, +\infty)$ we mean an infinite sequence

$$0 = t_0 < t_1 < t_2 < \dots$$

with $\lim_{i\to\infty} t_i = \infty$. We call

$$\mathbf{d}(\pi) := \sup_{i\geq 0}(t_{i+1} - t_i)$$

the *diameter* of π. Suppose that k is a given feedback law for system (1). For each π, the *π-trajectory starting from x^0* of system (3) is defined recursively on the intervals $[t_i, t_{i+1})$, $i = 0, 1, \dots$, as follows. On each interval $[t_i, t_{i+1})$, the initial state is measured, the control value $u_i = k(x(t_i))$ is computed, and the constant control $u \equiv u_i$ is applied until time t_{i+1}; the process is then iterated. That is, we start with $x(t_0) = x^0$ and solve recursively

$$\dot{x}(t) = f(x(t), k(x(t_i)))\,,\ t \in [t_i, t_{i+1})\,, \quad i = 0, 1, 2, \dots$$

using as initial value $x(t_i)$ the endpoint of the solution on the preceding interval. The ensuing π-trajectory, which we denote as $x_\pi(\cdot, x^0)$, is defined on some maximal nontrivial interval; it may fail to exist on the entire interval $[0, +\infty)$ due to a blow-up on one of the subintervals $[t_i, t_{i+1})$. We say that it is *well defined* if $x_\pi(t, x^0)$ is defined on all of $[0, +\infty)$.

Definition. The feedback $k : \mathbb{R}^n \to \mathcal{U}$ *stabilizes* the system (1) if there exists a function $\beta \in \mathcal{KL}$ so that the following property holds: For each

$$0 < \varepsilon < K$$

there exists a $\delta = \delta(\varepsilon, K) > 0$ such that, for every sampling schedule π with $\mathbf{d}(\pi) < \delta$, and for each initial state x^0 with $|x^0| \leq K$, the corresponding π-trajectory of (3) is well-defined and satisfies

$$|x_\pi(t, x^0)| \leq \max\{\beta(K, t)\ ,\ \varepsilon\} \quad \forall t \geq 0\,. \tag{9}$$

In particular, we have

$$|x_\pi(t, x^0)| \leq \max\{\beta(|x^0|, t)\ ,\ \varepsilon\} \quad \forall t \geq 0 \tag{10}$$

whenever $0 < \varepsilon < |x^0|$ and $\mathbf{d}(\pi) < \delta(\varepsilon, |x^0|)$ (just take $K := |x^0|$).

Observe that the role of δ is to specify a lower bound on intersampling times. Roughly, one is requiring that

$$t_{i+1} \leq t_i + \theta(|x(t_i)|)$$

for each i, where θ is an appropriate positive function.

Our definition of stabilization is physically meaningful, and is very natural in the context of sampled-data (computer control) systems. It says in essence that a feedback k stabilizes the system if it drives all states asymptotically to the origin and with small overshoot when using *any fast enough sampling schedule*. A high enough sampling frequency is generally required when close to the origin, in order to guarantee small displacements, and also at infinity, so as to preclude large excursions or even blow-ups in finite time. This is the reason for making δ depend on ε and K.

This concept of stabilization can be reinterpreted in various ways. One is as follows. Pick any initial state x^0, and consider any sequence of sampling schedules π_ℓ whose diameters $\mathbf{d}(\pi_\ell)$ converge to zero as $\ell \to \infty$ (for instance, constant sampling rates with $t_i = i/\ell$, $i = 0, 1, 2, \ldots$). Note that the functions $x_\ell := x_{\pi_\ell}(\cdot, x^0)$ remain in a bounded set, namely the ball of radius $\beta(|x^0|, 0)$ (at least for ℓ large enough, for instance, any ℓ so that $\mathbf{d}(\pi_\ell) < \delta(|x^0|/2, |x^0|)$). Because $f(x, k(x))$ is bounded on this ball, these functions are equicontinuous, and (Arzela-Ascoli's Theorem) we may take a subsequence, which we denote again as $\{x_\ell\}$, so that $x_\ell \to x$ as $\ell \to \infty$ (uniformly on compact time intervals) for some absolutely continuous (even Lipschitz) function $x : [0, \infty) \to \mathbb{R}^n$. *We may think of any limit function $x(\cdot)$ that arises in this fashion as a generalized solution of the closed-loop equation (3).* That is, generalized solutions are the limits of trajectories arising from arbitrarily high-frequency sampling when using the feedback law $u = k(x)$. Generalized solutions, for a given initial state x^0, may not be unique – just as may happen with continuous but non-Lipschitz feedback – but there is always existence, and, moreover, for any generalized solution, $|x(t)| \leq \beta(|x^0|, t)$ for all $t \geq 0$. This is precisely the defining estimate for the GAS property. Moreover, if k happens to be regular, then the unique solution of $\dot{x} = f(x, k(x))$ in the classical sense is also the unique generalized solution, so we have a reasonable extension of the concept of solution. (This type of interpretation is somewhat analogous, at least in spirit, to the way in which "relaxed" controls are interpreted in optimal trajectory calculations, namely through high-frequency switching of approximating regular controls.)

Remark. The definition of stabilization was given in [5] in a slightly different form. It was required there that there exist for each $R > 0$ a number $M(R) > 0$, with $\lim_{R \searrow 0} M(R) = 0$, and, for each $0 < r < R$, numbers $\delta_0(r, R) > 0$ and $T(r, R) \geq 0$, such that the following property holds: for each sampling schedule π with $\mathbf{d}(\pi) < \delta_0(r, R)$, each x^0 with $|x^0| \leq R$, and each $t \geq T(r, R)$, it holds that $|x_\pi(t, x^0)| \leq r$, and, in addition, $|x_\pi(t, x^0)| \leq M(R)$ for all $t \geq 0$. This definition is equivalent to the definition that we gave, based on an estimate of the type (9). See Section A.1 for a proof.

4.2 Stabilizing Feedbacks Exist

In the paper [5], the following result was proven by Clarke, Ledyaev, Subbotin, and the author:

Theorem C *The system (1) admits a stabilizing feedback if and only if it is asymptotically controllable.*

Necessity is clear. The sufficiency statement is proved by construction of k, and is based on the following ingredients:

- Existence of a nonsmooth control-Lyapunov function V.
- Regularization on shells of V.
- Pointwise minimization of a Hamiltonian for the regularized V

In order to sketch this construction, we start by quickly reviewing a basic concept from nonsmooth analysis.

Proximal Subgradients

Let V be any continuous function $\mathbb{R}^n \to \mathbb{R}$ (or even, just lower semicontinuous and with extended real values). A *proximal subgradient* of V at the point $x \in \mathbb{R}^n$ is any vector $\zeta \in \mathbb{R}^n$ such that, for some $\sigma > 0$ and some neighborhood $\mathcal{O}$ of x,

$$V(y) \geq V(x) + \zeta \cdot (y - x) - \sigma^2 |y - x|^2 \quad \forall y \in \mathcal{O}.$$

In other words, proximal subgradients are the possible gradients of supporting quadratics at the point x. The set of all proximal subgradients at x is denoted $\partial_P V(x)$. For example (see Figure 9), the function $V(x) = |x|$

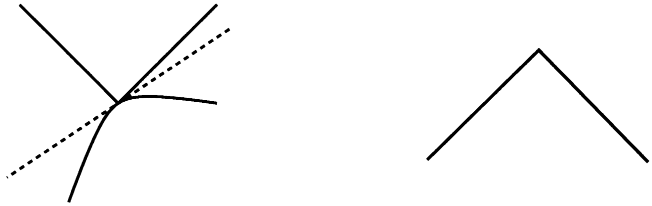

Figure 9: $\partial_P |x| (0) = [-1, 1]$ $\qquad$ $\partial_P(-|x|)(0) = \emptyset$

admits any $\zeta \in [-1, 1]$ as a proximal subgradient at $x = 0$ (elsewhere, $\partial_P V(x) = \{\nabla V(x)\}$), while for $V(x) = -|x|$ we have $\partial_P V(0) = \emptyset$ (because there are no possible quadrics that fit inside the graph and touch the corner).

Nonsmooth Control-Lyapunov Functions

A *continuous* (but not necessarily differentiable) $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a *control-Lyapunov function* (clf) if it is proper, positive definite, and infinitesimally decreasing in the following generalized sense: there exist a positive definite continuous $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and a $\sigma \in \mathcal{N}$ so that

$$\sup_{x \in \mathbb{R}^n} \max_{\zeta \in \partial_P V(x)} \min_{|u| \leq \sigma(|x|)} \zeta \cdot f(x, u) + W(x) \leq 0. \tag{11}$$

This is the obvious generalization of the differentiable case in (8); we are still asking that one should be able to make $\nabla V(x) \cdot f(x, u) < 0$ by an appropriate choice of $u = u_x$, for each $x \neq 0$, except that now we replace $\nabla V(x)$ by the proximal subgradient set $\partial_P V(x)$. An equivalent property is to ask that V be a viscosity supersolution of the corresponding Hamilton-Jacobi-Bellman equation.

In the paper [21], the following result was proven by the author:

Theorem D *The system (1) is asymptotically controllable if and only if it admits a continuous clf.*

Not surprisingly, the proof is based on first constructing an appropriate W, and then letting V be the optimal cost (Bellman function) for the problem $\min \int_0^\infty W(x(s))\, ds$. However, some care has to be taken to insure that V is continuous, and the cost has to be adjusted in order to deal with possibly unbounded minimizers. Actually, to be precise, the result as stated here is really a restatement (cf. [26], [5]) of the main theorem given in [21]. See Section A.2 for the details of this reduction.

Regularization

Once V is known to exist, the next step in the construction of a stabilizing feedback is to obtain Lipschitz approximations of V. For this purpose, one considers the Iosida-Moreau inf-convolution of V with a quadratic function:

$$V_\alpha(x) := \inf_{y\in\mathbb{R}^n}\left[V(y)+\frac{1}{2\alpha^2}\,|y-x|^2\right]$$

where the number $\alpha > 0$ is picked constant on appropriate regions. One has that $V_\alpha(x) \nearrow V(x)$, uniformly on compacts. Since V_α is locally Lipschitz, Rademacher's Theorem insures that V_α is differentiable almost everywhere. The feedback k is then made equal to a pointwise minimizer k_α of the Hamiltonian, at the points of differentiability (compare with (6) for the case of differentiable V):

$$k_\alpha(x) := \operatorname*{argmin}_{u\in\mathcal{U}_0} \nabla V_\alpha(x)\cdot f(x,u)\,,$$

where α and the compact $\mathcal{U}_0 = \mathcal{U}_0(\alpha)$ are chosen constant on certain compacts and this choice is made in between level curves, see Figure 10. The

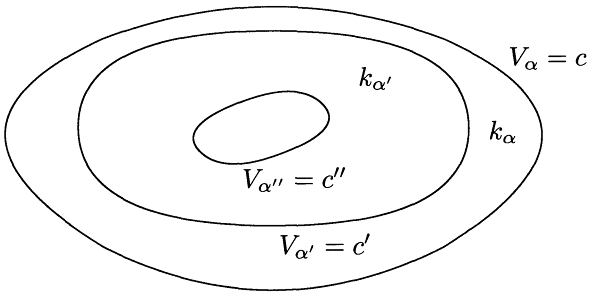

Figure 10: $k = k_\alpha$ on $\{x \mid V_\alpha(x) \le c, V_{\alpha'}(x) > c'\}$

critical fact is that V_α is itself a clf for the original system, at least when restricted to the region where it is needed. More precisely, on each shell of the form

$$C = \{x \in \mathbb{R}^n \mid r \le |x| \le R\}\,,$$

there are positive numbers m and α_0 and a compact subset $\mathcal{U}_0$ such that, for each $0 < \alpha \le \alpha_0$, each $x \in C$, and every $\zeta \in \partial_{\mathrm{P}} V_\alpha(x)$,

$$\min_{u\in\mathcal{U}_0} \zeta\cdot f(x,u) + m \le 0\,.$$

See Section A.3.

Actually, this description is oversimplified, and the proof is a bit more delicate. One must define, on appropriate compact sets

$$k(x) := \operatorname*{argmin}_{u \in \mathcal{U}_0} \zeta_\alpha(x) \cdot f(x,u)\,,$$

where $\zeta_\alpha(x)$ is carefully chosen. At points x of nondifferentiability, $\zeta_\alpha(x)$ is not a proximal subgradient of V_α, since $\partial_{\mathrm{P}} V_\alpha(x)$ may well be empty. One uses, instead, the fact that $\zeta_\alpha(x)$ happens to be in $\partial_{\mathrm{P}} V(x')$ for some $x' \approx x$.

An Example

As a simple example, we consider the system that was obtained from the two-dimensional reduction of the "shopping cart" problem, cf. Figure 8. For this example, continuous stabilization is not possible, and so no differentiable clf's can exist. On the other hand, the system is AC, so one can stabilize it using the techniques just described. A clf for this problem was obtained in [16]:

$$V(x_1,x_2) \;=\; \frac{x_1^2+x_2^2}{\sqrt{x_1^2+x_2^2}+|x_1|}$$

(0 if $x_1 = x_2 = 0$) and its level sets are as shown in Figure 11. Note

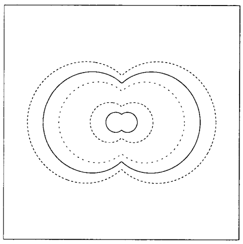

Figure 11: Clf Level Sets for system in Figure 8

that the nonsmoothness happens exactly on the x_2 axis (one has an empty subgradient at those points), and the clf inequality follows from the fact that

$$\inf_u \nabla V(x_1,x_2) \cdot f((x_1,x_2),u) \;\leq\; -\frac{1}{2}(x_1^2+x_2^2)$$

at points with $x_1 \neq 0$ (the proximal subgradient set is empty otherwise). The stabilizing feedback that results is the obvious one: if to the right of the x_2 axis, move clockwise, if to the left counterclockwise, and make an arbitrary decision (this arbitrariness corresponds to the choice of "$\zeta_\alpha(x)$" in the theory) on the x_2-axis.

5 Sensitivity to Small Measurement Errors

We have seen that every asymptotically controllable system admits a feedback stabilizer k, generally discontinuous, which renders the closed-loop system $\dot{x} = f(x, k(x))$ GAS. On the other hand, one of the main reasons for using feedback is to deal with uncertainty, and one possible source of uncertainty are measurement errors in state estimation. The use of discontinuous feedback means that undesirable behavior –chattering– may arise. In fact, one of the main reasons for the focus on continuous feedback is precisely in order to avoid such behaviors. Thus, we turn now to an analysis of the effect of measurement errors.

Suppose first that k is a continuous function of x. Then, if the error e is small, using the control $u' = k(x + e)$ instead of $u = k(x)$ results in behavior which remains close to the intended one, since $k(x + e) \approx k(x)$; moreover, if $e \ll x$ then stability is preserved. This property of robustness to small errors when k is continuous can be rigorously established by means of a Lyapunov proof, based on the observation that, if V is a Lyapunov function for the closed-loop system, then continuity of $f(x, k(x + e))$ on e means that

$$\nabla V(x) \cdot f(x, k(x + e)) \;\approx\; \nabla V(x) \cdot f(x, k(x)) \;<\; 0\,.$$

Unfortunately, when k is not continuous, this argument breaks down. However, it can be modified so as to avoid invoking continuity of k. Assuming that V is continuously differentiable, one can argue that

$$\nabla V(x) \cdot f(x, k(x + e)) \approx \nabla V(x + e) \cdot f(x, k(x + e)) \;<\; 0$$

(using the Lyapunov property at the point $x + e$ instead of at x). This observation leads to a theorem, formulated below, which says that a discontinuous feedback stabilizer, robust with respect to small observation errors, can be found provided that there is a $\mathcal{C}^1$ clf.

In general, as there are no $\mathcal{C}^1$, but only continuous, clf's, one may not be able to find any feedback law that is robust in this sense. We can see this fact intuitively with an example. Let us take once more the two-dimensional problem illustrated in Figure 8, and let us suppose that we are using the following control law: if to the right of, or exactly on, the x_2 axis, move clockwise, and if to the left move counterclockwise. See Figure 12, which indicates what happens on any circle. The main point that we wish to make is that *this feedback law is extremely sensitive to measurement errors*. Indeed, if the true state x is slightly to the left of the top point, but we mistakenly believe it to be to the right, we use a clockwise motion, in effect bringing the state towards the top, instead of downwards towards the target (the origin); see Figure 13. It is clear that, if we are unlucky enough to consistently make measurement errors that place us on the opposite side

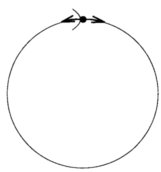

Figure 12: Feedback k on a typical integral manifold

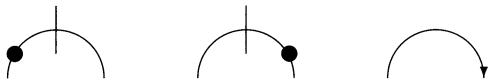

Figure 13: True state; measured state; erroneous motion commanded

of the true state, the result may well be an oscillation (chattering) around the top.

We might say, then, that the discontinuous feedback laws which are guaranteed to always exist by the general theorem in [5] lead to *fussy control*† –not to be confused, of course, with "fuzzy" control.

There are many well-known techniques for avoiding chattering, and a very common one is the introduction of deadzones where no action is taken. Indeed, in the above example, one may adopt the following modified control strategy: stay in the chosen direction (even if it might be "wrong") for some minimal time, until we are guaranteed to be far enough from the discontinuity; only after this minimal amount of time, we sample again. At this point, we know for sure on which side of the top we are. (This assumes that we have an upper bound on the magnitude of the error. Also, of course, observation errors when close to the origin will mean that we can only expect "practical" stability, meaning that we cannot be assured of convergence to the origin, but merely of convergence to a neighborhood of the origin whose size depends on the size of the observation errors.)

Such a control strategy is not a pure "continuous time" one, in that a minimum intersample time is required. It can be interpreted, rather, as constructing a *hybrid* (different time scales needed) and *dynamic* (requiring memory) controller. The paper [17] proves a general result showing the possibility of stabilization of every AC system, using an appropriate definition of general hybrid dynamic controllers. The controller given there incorporates an internal model of system. It compares, at appropriate sampling times, the state predicted by the internal model with the –noisy– observations of the state; whenever these differ substantially, a "reseting" is performed in the state of the controller.

Actually, already the feedback constructed in [5], with no modifications needed, can always be used in a manner robust with respect to small ob-

†Fussy (adjective): "... requiring ... close attention to details" (Webster).

servation errors, using the idea illustrated with the circle of not sampling again for some minimal period. Roughly speaking, the general idea is as follows.

Suppose that the true current state, let us say at time $t = t_i$, is x, but that the controller uses $u = k(\tilde{x})$, where $\tilde{x} = x + e$, and e is small. Call x' the state that results at the next sampling time, $t = t_{i+1}$. By continuity of solutions on initial conditions, $|x' - \tilde{x}'|$ is also small, where $\tilde{x}'$ is the state that would have resulted from applying the control u if the true state had been $\tilde{x}$. By continuity, it follows that $V_\alpha(x) \approx V_\alpha(\tilde{x})$ and also $V_\alpha(x') \approx V_\alpha(\tilde{x}')$. On the other hand, the construction in [5] provides that $V_\alpha(\tilde{x}') < V_\alpha(\tilde{x}) - d(t_{i+1} - t_i)$, where d is some positive constant (this is valid while we are far from the origin). Hence, if e is sufficiently small compared to the intersample time $t_{i+1} - t_i$, it will necessarily be the case that $V_\alpha(x')$ must also be smaller than $V_\alpha(x)$. See Figure 14. Thinking of

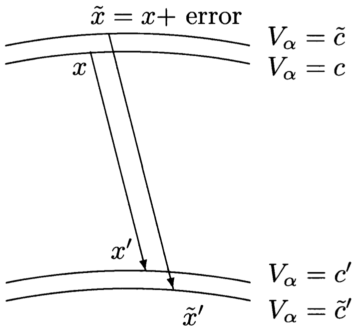

Figure 14: $t_{i+1} \gg t_i \;\Rightarrow\; \tilde{c}' \ll \tilde{c} \;\Rightarrow\; c' < c$

V_α as a Lyapunov function, this means that x' is made "smaller", even if the wrong control $u = k(\tilde{x})$, rather than $k(x)$, is applied.

This discussion may be formalized in several ways. We limit ourselves here to a theorem assuring semiglobal practical stability (i.e., driving all states in a given compact set of initial conditions into a specified neighborhood of zero). For any sampling schedule π, we denote

$$\underline{\mathbf{d}}(\pi) \;:=\; \inf_{i\geq 0}(t_{i+1} - t_i)\,.$$

If $e : [0,\infty) \to \mathbb{R}^n$ is any function ($e(t)$ is to be thought of as the state estimation error at time t), k is a feedback law, $x^0 \in \mathbb{R}^n$, and π is a sampling schedule, we define the solution of

$$\dot{x} \;=\; f(x, k(x+e))\,,\;\; x(0) = x^0 \tag{12}$$

as earlier, namely, recursively solving

$$\dot{x}(t) \;=\; f(x(t), k(x(t_i) + e(t_i)))$$

with initial condition $x(t_i)$ on the intervals $[t_i, t_{i+1}]$. The feedback stabilizer that was constructed in [5] was defined by patching together feedback laws, denoted there as k_ν (where $\nu = (\alpha, r, R)$ is a triple of positive numbers, with $r < R$). We use these same feedbacks in the statement of the result.

Theorem E *Suppose that the system (1) is asymptotically controllable. Then, there exists a function* $\Gamma \in \mathcal{K}_\infty$ *with the following property. For each* $0 < \varepsilon < K$, *there is a feedback of the type* k_ν, *and there exist positive* $\delta = \delta(\varepsilon, K)$, $\kappa = \kappa(\varepsilon, K)$, *and* $T = T(\varepsilon, K)$, *such that, for each sampling schedule* π *with* $\mathbf{d}(\pi) \leq \delta$, *each* $e : [0, \infty) \to \mathbb{R}^n$ *so that*

$$|e(t)| \leq \kappa\, \underline{\mathbf{d}}(\pi) \quad \forall\, t \geq 0\,,$$

and each x^0 *with* $|x^0| \leq K$, *the solution of the noisy system (12) satisfies*

$$|x(t)| \;\leq\; \Gamma(K) \quad \forall\, t \geq 0$$

and

$$|x(t)| \;\leq\; \varepsilon \quad \forall\, t \geq T\,.$$

See Section A.4.

5.1 A Necessary Condition

Theorem E insures that stabilization is possible if we sample "just right" (not too slow, so as to preserve stability, but also not too fast, so that observation errors do not cause chattering). It leaves open the theoretical question of precisely under what conditions is it possible to find a state feedback law which is robust with respect to small observation errors and which, on the other hand, is a continuous-time feedback, in the sense of arbitrarily fast sampling. The discussion preceding Theorem E suggests that this objective cannot always be met (e.g., for the circle problem), but that the existence of a C^1 clf might be sufficient for guaranteeing that it can. Indeed, this is what happens, as was proved in the recent paper [18]. We next present the main result from that paper.

We consider systems

$$\dot{x}(t) \;=\; f(x(t), k(x(t) + e(t)) + d(t)) \tag{13}$$

in which there are observation errors as well as, now, possible actuator errors $d(\cdot)$. (Actually, robustness to just small actuator errors is not a serious issue; the original paper [5] showed that the feedback laws obtained there stabilize even in the presence of such errors, or even model errors.) We assume that actuator errors $d(\cdot) : [0, \infty) \to \mathcal{U}$ are Lebesgue measurable and locally essentially bounded, and that observation errors $e(\cdot) : [0, \infty) \to \mathbb{R}^n$ are locally bounded.

We wish to define what it means for a feedback k to stabilize (13); roughly, we will ask that, for some $\beta \in \mathcal{KL}$, and for some "tolerance" function $\theta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, we have an estimate $|x(t,x^0)| \leq \beta(|x^0|,t)$ provided that $|e(t)| \leq \theta(|x(t)|)$ and $|d(t)| \leq \theta(|x(t)|)$ for all t (small enough errors). However, the definition is somewhat complicated by the need to appropriately choose sampling frequencies.

We define solutions of (13), for each sampling schedule π, in the usual manner, i.e., solving recursively on the intervals $[t_i, t_{i+1})$, $i = 0, 1, \ldots$, the differential equation

$$\dot{x}(t) = f\big(x(t)\,,\, k(x(t_i) + e(t_i)) + d(t)\big) \tag{14}$$

with $x(0) = x^0$. We write $x(t) = x_\pi(t, x^0, d, e)$ for the solution, and say it is *well-defined* if it is defined for all $t \geq 0$.

Definition. The feedback $k : \mathbb{R}^n \to \mathcal{U}$ *stabilizes* the system (13) if there exists a function $\beta \in \mathcal{KL}$ so that the following property holds: For each

$$0 < \varepsilon < K$$

there exist $\delta = \delta(\varepsilon, K) > 0$ and $\eta = \eta(\varepsilon, K)$ such that, for every sampling schedule π with $\mathbf{d}(\pi) < \delta$, each initial state x^0 with $|x^0| \leq K$, and each e, d such that $|e(t)| \leq \eta$ for all $t \geq 0$ and $|d(t)| \leq \eta$ for almost all $t \geq 0$, the corresponding π-trajectory of (13) is well-defined and satisfies

$$|x_\pi(t, x^0, d, e)| \leq \max\{\beta(K,t)\ ,\ \varepsilon\} \quad \forall\, t \geq 0\,. \tag{15}$$

In particular, taking $K := |x^0|$, one has that

$$|x_\pi(t, x^0, d, e)| \leq \max\{\beta(|x^0|,t)\ ,\ \varepsilon\} \quad \forall\, t \geq 0$$

whenever $0 < \varepsilon < |x^0|$, $\mathbf{d}(\pi) < \delta(\varepsilon, |x^0|)$, and for all t, $|e(t)| \leq \eta(\varepsilon, |x^0|)$, and $|d(t)| \leq \eta(\varepsilon, |x^0|)$.

The main result in [18] is as follows.

Theorem F *There is a feedback which stabilizes the system (13) if and only if there is a $\mathcal{C}^1$ clf for the unperturbed system (1).*

This result is somewhat analogous to a result obtained, for classical solutions, by Hermes in [11]; see also [10].

It is interesting to note that, as a corollary of Artstein's Theorem, for control-affine systems $\dot{x} = g_0(x) + \sum u_i g_i(x)$ we may conclude that if there is a discontinuous feedback stabilizer that is robust with respect to small noise, then there is also a regular one, and even one that is smooth on $\mathbb{R}^n \setminus \{0\}$.

For non control-affine systems, however, there may exist a discontinuous feedback stabilizer that is robust with respect to small noise, yet there is no regular feedback. For example, consider the following system with $n = 3$ and $m = 1$:

$$\begin{aligned} \dot{x}_1 &= u_2 u_3 \\ \dot{x}_2 &= u_1 u_3 \\ \dot{x}_3 &= u_1 u_2 \,. \end{aligned}$$

Here, there is a $\mathcal{C}^1$ clf, namely the squared norm $(x_1^2 + x_2^2 + x_3^2)$, but there is no possible regular feedback stabilizer, since Brockett's condition fails because points $(0, \neq 0, \neq 0)$ cannot be in the image of $(x, u) \mapsto f(x, u)$. (Because this is a homogeneous system with no drift, Brockett's condition rules out even feedbacks that are not continuous at the origin, see [18] for a remark to that effect.)

The sufficiency part of Theorem F proceeds by taking a pointwise minimization of the Hamiltonian, for a given $\mathcal{C}^1$ clf, i.e. $k(x)$ is defined as any u with $|u| \leq \sigma(|x|)$ which minimizes $\nabla V(x) \cdot f(x, u)$. The necessity part is based on the following technical fact: if the perturbed system can be stabilized, then the differential inclusion

$$\dot{x} \in F(x) := \bigcap_{\varepsilon > 0} \overline{\mathrm{co}}\, f(x, k(x + \varepsilon B))$$

(where B denotes the unit ball in $\mathbb{R}^n$) is strongly asymptotically stable. One may then apply the recent converse Lyapunov theorem of [3] for upper semicontinuous compact convex differential inclusions (which generalized to differential inclusions the theorem from [19] characterizing uniform asymptotic stability of systems with disturbances $\dot{x} = f(x, d)$) to deduce the existence of V.

We can now summarize exactly which implications hold. We write "robust" to mean stabilization of the system subject to observation and actuator noise:

$$\begin{array}{ccccc} \mathcal{C}^1\, V & \Longleftrightarrow & \exists \text{ robust } k & & \\ \Downarrow & & \Downarrow & & \\ \mathcal{C}^0\, V & \Longleftrightarrow & \exists\, k & \Longleftrightarrow & \text{AC} \end{array}$$

Remark. The definition of stabilization of (13) was given in [18] in a slightly different form than here. There, it was required that for each $0 < r < R$ there exist $M = M(R) > 0$ with $\lim_{R \searrow 0} M(R) = 0$, $\delta = \delta(r, R) > 0$, $T = T(r, R) > 0$, and $\eta = \eta(r, R)$, such that, for every partition π with $\mathbf{d}(\pi) < \delta$, each initial state with $|x^0| \leq R$, and each e, d such that $|e(t)| \leq \eta$ for all $t \geq 0$ and $|d(t)| \leq \eta$ for almost all $t \geq 0$, the π-trajectory of $\dot{x} = f(x, k(x + e) + d)$ is defined for all $t \geq 0$ and $|x(t)| \leq r \;\forall t \geq T$ and $|x(t)| \leq M(R) \;\forall t \geq 0$. This definition is equivalent to the definition that we just gave. The same proof as in Section A.1 applies.

Appendix: Proofs

We fill-in here the proofs of several technical points regarding stabilization.

A.1 Estimates for Stabilization

We prove here that if a feedback k stabilizes in the sense of the definition of stabilization given in [5], then it is also stabilizing in the sense of the definition given here, by means of an estimate of the type (9). (The converse implication is obvious.) Suppose $M(R) > 0$, $\delta_0(r, R) > 0$, and $T(r, R) \geq 0$, are such that $\lim_{R \searrow 0} M(R) = 0$, and whenever $\mathbf{d}(\pi) < \delta_0(r, R)$ and $|x^0| \leq R$, necessarily $|x_\pi(t, x^0)| \leq M(R)$ for all $t \geq 0$ and $|x_\pi(t, x^0)| \leq r$ for all $t \geq T(r, R)$. We first define $\delta = \delta(\varepsilon, R) > 0$, for each $0 < \varepsilon < R$, as follows: pick the smallest positive integer k (necessarily ≥ 2) such that $\frac{R}{k} \leq \varepsilon$, and let

$$\delta(\varepsilon, R) := \min\left\{\delta_0\left(\frac{R}{2}, R\right), \ldots, \delta_0\left(\frac{R}{k}, R\right)\right\}.$$

Next, we define a function

$$\varphi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$$

as follows. Pick any $R > 0$ and any $t \geq 0$. Let $0 < t_1 < t_2 < \ldots$ be a sequence of real numbers (depending on R) so that $t_i \to \infty$ as $i \to \infty$ and such that

$$t_i \geq T\left(\frac{R}{i+1}, R\right), \quad i = 1, 2, \ldots.$$

Now define $\varphi(R, t) := M(R)$ for $t \in [0, t_1)$ and $\varphi(R, t) := \frac{R}{i+1}$ if $t \in [t_i, t_{i+1})$ for some $i \geq 1$.

We claim now: for each $0 < \varepsilon < R$, each $|x^0| \leq R$, each $t \geq 0$, and each sampling schedule such that $\mathbf{d}(\pi) \leq \delta(\varepsilon, R)$,

$$|x_\pi(t, x^0)| \leq \max\{\varphi(R, t), \varepsilon\}.$$

Pick any such ε, R, x^0, π. Define the sequence $\{t_i\}$ as above, for this R. Let k be the smallest positive integer such that $\frac{R}{k} \leq \varepsilon$. By definition of δ, $\delta(\varepsilon, R) \leq \delta_0(R/j, R)$ for all $j = 2, \ldots, k$. We consider three cases: (i) $t < t_1$, (ii) $t \in [t_1, t_{k-1})$, and (iii) $t \geq t_{k-1}$. In the first case, we know that $|x_\pi(t, x^0)| \leq M(R) = \varphi(R, t)$. In the last case, we know that $|x_\pi(t, x^0)| \leq \varepsilon$, because $\mathbf{d}(\pi) \leq \delta_0(R/k, R)$ implies that $|x_\pi(t, x^0)| \leq R/k \leq \varepsilon$ for all $t \geq T(R/k, R)$, and $t_{k-1} \geq T(R/k, R)$ by definition of the t_i's. In case (ii), we have that there is some $j \in \{2, \ldots, k-1\}$ so that $t \in [t_{j-1}, t_j)$. Since $t_{j-1} \geq T(R/j, R)$ and $\mathbf{d}(\pi) \leq \delta_0(R/j, R)$, $|x_\pi(t, x^0)| \leq R/j = \varphi(R, t)$. The claim is then established.

It only remains to show that there is some function $\beta \in \mathcal{KL}$ so that $\varphi(s, t) \leq \beta(s, t)$ for all s, t. The constructions given in the first section of [19] show that there exists such a β, provided that the following properties hold for φ:

1. For some $\gamma \in \mathcal{K}_\infty$, $\sup_{t>0} \varphi(R,t) \leq \gamma(R)$ for all $R > 0$.
2. For each $\varepsilon > 0$ and each $R > 0$ there is some $T(\varepsilon, R)$ such that

$$t \geq T(\varepsilon, R) \;\Rightarrow\; \varphi(R,t) \leq \varepsilon\,.$$

The first property is satisfied because M could, without loss of generality, be taken to be of class $\mathcal{K}_\infty$ (if necessary, first use a step function M; then majorize it by a $\mathcal{K}_\infty$ function). The second property holds as well: take without loss of generality $\varepsilon < R$, then pick a positive integer k minimal with $R/k \leq \varepsilon$, let the sequence $\{t_i\}$ be as in the construction of φ, and define $T(\varepsilon, R) := t_{k-1}$. Observe that $t \geq t_{k-1}$ implies, by definition of φ, that $\varphi(R,t) \leq R/k \leq \varepsilon$, as wanted.

A.2 Proximal Form of clf Theorem

We fill-in the details here to show how the proximal subgradient form of the continuous clf existence Theorem D follows from the result in [21]. Before stating the original form of the result, we recall the notion of relaxed control. For each real $s > 0$, let us denote by $\mathcal{U}_s$ the radius-s ball in $\mathcal{U} = \mathbb{R}^m$. A *relaxed* $\mathcal{U}_s$*-valued control* is a measurable map $\omega : I \to \mathbb{P}(\mathcal{U}_s)$, where I is an interval containing zero and $\mathbb{P}(\mathcal{U}_s)$ denotes the set of all Borel probability measures on $\mathcal{U}_s$. Note that ordinary controls can be seen also as relaxed controls, via the natural embedding of $\mathcal{U}_s$ into $\mathbb{P}(\mathcal{U}_s)$ (map any point $u \in \mathcal{U}_s$ into the Dirac delta measure supported at u). Given any $\mu \in \mathbb{P}(\mathcal{U}_s)$, we write $\int_{\mathcal{U}_s} f(x,u)\, d\mu(u)$ simply as $f(x,\mu)$. As with ordinary controls, we also denote by $x(t, x^0, \omega)$ the solution of the initial value problem that obtains from initial state x^0 and relaxed control ω, and we consider the supremum norm $\|\omega\|$, defined as the infimum of the set of s such that $\omega(t) \in \mathbb{P}(\mathcal{U}_s)$ for almost all $t \in I$.

The main result in [21] says that if (and only if) a system is AC, there exist two continuous, positive definite functions $V, W : \mathbb{R}^n \to \mathbb{R}$, with V proper, and a nondecreasing function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, so that the following property holds: for each $x^0 \in \mathbb{R}^n$ there are a $T > 0$ and a relaxed control $\omega : [0,T) \to \mathbb{P}(\mathcal{U}_{\sigma(|x^0|)})$, so that $x(t) := x(t, x^0, \omega)$ is defined for all $0 \leq t < T$ and

$$V(x(t)) - V(x^0) \leq -\int_0^t W(x(\tau))\, d\tau \quad \forall\, t \in [0,T)\,. \tag{16}$$

To simplify notations, let us write $s := \sigma(|x^0|)$. In order to obtain the proximal version of the result, we show that

$$\min_{|u| \leq s} \zeta \cdot f(x^0, u) \;\leq\; -W(x^0) \tag{17}$$

for all $\zeta \in \partial_{\mathrm{P}} V(x^0)$.

As in [26], we first make an intermediate reduction, showing that

$$\min_{v \in F(x^0,s)} DV(x^0;v) \;\leq\; -W(x^0)\,, \tag{18}$$

where $F(x^0,s)$ denotes the (closed) convex hull of $\{f(x^0,u), u \in \mathcal{U}_s\}$, and $DV(x^0;v)$ is the directional subderivate, or contingent epiderivative, of V in the direction of v at x^0, defined as

$$DV(x^0;v) \;:=\; \liminf_{\substack{t \searrow 0 \\ v' \to v}} \;\frac{1}{t}\left[V(x^0+tv') - V(x^0)\right] .$$

(The minimum in (18) is achieved, because the map $v \mapsto DV(x^0;v)$ is lower semicontinuous, see e.g. [4], ex.3.4.1e.) So, let $x(t) = x(t,x^0,\omega)$ be as above, and consider for each $t \in [0,T)$ the vectors

$$r_t \;:=\; \frac{1}{t}\left(x(t) - x^0\right) \;=\; \frac{1}{t}\int_0^t f(x(\tau),\omega(\tau))\,d\tau \;=\; q_t + p_t\,,$$

with

$$p_t \;:=\; \frac{1}{t}\int_0^t f(x^0,\omega(\tau))\,d\tau\,,$$

where $q_t \to 0$ as $t \searrow 0$ (the existence of such a q_t, for each t, is an easy consequence of the fact that f is locally Lipschitz on x, uniformly on $u \in \mathcal{U}_s$). Moreover, $p_t \in F(x^0,s)$ for all t (because $F(x^0,s)$ is convex, so $f(x^0,\omega(\tau)) \in F(x^0,s)$ for each τ, and then using convexity once more). By compactness, we have that there is some $v \in F(x^0,s)$ and some subsequence $p_{t_j} \to v$ with $t_j \searrow 0$; as $q_t \to 0$, also $v_j := r_{t_j} \to v$. For this v,

$$\begin{aligned} DV(x^0;v) \;&\leq\; \liminf_{j\to\infty} \frac{1}{t_j}\left[V(x^0+t_jv_j) - V(x^0)\right] \\ &=\; \liminf_{j\to\infty} \frac{1}{t_j}\left[V(x(t_j)) - V(x^0)\right] \;\leq\; -W(x^0) \end{aligned}$$

(using (16)), so (18) indeed holds.

Finally, we show that (18) implies (17). Let $v \in F(x^0,s)$ achieve the minimum, and pick any $\zeta \in \partial_{\mathrm{P}} V(x^0)$. By definition of proximal subgradient, there is some $\mu > 0$ so that, for each $v' \in F(x^0,s)$ and each small $t \geq 0$,

$$\zeta \cdot v' \;\leq\; \frac{1}{t}\left[V(x^0+tv') - V(x^0)\right] \;+\; \mu\, t\, |v'|^2$$

so taking limits along any sequence $t \searrow 0$ and $v' \to v$ shows that $\zeta \cdot v \leq DV(x^0;v) \leq -W(x^0)$. By definition of $F(x^0,s)$, this means that there must exist a $u \in \mathcal{U}_0$ so that also $\zeta \cdot f(x^0,u) \leq -W(x^0)$, as desired.

Observe that the proximal condition (17), being linear on the velocities $f(x,u)$, has the great advantage of not requiring convex hulls in order to state, and in that sense is far more elegant than (18).

A.3 V_α is a (Local) clf

We prove here that, on each set $C_{r,R} = \{x \in \mathbb{R}^n \mid r \leq |x| \leq R\}$, there is an $\alpha_0(r,R) > 0$ such that, for each positive $\alpha \leq \alpha_0(r,R)$, the (locally Lipschitz) function V_α behaves like a clf on the set $C_{r,R}$. In order to simplify referencing, we write "([5]n)" to refer to Equation (n) in [5] and do not redefine notations given there. Let $\mathcal{U}_0$ be a compact subset so that $\min_{u\in\mathcal{U}_0} \zeta \cdot f(x,u) \leq -W(x)$ for every $\zeta \in \partial_{\mathrm{P}} V(x)$ and every x in the ball of radius $R + \sqrt{2\beta(R)}$. Let $m_{r,R} := \frac{1}{2}\min\{W(x) \mid x \in C_{r,R}\}$, and let ℓ be so that $|f(x,u) - f(x',u)| \leq \ell\,|x - x'|$ for all $u \in \mathcal{U}_0$ and all x in the ball of radius $R + \sqrt{2\beta(R)}$. (This is almost as in ([5]29), except that, there, ℓ was a Lipschitz constant only with respect to $|x| \leq R$.) Finally, as in [5], $\omega_R(\cdot)$ denotes the modulus of continuity of V on the ball of radius $R + \sqrt{2\beta(R)}$.

Proposition. Let $\alpha \in (0,1]$ satisfy $2\ell\,\omega_R\left(\alpha\sqrt{2\beta(R)}\right) \leq m_{r,R}$. Then, for all $x \in C_{r,R}$ and all $\zeta \in \partial_{\mathrm{P}} V_\alpha(x)$,

$$\min_{u\in\mathcal{U}_0} \zeta \cdot f(x,u) \;\leq\; -m_{r,R}\,.$$

Proof. We start by remarking that, for all $\alpha \in (0,1]$ and all $x \in B_R$,

$$|y_\alpha(x) - x|^2 \;\leq\; 2\alpha^2\,\omega_R\left(\alpha\sqrt{2\beta(R)}\right)\,. \tag{19}$$

This follows from (cf. ([5]19)):

$$\frac{1}{2\alpha^2}\,|y_\alpha(x) - x|^2 \;\leq\; V(x) - V(y_\alpha(x)) \;\leq\; \omega_R\left(|y_\alpha(x) - x|\right)$$

and using that $|y_\alpha(x) - x| \leq \alpha\sqrt{2\beta(R)}$ (cf. Lemma III.3 in [5]) plus the fact that $\omega_R(\cdot)$ is nondecreasing. So

$$|\zeta_\alpha(x)| \cdot |y_\alpha(x) - x| \;=\; \frac{|y_\alpha(x) - x|^2}{\alpha^2} \;\leq\; 2\,\omega_R\left(\alpha\sqrt{2\beta(R)}\right)$$

and hence

$$\zeta_\alpha(x)\cdot[f(y_\alpha(x),u) - f(x,u)] \;\leq\; \ell\,|\zeta_\alpha(x)|\cdot|y_\alpha(x) - x| \;\leq\; 2\ell\,\omega_R\left(\alpha\sqrt{2\beta(R)}\right)$$

for all $u \in \mathcal{U}_0$ and all $x \in B_R$, because $y_\alpha(x) \in B_{R+\sqrt{2\beta(R)}}$ (cf. ([5]20)). Thus,

$$\zeta_\alpha(x) \cdot f(x,u) \;\leq\; \zeta_\alpha(x) \cdot f(y_\alpha(x),u) \,+\, 2\ell\,\omega_R\left(\alpha\sqrt{2\beta(R)}\right)\,.$$

For each $x \in C_{r,R}$, we have that $y_\alpha(x) \in B_{R+\sqrt{2\beta(R)}}$ and (cf. ([5]12)) $\zeta_\alpha(x) \in \partial_{\mathrm{P}} V(y_\alpha(x))$, so we can pick a $u_x \in \mathcal{U}_0$ so that $\zeta_\alpha(x) \cdot f(y_\alpha(x), u_x) \leq -W(x) \leq -2m_{r,R}$. Using now the fact that $2\ell\,\omega_R\left(\alpha\sqrt{2\beta(R)}\right) \leq m_{r,R}$, we conclude that

$$\zeta_\alpha(x) \cdot f(x,u) \;\leq\; -m_{r,R}\,.$$

So we only need to prove that $\partial_{\mathrm{P}} V_\alpha(x) \subseteq \{\zeta_\alpha(x)\}$ for all x. This is given in [4], Theorem 1.'5.1, but is easy to prove directly: pick any $\zeta \in \partial_{\mathrm{P}} V_\alpha(x)$; by definition, this means that exists some $\sigma > 0$ such that, for all y near x, $\zeta \cdot (y-x) \leq V_\alpha(y) - V_\alpha(x) + \sigma\,|y-x|^2$, so

$$\zeta \cdot (y-x) \;\leq\; V_\alpha(y) - V_\alpha(x) + \gamma\left(|y-x|\right) \tag{20}$$

for some $\gamma(r) = o(r)$. Adding

$$-\zeta_\alpha(x) \cdot (y-x) \;\leq\; -V_\alpha(y) + V_\alpha(x) + \frac{1}{2\alpha^2}\,|y-x|^2$$

(cf. ([5]13)), we conclude

$$(\zeta - \zeta_\alpha(x)) \cdot (y-x) \;\leq\; o\left(|y-x|\right)\,.$$

Substituting $y = x + h(\zeta - \zeta_\alpha(x))$ and letting $h \searrow 0$ shows that $\zeta = \zeta_\alpha(x)$. (Observe that we have proved more that claimed: Equation (20) is satisfied by any viscosity subgradient ζ; in particular, the gradient of V, if V happens to be differentiable at the point x, must coincide with $\zeta_\alpha(x)$.) ∎

Observe that the feedback constructed in [5] was $k(x) =$ any $u \in \mathcal{U}_0$ minimizing $\zeta_\alpha(x) \cdot f(x,u)$ (where α and $\mathcal{U}_0$ are chosen constant on certain compacts). As V_α is locally Lipschitz, it is differentiable almost everywhere. Thus, the Proposition (see the end of the proof) insures that $\zeta_\alpha(x) = \nabla V_\alpha(x)$ for almost all x. So $k(x) = u$ is, at those points, the pointwise minimizer of the Hamiltonian $\nabla V_\alpha(x) \cdot f(x,u)$ associated to the regularized clf V_α.

A.4 Proof of Theorem E

We will prove the following more precise result. All undefined notations, including the definitions of the functions γ and ρ, can be found in [5].

Theorem G *Pick any $0 < r < R$ so that $2\gamma(r) < \gamma(R)$. Then there exist positive numbers $\alpha, \delta, \kappa, T$ such that, for each partition π with $\mathbf{d}(\pi) \leq \delta$, and each $e : [0,\infty) \to \mathbb{R}^n$ which satisfies $|e(t)| \leq \kappa\,\underline{\mathbf{d}}(\pi)$ for all t, the following property holds: if $x(\cdot)$ satisfies*

$$\dot{x} = f(x, k(x+e))\,, \quad |x(0)| \leq \tfrac{1}{2}\rho(R)\,, \tag{21}$$

where k is the feedback $k_{\alpha,r,R}$, then

$$x(t) \in B_R \quad \forall t \geq 0 \tag{22}$$

and

$$x(t) \in B_r \quad \forall t \geq T\,. \tag{23}$$

Theorem E is a corollary: we first pick any Γ such that $s \leq (1/2)\rho(\Gamma(s))$ for all $s \geq 0$. Then, given ε and K, we can let $R := \Gamma(K)$ (so, $K \leq (1/2)\rho(R)$), we then take any $0 < r < \varepsilon$ such that $2\gamma(r) \leq \gamma(R)$, and apply the above result to $0 < r < R$.

We prove Theorem G through a series of technical steps. Let $0 < r < R$ be given, with $2\gamma(r) < \gamma(R)$. In order to simplify referencing, we write "([5]n)" to refer to Equation (n) in [5].

We start by picking α as any positive number which satisfies ([5]23), ([5]30), and, also, instead of ([5]40), the slightly stronger condition

$$\omega_R\left(\sqrt{2\beta(R)}\alpha\right) < \frac{1}{16}\gamma(r)\,. \tag{24}$$

The function V_α is defined as in ([5]9); because of ([5]13), ([5]11), and ([5]19), the number

$$c := \frac{\sqrt{2\beta(R)}}{\alpha} + \frac{R}{\alpha^2}$$

is a Lipschitz constant for V_α on the set B_R. Without loss of generality, we assume $c \geq 2$. We let $\mathcal{U}_0$ be as in [5], and take the feedback $k = k_{\alpha,r,R}$. The numbers ℓ, m, and Δ are as in ([5]29) and the equation that follows it.

Next, we pick any $\varepsilon_0 > 0$ so that all the following – somewhat redundant – properties hold:

$$B_{\frac{1}{2}\rho(R)} + \varepsilon_0 B \subseteq G_R^\alpha \tag{25}$$

(this is possible because $B_{\rho(R)} \subseteq G_R^\alpha$ by ([5]22)),

$$G_R^\alpha + 2\varepsilon_0 B \subseteq B_R \tag{26}$$

(possible because $G_R^\alpha \subseteq \operatorname{int} B_R$ by ([5]24)),

$$\varepsilon_0 \leq \frac{\gamma(r)}{8c} \leq \frac{\gamma(r)}{16}\,, \tag{27}$$

$$\gamma(r) + c\varepsilon_0 < \tfrac{1}{2}\gamma(R) \tag{28}$$

(recall that $2\gamma(r) < \gamma(R)$), and

$$G_r^\alpha + 2\varepsilon_0 B \subseteq G_R^\alpha\,. \tag{29}$$

We let δ_0 be any positive number so that, for every initial state x^0 in the compact set G_R^α, and for each control $u : [0, \delta_0] \to \mathcal{U}_0$, the solution of $\dot{x} = f(x,u)$ with $x(0) = x^0$ is defined on the entire interval $[0, \delta_0]$ and satisfies $x(t) \in G_R^\alpha + \varepsilon_0 B$ for all t.

Finally, we pick any $\delta > 0$ which satisfies ([5]33), ([5]41), as well as

$$\delta \le \min\left\{1\,,\,\delta_0\,,\,\frac{2}{\Delta}\varepsilon_0\,,\,\frac{\gamma(r)}{8cm}\right\}, \tag{30}$$

and we let

$$\kappa := \frac{\Delta}{4c}e^{-\ell} \tag{31}$$

and

$$T := \frac{\gamma(R)}{\Delta} \tag{32}$$

(this is twice the value used in ([5]39)).

The main technical fact needed is as follows.

Proposition A.4.1 *Let $0 < r < R$ satisfy $2\gamma(r) < \gamma(R)$. Let $\alpha, \delta, \kappa, T$ be defined as above, and k be the feedback $k_{\alpha,r,R}$. Pick any $\varepsilon > 0$, and consider the following set:*

$$P = P_{r,R,\varepsilon} := \{x \mid x + \varepsilon B \subseteq G_R^\alpha\}\,.$$

Let π be a partition which satisfies

$$\frac{\varepsilon}{\kappa} \le t_{i+1} - t_i \le \delta \quad \forall i = 0, 1, \ldots \tag{33}$$

(that is, $\mathbf{d}(\pi) \le \delta$ and $\varepsilon \le \kappa\,\underline{\mathbf{d}}(\pi)$). Then, for any $e : [0,\infty) \to \mathbb{R}^n$ such that $|e(t)| \le \varepsilon$ for all t, and any $x^0 \in P$, the solution of $\dot{x} = f(x, k(x+e))$ is defined for all $t \ge 0$, it satisfies (22) and (23), and $x(t_i) \in P$ for all i.

We will prove this via a couple of lemmas, but let us first point out how Theorem G follows from Proposition A.4.1. Suppose given a partition π with $\mathbf{d}(\pi) \le \delta$, an error function e so that $|e(t)| \le \kappa\,\underline{\mathbf{d}}(\pi)$ for all t, and an initial state x^0 with $|x^0| \le \frac{1}{2}\rho(R)$. Let $\varepsilon := \sup_{t\ge 0}|e(t)| \le \kappa\,\underline{\mathbf{d}}(\pi)$. Note that

$$\varepsilon \le \kappa\,\underline{\mathbf{d}}(\pi) \le \kappa\,\mathbf{d}(\pi) \le \kappa\,\delta \le \kappa\frac{2}{\Delta}\varepsilon_0 \le \kappa\frac{4c}{\Delta}e^{\ell}\varepsilon_0 = \varepsilon_0\,. \tag{34}$$

Thus, by (25),

$$B_{\frac{1}{2}\rho(R)} \subseteq P\,, \tag{35}$$

so $x^0 \in P$. Therefore, (22) and (23) hold for the solution of $\dot{x} = f(x, k(x+e))$, as wanted for Theorem G.

We now prove Proposition A.4.1. Observe that (33) implies $\varepsilon \le \kappa\,\underline{\mathbf{d}}(\pi)$, so, arguing as in (34), $\varepsilon \le \varepsilon_0$. It is useful to introduce the following set as well:

$$Q = Q_{r,R,\varepsilon} := G_r^\alpha + \varepsilon B\,.$$

Observe that $Q \subseteq P$, by (29).

We start the proof by establishing an analogue of Lemma IV.2 in [5]:

Lemma A.4.1 *If, for some index i, $x_i := x(t_i) \in P \setminus Q$, then $x(t)$ is defined for all $t \in [t_i, t_{i+1}]$,*

$$x(t) \in B_R \quad \forall t \in [t_i, t_{i+1}], \tag{36}$$

$$V_\alpha(x(t)) \leq V_\alpha(x_i) + \varepsilon_0 \quad \forall t \in [t_i, t_{i+1}], \tag{37}$$

and, letting $x_{i+1} := x(t_{i+1})$:

$$x_{i+1} \in P, \tag{38}$$

$$V_\alpha(x_{i+1}) - V_\alpha(x_i) \leq -\frac{\Delta}{2}(t_{i+1} - t_i). \tag{39}$$

Proof. By definition of P, we have that $\tilde{x}_i := x_i + e(t_i) \in G_R^\alpha$. Also, $\tilde{x}_i \notin G_r^\alpha$, since otherwise x_i would belong to Q. In particular, $\tilde{x}_i \notin B_{\rho(r)}$. Let $\tilde{x}(\cdot)$ be the solution of $\dot{x} = f(x, k(\tilde{x}_i))$ with $\tilde{x}(t_i) = \tilde{x}_i$ on $[t_i, t_{i+1}]$. By Lemma IV.2 in [5], this solution is well-defined and it holds that $\tilde{x}(t_{i+1}) \in G_R^\alpha$ and $V_\alpha(\tilde{x}(t)) - V_\alpha(\tilde{x}_i) \leq -\Delta\,(t - t_i)$ for all t. As $x_i \in P \subseteq G_R^\alpha$, and $t_{i+1} - t_i \leq \mathbf{d}(\pi) \leq \delta \leq \delta_0$, the definition of δ_0 insures that the solution $x(\cdot)$ of $\dot{x} = f(x, k(\tilde{x}_i))$ with $x(t_i) = x_i$ is indeed well-defined, and it stays in $G_R^\alpha + \varepsilon_0 B \subseteq B_R$ for all t. So, by Gronwall's inequality, we know that

$$|x(t) - \tilde{x}(t)| \leq e^{(t-t_i)\ell}\,|x_i - \tilde{x}_i| \leq e^{\delta\ell}\varepsilon$$

for all $t \in [t_i, t_{i+1}]$. Since V_α has Lipschitz constant c on B_R, we have

$$\begin{aligned}
&V_\alpha(x(t)) - V_\alpha(x_i) \\
&\quad = \; V_\alpha(x(t)) - V_\alpha(\tilde{x}(t)) + V_\alpha(\tilde{x}(t)) - V_\alpha(\tilde{x}_i) + V_\alpha(\tilde{x}_i) - V_\alpha(x_i) \\
&\quad \leq \; ce^{\delta\ell}\varepsilon - \Delta\,(t - t_i) + c\varepsilon \\
&\quad \leq \; \frac{\Delta}{2}(t_{i+1} - t_i) - \Delta\,(t - t_i)
\end{aligned}$$

for all $t \in [t_i, t_{i+1}]$, where we have used that $\delta \leq 1$, $\varepsilon \leq \kappa\underline{\mathbf{d}}(\pi)$, the definition $\kappa = (\Delta/4c)e^{-\ell}$, and the fact that $\underline{\mathbf{d}}(\pi) \leq t_{i+1} - t_i$. In particular, the estimate (39) results at $t = t_{i+1}$, and (37) holds because $(\Delta/2)\delta \leq \varepsilon_0$ by (30).

We are only left to prove that $x_{i+1} \in P$. By definition of P, this means that for any given $\eta \in \mathbb{R}^n$ with $|\eta| \leq \varepsilon$ it must hold that

$$\eta + x_{i+1} \in G_R^\alpha = \{x \mid V_\alpha(x) \leq (1/2)\gamma(R)\}.$$

Pick such an η; then $|\eta + x_{i+1} - \tilde{x}(t_{i+1})| \leq \varepsilon + e^{\delta\ell}\varepsilon \leq 2e^\ell\varepsilon$, and so, since $\eta + x_{i+1} \in G_R^\alpha + 2\varepsilon_0 B \subseteq B_R$,

$$\begin{aligned}
V_\alpha(\eta + x_{i+1}) &\leq 2ce^\ell\varepsilon + V_\alpha(\tilde{x}(t_{i+1})) \\
&\leq 2ce^\ell\varepsilon + V_\alpha(\tilde{x}_i) - \Delta\,(t_{i+1} - t_i) \\
&\leq V_\alpha(\tilde{x}_i) - \frac{\Delta}{2}(t_{i+1} - t_i) \leq \frac{1}{2}\gamma(R)
\end{aligned}$$

where we again used $\varepsilon \leq \kappa\,\underline{\mathbf{d}}(\pi)$ as well as the fact that $V_\alpha(\tilde{x}_i) \leq (1/2)\gamma(R)$ (since $\tilde{x}_i \in G_R^\alpha$). ∎

We also need another observation, this one paralleling the proof of Lemma IV.4 in [5].

Lemma A.4.2 *If, for some index i, $x_i := x(t_i) \in Q$, then $x(t)$ is defined for all $t \in [t_i, t_{i+1}]$,*

$$V_\alpha(x(t)) \leq \frac{3}{4}\gamma(r) \quad \forall t \in [t_i, t_{i+1}]\,, \tag{40}$$

and

$$V(x(t)) \leq \frac{7}{8}\gamma(r) \quad \forall t \in [t_i, t_{i+1}]\,. \tag{41}$$

In particular, $x(t) \in B_r$ for all $t \in [t_i, t_{i+1}]$ and $x(t_{i+1}) \in P$.

Proof. The fact that x is defined follows from the choice of δ_0, and we know that $x(t) \in B_R$ for all $t \in [t_i, t_{i+1}]$. So

$$|x(t) - x_i| \leq m\delta \quad \forall t \in [t_i, t_{i+1}]\,. \tag{42}$$

By definition of Q, we may write $x_i = x' + \eta$, for some $x' \in G_r^\alpha$ and some $|\eta| \leq \varepsilon$. Thus, $V_\alpha(x_i) \leq V_\alpha(x') + c\varepsilon_0 \leq \frac{1}{2}\gamma(r) + c\varepsilon_0$ (second inequality by definition of G_r^α). Together with (42), this gives

$$V_\alpha(x(t)) \leq \frac{1}{2}\gamma(r) + c\varepsilon_0 + cm\delta \leq \frac{3}{4}\gamma(r) \quad \forall t \in [t_i, t_{i+1}]$$

(using (27) and (30)). So, using (24) and ([5]21),

$$V(x(t)) \leq V_\alpha(x(t)) + \omega_R\left(\sqrt{2\beta(R)\alpha}\right) \leq \frac{3}{4}\gamma(r) + \frac{1}{16}\gamma(r) < \frac{7}{8}\gamma(r)$$

for all $t \in [t_i, t_{i+1}]$, as wanted. By the definition of γ (see [5]), this means that $x(t) \in B_r$ for all $t \in [t_i, t_{i+1}]$. Finally, if $|\eta| \leq \varepsilon$ then

$$V_\alpha(x(t_{i+1}) + \eta) \leq c\varepsilon + \frac{3}{4}\gamma(r) \leq \frac{1}{2}\gamma(R)$$

(the last inequality by (28)), which means that $x(t_{i+1}) + \eta \in G_R^\alpha$; this implies that $x(t_{i+1}) \in P$. ∎

Back to the proof of Proposition A.4.1, since $x^0 \in P$, Lemmas A.4.1 and A.4.2 guarantee that the solution exists for all t and remains in B_R, and that $x_i := x(t_i) \in P$ for all i.

Moreover, if there is some j so that $x(t_j) \in Q$, then it holds that $V_\alpha(x_i) \leq \frac{3}{4}\gamma(r)$ for all $i > j$. This is because on intervals in which $x_{i-1} \in Q$, we already know that $V_\alpha(x(t_i)) \leq \frac{3}{4}\gamma(r)$, and if instead $x_{i-1} \in P \setminus Q$, then we have $V_\alpha(x_i) < V_\alpha(x_{i-1})$. So, for any such $i > j$, either $x(t) \in B_r$ for all

$t \in [t_i, t_{i+1}]$ (first case) or $V_\alpha(x(t)) \leq \frac{3}{4}\gamma(r) + \varepsilon_0$ for all $t \in [t_i, t_{i+1}]$ (second case). Actually, in this last case we also have

$$\begin{aligned} V(x(t)) &\leq V_\alpha(x(t)) + \omega_R\left(\sqrt{2\beta(R)}\alpha\right) \\ &\leq \frac{3}{4}\gamma(r) + \varepsilon_0 + \omega_R\left(\sqrt{2\beta(R)}\alpha\right) \\ &\leq \frac{3}{4}\gamma(r) + \frac{1}{16}\gamma(r) + \frac{1}{16}\gamma(r) < \gamma(r), \end{aligned}$$

(using (27) and (24)), so, again by definition of γ, also $x(t) \in B_r$ for all $t \in [t_i, t_{i+1}]$. In conclusion, trajectories stay in B_r after the first time that $x(t_j) \in Q$. So we only need to show that there is such a j, with $t_j \leq T$.

Suppose instead that for $i = 0, \ldots, k$ it holds that $x(t_i) \notin Q$, and $t_k > T$. Applying (39) repeatedly,

$$0 \leq V_\alpha(x(t_k)) \leq V_\alpha(x^0) - \frac{\Delta}{2}t_k < \frac{\gamma(R)}{2} - \frac{\Delta}{2}T,$$

(recall that $x^0 \in P$ implies that $x \in G_R^\alpha$), and this contradicts (32). The proof of Proposition A.4.1 is then complete. ∎

This completes the proof. We remark that a more global result is also possible, as follows. We start by picking two sequences $\{r_j, j \in \mathbb{Z}\}$ and $\{R_j, j \in \mathbb{Z}\}$ such that $r_j, R_j \to 0$ as $j \to -\infty$, $r_j, R_j \to \infty$ as $j \to \infty$, and $2R_j \leq \rho(R_{j+1})$, $2\gamma(r_j) < \gamma(R_j)$, and $2r_j \leq \rho(R_{j-1})$ for all j. Next we pick, for each j, positive numbers $\alpha_j, \delta_j, \kappa_j, T_j$ associated as per Proposition A.4.1 to r_j and R_j, and let $k_j := k_{\alpha_j, r_j, R_j}$. (We may assume that values of k_j belong to some fixed $\mathcal{U}_0$ for all $j \leq 0$, and to $\mathcal{U}_j$ for $j > 0$, with all the $\mathcal{U}_j$ compact and forming an increasing sequence.) Since $G_{R_j}^{\alpha_j} \subseteq \operatorname{int} G_{R_{j+1}}^{\alpha_{j+1}}$ for all j (this is proved just as in ([5]48)), there is some sequence of positive numbers $\{\varepsilon_j, j \in \mathbb{Z}\}$ so that $\varepsilon_j < \kappa_j\delta_j$ for all j and also, denoting $P_j := P_{r_j, R_j, \varepsilon_j}$, so that $P_j \subseteq P_{j+1}$ for all j. Note that the choice of the r_j's and R_j's assures that $\bigcup P_j = \mathbb{R}^n \setminus \{0\}$. Since $2r_j \leq \rho(R_{j-1})$ for all j, and using (35), we know that $B_{r_j} \subseteq P_{j-1}$ for all j.

Finally, we define the feedback $k : \mathbb{R}^n \to \mathbb{R}^m$ via:

$$k(x) := k_j(x) \text{ if } x \in P_j \setminus P_{j-1}$$

(and $k(0) = 0$), and let $\varepsilon(x) := \varepsilon_j$, $\mu(x) := \frac{\varepsilon_j}{\kappa_j}$, and $\delta(x) := \delta_j$ for $x \in P_j \setminus P_{j-1}$, for each j (let $\varepsilon_j(0) = 0$ and $0 < m(0) < \delta(0)$ be arbitrary). Observe that, since $\varepsilon_j < \kappa_j\delta_j$ for all j, it holds that $\mu(x) < \delta(x)$ for all x.

Now suppose that x^0 is given, and a partition π and a function $e(\cdot)$ are given so that, recursively along the solution of $\dot{x} = f(x, k(x+e))$,

$$\mu(x(t_i)) \leq t_{i+1} - t_i \leq \delta(x(t_i))$$

and

$$e(t_i) \leq \varepsilon(x(t_i))$$

for all i. Note that, if for some i and j, $x(t_i) \in P_j \setminus P_{j-1}$, then the inequality in (33) holds with $\varepsilon = \varepsilon_j$, $\kappa = \kappa_j$, $\delta = \delta_j$, and this i.

Let $j \in \mathbb{Z}$ be maximal so that $x^0 \in P_j$. Then, it will hold that the solution stays in the bounded set B_{R_j}, and every $x(t_i)$ is again in P_j, until the first sampling time t_i that $x(t_i) \in P_\ell$ for some $\ell < j$. This first time, say t_q, is at most T_j, because $B_{r_j} \subseteq P_{j-1}$. If $x(t_q) = 0$, then the solution stays there forever (since $f(0,0) = 0$). Otherwise, $x(t_q) \in P_\ell \setminus P_{\ell-1}$ for some $\ell < j$, and we may repeat the argument. The conclusion is that the trajectory keeps visiting smaller sets P_ℓ (or it becomes 0 in finite time), with an upper bound $T(i,j)$ (the sum of the corresponding T_ℓ's) on the time required for entering a given P_i, if the initial state x^0 was in a given P_j. Furthermore, given any $0 < r < R$, there are $i < j$ so that $P_i \subseteq B_r \subseteq B_R \subseteq P_j$. Thus, all trajectories in B_R are taken into B_r in a uniform time $T(r,R)$, with bounded overshoot (since trajectories stay in B_{R_j}).

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Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA; Email: `sontag@control.rutgers.edu`