Input-to-State Stability

Eduardo D. Sontag
Departments of Bioengineering and Electrical and Computer Engineering, Northeastern University, Boston, MA, USA

Abstract

The notion of input-to-state stability (ISS) qualitatively describes stability of the mapping from initial states and inputs to internal states (and more generally outputs). The entry focuses on the definition of ISS and a discussion of equivalent characterizations.

Keywords

Input-to-state stability · Lyapunov functions · Dissipation · Asymptotic stability

Introduction

We consider here systems with inputs in the usual sense of control theory:

$$\dot{x}(t) = f(x(t), u(t))$$

(the arguments "t" is often omitted). There are n state variables and m input channels. States $x(t)$ take values in Euclidean space $\mathbb{R}^n$, and the inputs (also called "controls" or "disturbances" depending on the context) are measurable locally essentially bounded maps $u(\cdot) : [0, \infty) \to \mathbb{R}^m$. The map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be locally Lipschitz with $f(0,0) = 0$. The solution, defined on some maximal interval $[0, t_{\max}(x^0, u))$, for each initial state x^0 and input u, is denoted as $x(t, x^0, u)$, and in particular, for systems with no inputs $\dot{x}(t) = f(x(t))$, just as $x(t, x^0)$. The *zero-system* associated with $\dot{x} = f(x, u)$ is by definition the system with no inputs $\dot{x} = f(x, 0)$. Euclidean norm is written as $|x|$. For a function of time, typically an input or a state trajectory, $\|u\|$, or $\|u\|_\infty$ for emphasis, is the (essential) supremum or "sup" norm (possibly $+\infty$, if u is not bounded). The norm of the restriction of a signal to an interval I is denoted by $\|u_I\|_\infty$ (or just$\|u_I\|$).

Input-to-State Stability

It is convenient to introduce "comparison functions" to quantify stability. A *class* $\mathcal{K}_\infty$ *function* is a function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which is continuous, strictly increasing, and unbounded and satisfies $\alpha(0) = 0$, and a *class* $\mathcal{KL}$ *function* is a function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each t and $\beta(r, t)$ decreases to zero as $t \to \infty$, for each fixed r.

For a system with no inputs $\dot{x} = f(x)$, there is a well-known notion of global asymptotic stability (for short from now on, *GAS* or "*0-GAS*" when referring to the zero-system $\dot{x} = f(x, 0)$ associated with a given system with inputs

© Springer-Verlag London Ltd., part of Springer Nature 2020
J. Baillieul, T. Samad (eds.), *Encyclopedia of Systems and Control*,
https://doi.org/10.1007/978-1-4471-5102-9_78-2

$\dot{x} = f(x,u)$) due to Lyapunov, and usually defined in "ε-δ" terms. It is an easy exercise to show that this standard definition is in fact equivalent to the following statement:

$$(\exists\, \beta \in \mathcal{KL}) \quad |x(t,x^0)| \;\le\; \beta\left(|x^0|, t\right) \quad \forall\, x^0,\ \forall\, t \ge 0\,.$$

The notion of input-to-state stability (ISS) was introduced in Sontag (1989), and it provides theoretical concepts used to describe stability features of a mapping $(u(\cdot), x(0)) \mapsto x(\cdot)$ that sends initial states and input functions into states (or, more generally, outputs). Prominent among these features are that inputs that are bounded, "eventually small," "integrally small," or convergent, should lead to outputs with the respective property. In addition, ISS and related notions quantify in what manner initial states affect transient behavior. The formal definition is as follows.

A system is said to be *input to state stable (ISS)* if there exist some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$|x(t)| \;\le\; \beta(\left|x^0\right|, t) + \gamma\left(\|u\|_\infty\right) \qquad \text{(ISS)}$$

holds for all solutions (meaning that the estimate is valid for all inputs $u(\cdot)$, all initial conditions x^0, and all $t \ge 0$). Note that the supremum $\sup_{s\in[0,t]} \gamma(|u(s)|)$ over the interval $[0,t]$ is the same as $\gamma(\|u_{[0,t]}\|_\infty) = \gamma(\sup_{s\in[0,t]}(|u(s)|))$, because the function γ is increasing, so one may replace this term by $\gamma(\|u\|_\infty)$, where $\|u\|_\infty = \sup_{s\in[0,\infty)} \gamma(|u(s)|)$ is the sup norm of the input, because the solution $x(t)$ depends only on values $u(s), s \le t$ (so, one could equally well consider the input that has values $\equiv 0$ for all $s > t$).

Since, in general, $\max\{a,b\} \le a + b \le \max\{2a, 2b\}$, one can restate the ISS condition in a slightly different manner, namely, asking for the existence of some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ (in general different from the ones in the ISS definition) such that

$$|x(t)| \;\le\; \max\left\{\beta(|x^0|, t)\,,\ \gamma\left(\|u\|_\infty\right)\right\}$$

holds for all solutions. Such redefinitions, using "max" instead of sum, are also possible for each of the other concepts to be introduced later.

Intuitively, the definition of ISS requires that, for t large, the size of the state must be bounded by some function of the sup norm – that is to say, the amplitude – of inputs (because $\beta(|x^0|, t) \to 0$ as $t \to \infty$). On the other hand, the $\beta(|x^0|, 0)$ term may dominate for small t, and this serves to quantify the magnitude of the transient (overshoot) behavior as a function of the size of the initial state x^0 (Fig. 1). The *ISS superposition theorem*, discussed later, shows that ISS is, in a precise mathematical sense, the conjunction of two properties, one of them dealing with asymptotic bounds on $|x^0|$ as a function of the magnitude of the input and the other one providing a transient term obtained when one ignores inputs.

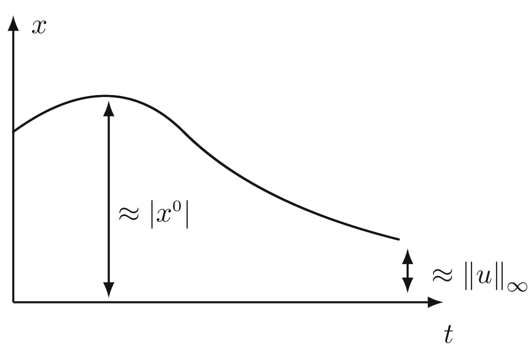
For internally stable linear systems $\dot{x} = Ax + Bu$, the variation of parameters formula gives immediately the following inequality:

$$|x(t)| \;\le\; \beta(t)\left|x^0\right| + \gamma\,\|u\|_\infty\,,$$

where

$$\beta(t) = \left\|e^{tA}\right\| \;\to\; 0 \text{ and}$$

$$\gamma = \|B\| \int_0^\infty \left\|e^{sA}\right\| ds \;<\; \infty\,.$$



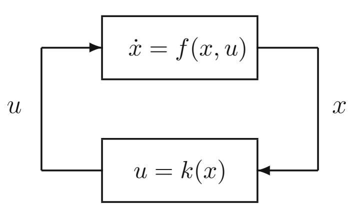
Input-to-State Stability, Fig. 1 ISS combines overshoot and asymptotic behavior

This is a particular case of the ISS estimate, $|x(t)| \le \beta(|x^0|, t) + \gamma(\|u\|_\infty)$, with linear comparison functions.

Feedback Redesign

The notion of ISS arose originally as a way to precisely formulate, and then answer, the following question. Suppose that, as in many problems in control theory, a system $\dot{x} = f(x, u)$ has been stabilized by means of a feedback law $u = k(x)$ (Fig. 2), that is to say, k was chosen such that the origin of the closed-loop system $\dot{x} = f(x, k(x))$ is globally asymptotically stable. (See, e.g., Sontag 1999 for a discussion of mathematical aspects of state feedback stabilization.) Typically, the design of k was performed by ignoring the effect of possible *input disturbances* $d(\cdot)$ (also called actuator disturbances). These "disturbances" might represent true noise or perhaps errors in the calculation of the value $k(x)$ by a physical controller or modeling uncertainty in the controller or the system itself. What is the effect of considering disturbances? In order to analyze the problem, d is incorporated into the model, and one studies the new system $\dot{x} = f(x, k(x) + d)$, where d is seen as an input (Fig. 3). One may then ask what is the effect of d on the behavior of the system. Disturbances d may well destabilize the system, and the problem may arise even when using a routine technique for control design, feedback linearization. To appreciate this issue, take the following very simple example. Given is the system

$$\dot{x} = f(x, u) = x + (x^2 + 1)u \,.$$



Input-to-State Stability, Fig. 2 Feedback stabilization, closed-loop system $\dot{x} = f(x, k(x))$

In order to stabilize it, substitute $u = \frac{\tilde{u}}{x^2+1}$ (a preliminary feedback transformation), rendering the system linear with respect to the new input $\tilde{u}$: $\dot{x} = x + \tilde{u}$, and then use $\tilde{u} = -2x$ in order to obtain the closed-loop system $\dot{x} = -x$. In other words, in terms of the original input u, the feedback law is

$$k(x) = \frac{-2x}{x^2 + 1}$$

so that $f(x, k(x)) = -x$. This is a GAS system. The effect of the disturbance input d is analyzed as follows. The system $\dot{x} = f(x, k(x) + d)$ is:

$$\dot{x} = -x + (x^2 + 1)\, d \,.$$

This system has solutions which diverge to infinity even for inputs d that converge to zero; moreover, the constant input $d \equiv 1$ results in solutions that explode in finite time. Thus $k(x) = \frac{-2x}{x^2+1}$ was not a good feedback law, in the sense that its performance degraded drastically once actuator disturbances were taken into account.

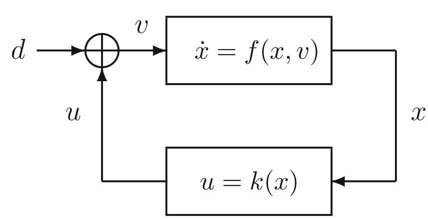
The key observation for what follows is that if one adds a correction term "$-x$" to the above formula for $k(x)$, so that now:

$$\tilde{k}(x) = \frac{-2x}{x^2 + 1} \boldsymbol{-x}$$

then the system $\dot{x} = f(x, \tilde{k}(x) + d)$ with disturbance d as input becomes, instead:

$$\dot{x} = -2x - x^3 + (x^2 + 1)\, d$$

and this system is much better behaved: it is still GAS when there are no disturbances (it reduces



Input-to-State Stability, Fig. 3 Actuator disturbances, closed-loop system $\dot{x} = f(x, k(x) + d)$

to $\dot{x} = -2x - x^3$) but, in addition, it is ISS (easy to verify directly, or appealing to some of the characterizations mentioned later). Intuitively, for large x, the term $-x^3$ serves to dominate the term $(x^2+1)d$, for all bounded disturbances $d(\cdot)$, and this prevents the state from getting too large.

This example is an instance of a general result, which says that, whenever there is some feedback law that stabilizes a system, there is also a (possibly different) feedback so that the system with external input d is ISS.

Theorem 1 (Sontag 1989) *Consider a system affine in controls*

$$\begin{aligned}\dot{x} &= f(x,u) \\ &= g_0(x) + \sum_{i=1}^{m} u_i g_i(x) \qquad (g_0(0)=0)\end{aligned}$$

and suppose that there is some differentiable feedback law $u = k(x)$ *so that*

$$\dot{x} = f(x, k(x))$$

has $x = 0$ *as a GAS equilibrium. Then, there is a feedback law* $u = \widetilde{k}(x)$ *such that*

$$\dot{x} = f(x, \widetilde{k}(x) + d)$$

is ISS with input $d(\cdot)$

The reader is referred to the book Krstić et al. (1995), and the references given later, for many further developments on the subjects of recursive feedback design, the "backstepping" approach, and other far-reaching extensions.

Equivalences for ISS

This section reviews results that show that ISS is equivalent to several other notions, including asymptotic gain, existence of robustness margins, dissipativity, and an energy-like stability estimate.

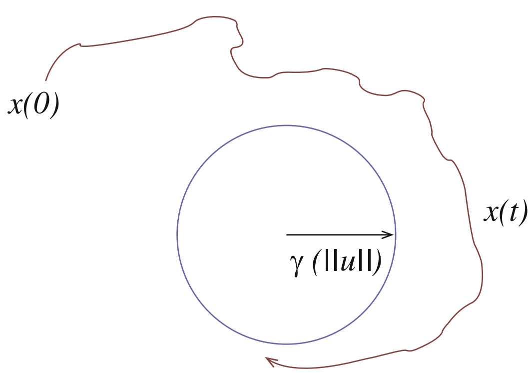
Nonlinear Superposition Principle

Clearly, if a system is ISS, then the system with no inputs $\dot{x} = f(x,0)$ is GAS: the term $\|u\|_\infty$ vanishes, leaving precisely the GAS property. In particular, then, the system $\dot{x} = f(x,u)$ is *0-stable*, meaning that the origin of the system without inputs $\dot{x} = f(x,0)$ is stable in the sense of Lyapunov: for each $\varepsilon > 0$, there is some $\delta > 0$ such that $|x^0| < \delta$ implies $|x(t,x^0)| < \varepsilon$. (In comparison-function language, one can restate 0-stability as there is some $\gamma \in \mathcal{K}$ such that $|x(t,x^0)| \le \gamma(|x^0|)$ holds for all small x^0.)

On the other hand, since $\beta(|x^0|, t) \to 0$ as $t \to \infty$, for t large one has that the first term in the ISS estimate $|x(t)| \le \max\{\beta(|x^0|,t), \gamma(\|u\|_\infty)\}$ vanishes. Thus an ISS system satisfies the following *asymptotic gain property ("AG")*: there is some $\gamma \in \mathcal{K}_\infty$ so that:

$$\overline{\lim_{t\to+\infty}} \left|x(t,x^0,u)\right| \le \gamma(\|u\|_\infty) \quad \forall\, x^0,\ u(\cdot) \quad \text{(AG)}$$

(see Fig. 4). In words, for all large enough t, the trajectory exists, and it gets arbitrarily close to a sphere whose radius is proportional, in a possibly nonlinear way quantified by the function γ, to the amplitude of the input. In the language of robust control, the estimate (AG) would be called an "ultimate boundedness" condition; it is a generalization of attractivity (all trajectories converge to zero, for a system $\dot{x} = f(x)$ with no inputs) to the case of systems with



Input-to-State Stability, Fig. 4 Asymptotic gain property inputs; the "limsup" is required since the limit of $x(t)$ as $t \to \infty$ may well not exist. From now on (and analogously when defining other properties), we will just say "the system is AG" instead of the more cumbersome "satisfies the AG property".

Observe that, since only large values of t matter in the limsup, one can equally well consider merely tails of the input u when computing its sup norm. In other words, one may replace $\gamma(\|u\|_\infty)$ by $\gamma(\overline{\lim}_{t\to+\infty}|u(t)|)$, or (since γ is increasing), $\overline{\lim}_{t\to+\infty}\gamma(|u(t)|)$.

The surprising fact is that these two necessary conditions are also sufficient. This is summarized by the *ISS superposition theorem*:

Theorem 2 (Sontag and Wang 1996) *A system is ISS if and only if it is 0-stable and AG.*

A minor variation of the above superposition theorem is as follows. Let us consider the *limit property (LIM)*:

$$\inf_{t\geq 0}|x(t,x^0,u)| \leq \gamma(\|u\|_\infty) \quad \forall\, x^0,\ u(\cdot) \qquad \text{(LIM)}$$

(for some $\gamma \in \mathcal{K}_\infty$).

Theorem 3 (Sontag and Wang 1996) *A system is ISS if and only if it is 0-stable and LIM.*

Robust Stability

In this entry, a system is said to be *robustly stable* if it admits a *margin of stability* ρ, that is, a smooth function $\rho \in \mathcal{K}_\infty$ so the system

$$\dot{x} = g(x,d) := f(x, d\rho(|x|))$$

is GAS uniformly in this sense: for some $\beta\in\mathcal{KL}$,

$$\left|x(t,x^0,d)\right| \leq \beta(\left|x^0\right|, t)$$

for all possible $d(\cdot) : [0,\infty) \to [-1,1]^m$. An alternative way to interpret this concept (cf. Sontag and Wang 1995) is as uniform global asymptotic stability of the origin with respect to all possible time-varying feedback laws Δ bounded by ρ: $|\Delta(t,x)| \leq \rho(|x|)$. In other words, the system

$$\dot{x} = f(x, \Delta(t,x))$$

(Fig. 5) is stable uniformly over all such perturbations Δ. In contrast to the ISS definition, which deals with all possible "open-loop" inputs, the present notion of robust stability asks about all possible closed-loop interconnections. One may think of Δ as representing uncertainty in the dynamics of the original system, for example.

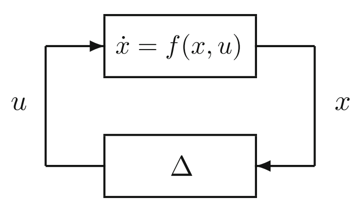
Theorem 4 (Sontag and Wang 1995) *A system is ISS if and only if it is robustly stable.*

Intuitively, the ISS estimate $|x(t)| \leq \max\{\beta(|x^0|,t), \gamma(\|u\|_\infty)\}$ says that the β term dominates as long as $|u(t)| \ll |x(t)|$ for all t, but $|u(t)| \ll |x(t)|$ amounts to $u(t) = d(t).\rho(|x(t)|)$ with an appropriate function ρ. This is an instance of a "small-gain" argument; see below. One analog for linear systems is as follows: if A is a Hurwitz matrix, then $A + Q$ is also Hurwitz, for all small enough perturbations Q; note that when Q is a nonsingular matrix, $|Qx|$ is a $\mathcal{K}_\infty$ function of $|x|$.

Dissipation

Another characterization of ISS is as a dissipation notion stated in terms of a Lyapunov-like function. A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a *storage function* if it is positive definite, that is, $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$, and proper, that is, $V(x) \to \infty$ as $|x| \to \infty$. This last property is equivalent to the requirement that the sets $V^{-1}([0,A])$ should be compact subsets of $\mathbb{R}^n$, for each $A > 0$, and in the engineering literature, it is usual to call such functions *radially unbounded*. It is an easy exercise to show that $V : \mathbb{R}^n \to \mathbb{R}$ is a storage function if and only if there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \quad \forall\, x \in \mathbb{R}^n$$



Input-to-State Stability, Fig. 5 Margin of robustness (the lower bound amounts to properness and $V(x) > 0$ for $x \neq 0$, while the upper bound guarantees $V(0) = 0$). For convenience, $\dot{V} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is the function:

$$\dot{V}(x,u) := \nabla V(x).f(x,u)$$

which provides, when evaluated at $(x(t), u(t))$, the derivative $dV(x(t))/dt$ along solutions of $\dot{x} = f(x,u)$.

An *ISS-Lyapunov function* for $\dot{x} = f(x,u)$ is by definition a smooth storage function V for which there exist functions $\gamma, \alpha \in \mathcal{K}_\infty$ so that

$$\dot{V}(x,u) \;\leq\; -\alpha(|x|) + \gamma(|u|) \quad \forall\, x, u\,. \qquad \text{(L-ISS)}$$

Integrating, an equivalent statement is that, along all trajectories of the system, there holds the following dissipation inequality:

$$V(x(t_2)) - V(x(t_1)) \;\leq\; \int_{t_1}^{t_2} w(u(s), x(s))\, ds$$

where, using the terminology of Willems (1976), the "supply" function is $w(u,x) = \gamma(|u|) - \alpha(|x|)$. For systems with no inputs, an ISS-Lyapunov function is precisely the same object as a Lyapunov function in the usual sense.

Theorem 5 (Sontag and Wang 1995) *A system is ISS if and only if it admits a smooth ISS-Lyapunov function.*

Since $-\alpha(|x|) \leq -\alpha(\overline{\alpha}^{-1}(V(x)))$, the ISS-Lyapunov condition can be restated as

$$\dot{V}(x,u) \;\leq\; -\tilde{\alpha}(V(x)) \;+\; \gamma(|u|) \quad \forall\, x, u$$

for some $\tilde{\alpha} \in \mathcal{K}_\infty$. In fact, one may strengthen this a bit (Praly and Wang 1996): for any ISS system, there is always a smooth ISS-Lyapunov function satisfying the "exponential" estimate $\dot{V}(x,u) \leq -V(x) + \gamma(|u|)$.

The sufficiency of the ISS-Lyapunov condition is easy to show and was already in the original paper (Sontag 1989). A sketch of proof is as follows, assuming for simplicity a dissipation estimate in the form $\dot{V}(x,u) \leq -\alpha(V(x)) + \gamma(|u|)$. Given any x and u, either $\alpha(V(x)) \leq 2\gamma(|u|)$ or $\dot{V} \leq -\alpha(V)/2$. From here, one deduces by a comparison theorem that, along all solutions,

$$V(x(t)) \;\leq\; \max\left\{\beta(V(x^0), t)\,,\, \alpha^{-1}(2\gamma(\|u\|_\infty))\right\}\,,$$

where the $\mathcal{KL}$ function $\beta(s,t)$ is the solution $y(t)$ of the initial value problem

$$\dot{y} = -\frac{1}{2}\alpha(y) + \gamma(u)\,, \quad y(0) = s\,.$$

Finally, an ISS estimate is obtained from $V(x^0) \leq \overline{\alpha}(x^0)$.

The proof of the converse part of the theorem is based upon first showing that ISS implies robust stability in the sense already discussed and then obtaining a converse Lyapunov theorem for robust stability for the system $\dot{x} = f(x, d\rho(|x|)) = g(x,d)$, which is asymptotically stable uniformly on all Lebesgue-measurable functions $d(\cdot) : \mathbb{R}_{\geq 0} \to B(0,1)$. This last theorem was given in Lin et al. (1996) and is basically a theorem on Lyapunov functions for differential inclusions. The classical result of Massera (1956) for differential equations (with no inputs) becomes a special case.

Using "Energy" Estimates Instead of Amplitudes

In linear control theory, H_∞ theory studies $L^2 \to L^2$ induced norms, which under coordinate changes leads to the following type of estimate:

$$\int_0^t \alpha\left(|x(s)|\right) ds \;\leq\; \alpha_0(\left|x^0\right|) + \int_0^t \gamma(|u(s)|)\, ds$$

along all solutions, and for some $\alpha, \alpha_0, \gamma \in \mathcal{K}_\infty$. Just for the statement of the next result, a system is said to *satisfy an integral-integral estimate* if for every initial state x^0 and input u, the solution $x(t, x^0, u)$ is defined for all $t > 0$ and an estimate as above holds. (In contrast to ISS, this definition explicitly demands that $t_{\max} = \infty$.)

Theorem 6 (Sontag 1998) *A system is ISS if and only if it satisfies an integral-integral estimate.*

This theorem is quite easy to prove, in view of previous results. A sketch of proof is as follows. If the system is ISS, then there is an

ISS-Lyapunov function satisfying $\dot{V}(x,u) \leq -V(x) + \gamma(|u|)$, so, integrating along any solution:

$$\int_0^t V(x(s))\,ds \leq \int_0^t V(x(s))\,ds + V(x(t)) \leq V(x(0)) + \int_0^t \gamma(|u(s)|)\,ds$$

and thus an integral-integral estimate holds. Conversely, if such an estimate holds, one can prove that $\dot{x} = f(x,0)$ is stable and that an asymptotic gain exists.

Integral Input-to-State Stability

A concept of nonlinear stability that is truly distinct from ISS arises when considering a mixed notion which combines the "energy" of the input with the amplitude of the state. A system is said to be *integral input to state stable (iISS)* provided that there exist $\alpha, \gamma \in \mathcal{K}_\infty$, and $\beta \in \mathcal{KL}$ such that the estimate

$$\alpha(|x(t)|) \leq \beta(\left|x^0\right|, t) + \int_0^t \gamma(|u(s)|)\,ds \quad \text{(iISS)}$$

holds along all solutions. Just as with ISS, one could state this property merely for all times $t \in t_{\max}(x^0, u)$. Since the right-hand side is bounded on each interval $[0,t]$ (because, recall, inputs are by definition assumed to be bounded on each finite interval), it is automatically true that $t_{\max}(x^0, u) = +\infty$ if such an estimate holds along maximal solutions. So forward completeness (solution exists for all $t > 0$) can be assumed with no loss of generality.

One might also consider the following type of "weak integral to integral" mixed estimate:

$$\int_0^t \underline{\alpha}(|x(s)|)\,ds \leq \kappa(|x^0|) + \alpha\left(\int_0^t \gamma(|u(s)|)\,ds\right)$$

for appropriate $\mathcal{K}_\infty$ functions (note the additional "α").

Theorem 7 (Angeli et al. 2000) *A system satisfies a weak integral to integral estimate if and only if it is iISS.*

Another interesting variant is found when considering mixed *integral/supremum* estimates:

$$\underline{\alpha}(|x(t)| \leq \beta(|x^0|, t) + \int_0^t \gamma_1(|u(s)|)\,ds + \gamma_2(\|u\|_\infty)$$

for suitable $\beta \in \mathcal{KL}$ and $\underline{\alpha}, \gamma_i \in \mathcal{K}_\infty$. One then has:

Theorem 8 (Angeli et al. 2000) *A system satisfies a mixed estimate if and only if it is iISS.*

Dissipation Characterization of iISS

A smooth storage function V is an *iISS-Lyapunov function* for the system $\dot{x} = f(x,u)$ if there are a $\gamma \in \mathcal{K}_\infty$ and an $\alpha : [0,+\infty) \to [0,+\infty)$ which is merely *positive definite* (i.e., $\alpha(0) = 0$ and $\alpha(r) > 0$ for $r > 0$) such that the inequality:

$$\dot{V}(x,u) \leq -\alpha(|x|) + \gamma(|u|) \quad \text{(L-iISS)}$$

holds for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$. To compare, recall that an ISS-Lyapunov function is required to satisfy an estimate of the same form but where α is required to be of class $\mathcal{K}_\infty$; since every $\mathcal{K}_\infty$ function is positive definite, an ISS-Lyapunov function is also an iISS-Lyapunov function.

Theorem 9 (Angeli et al. 2000) *A system is iISS if and only if it admits a smooth iISS-Lyapunov function.*

Since an ISS-Lyapunov function is also an iISS one, ISS implies iISS. However, iISS is a strictly weaker property than ISS, because α may be bounded in the iISS-Lyapunov estimate, which means that V may increase, and the state become unbounded, even under bounded inputs, so long as $\gamma(|u(t)|)$ is larger than the range of α. This is also clear from the iISS definition, since a constant input with $|u(t)| = r$ results in a term in the right-hand side that grows like rt.

An interesting general class of examples is given by *bilinear* systems

$$\dot{x} = \left(A + \sum_{i=1}^{m} u_i A_i\right) x + Bu$$

for which the matrix A is Hurwitz. Such systems are always iISS (see Sontag 1998), but they are not in general ISS. For instance, in the case when $B = 0$, boundedness of trajectories for all constant inputs already implies that $A + \sum_{i=1}^{m} u_i A_i$ must have all eigenvalues with nonpositive real part, for all $u \in \mathbb{R}^m$, which is a condition involving the matrices A_i (e.g., $\dot{x} = -x + ux$ is iISS but it is not ISS).

The notion of iISS is useful in situations where an appropriate notion of detectability can be verified using LaSalle-type arguments. There follow two examples of theorems along these lines.

Theorem 10 (Angeli et al. 2000) *A system is iISS if and only if it is 0-GAS and there is a smooth storage function V such that, for some $\sigma \in \mathcal{K}_\infty$:*

$$\dot{V}(x,u) \le \sigma(|u|)$$

for all (x,u).

The sufficiency part of this result follows from the observation that the 0-GAS property by itself already implies the existence of a smooth and positive definite, but not necessarily proper, function V_0 such that $\dot{V}_0 \le \gamma_0(|u|) - \alpha_0(|x|)$ for all (x,u), for some $\gamma_0 \in \mathcal{K}_\infty$ and positive definite α_0 (if V_0 were proper, then it would be an iISS-Lyapunov function). Now one uses $V_0 + V$ as an iISS-Lyapunov function (V provides properness).

Theorem 11 (Angeli et al. 2000) *A system is iISS if and only if there exists an output function $y = h(x)$ (continuous and with $h(0) = 0$) which provides zero detectability ($u \equiv 0$ and $y \equiv 0 \Rightarrow x(t) \to 0$) and dissipativity in the following sense: there exists a storage function V and $\sigma \in \mathcal{K}_\infty$, α positive definite, so that:*

$$\dot{V}(x,u) \le \sigma(|u|) - \alpha(h(x))$$

holds for all (x,u).

The paper Angeli et al. (2000) contains several additional characterizations of iISS.

Superposition Principles for iISS

There are also asymptotic gain characterizations for iISS. A system is *bounded energy weakly converging state (BEWCS)* if there exists some $\sigma \in \mathcal{K}_\infty$ so that the following implication holds:

$$\int_0^{+\infty} \sigma(|u(s)|)\, ds < +\infty \quad \Rightarrow \quad \liminf_{t\to+\infty} \left|x(t,x^0,u)\right| = 0 \qquad \text{(BEWCS)}$$

(more precisely, if the integral is finite, then $t_{\max}(x^0,u) = +\infty$, and the liminf is zero). It is *bounded energy frequently bounded state (BEFBS)* if there exists some $\sigma \in \mathcal{K}_\infty$ so that the following implication holds:

$$\int_0^{+\infty} \sigma(|u(s)|)\, ds < +\infty \quad \Rightarrow \quad \liminf_{t\to+\infty} \left|x(t,x^0,u)\right| < +\infty \qquad \text{(BEFBS)}$$

(again, meaning that $t_{\max}(x^0,u) = +\infty$ and the liminf is finite).

Theorem 12 (Angeli et al. 2004) *The following three properties are equivalent for any given system $\dot{x} = f(x,u)$:*

- *The system is iISS.*
- *The system is BEWCS and zero stable.*
- *The system is BEFBS and zero GAS.*

Summary, Extensions, and Future Directions

This entry focuses on stability notions relative to steady states, but a more general theory is also possible, that allows consideration of more arbitrary attractors, as well as robust and/or adaptive concepts. Much else has been omitted from this entry. Most importantly, one of the key results is the *ISS small-gain theorem* due to Jiang, Teel, and Praly (Jiang et al. 1994), which provides a powerful sufficient condition for the interconnection of ISS systems being itself ISS. The extension of ISS to PDEs, especially to PDEs with inputs entering the boundary conditions, long remained elusive because of the need to consider unbounded input operators. However, ISS theory is now well-established for both parabolic and hyperbolic PDEs (Karafyllis and Krstić 2019), including stability results, obtained via small-gain arguments, for interconnections or parabolic and/or hyperbolic systems, in various physically meaningful norms.

Other topics not treated include, among many others, all notions involving outputs; ISS properties of time-varying (and in particular periodic) systems; ISS for discrete-time systems; questions of sampling, relating ISS properties of continuous and discrete-time systems; ISS with respect to a closed subset K; stochastic ISS; applications to tracking, vehicle formations ("leader to followers" stability); and averaging of ISS systems. The paper Sontag (2006) may also be consulted for further references, a detailed development of some of these ideas, and citations to the literature for others. In addition, the textbooks Isidori (1999), Krstić et al. (1995), Khalil (1996), Sepulchre et al. (1997), Krstić and Deng (1998), Freeman and Kokotović (1996), Isidori et al. (2003) contain many extensions of the theory as well as applications.

Bibliography

Angeli D, Ingalls B, Sontag ED, Wang Y (2004) Separation principles for input-output and integral-input-to-state stability. SIAM J Control Optim 43(1):256–276

Angeli D, Sontag ED, Wang Y (2000) A characterization of integral input-to-state stability. IEEE Trans Autom Control 45(6):1082–1097

Angeli D, Sontag ED, Wang Y (2000) Further equivalences and semiglobal versions of integral input to state stability. Dyn Control 10(2):127–149

Freeman RA, Kokotović PV (1996) Robust nonlinear control design, state-space and Lyapunov techniques. Birkhauser, Boston

Isidori A (1999) Nonlinear control systems II. Springer, London

Isidori A, Marconi L, Serrani A (2003) Robust autonomous guidance: an internal model-based approach. Springer, London

Jiang Z-P, Teel A, Praly L (1994) Small-gain theorem for ISS systems and applications. Math Control Signals Syst 7:95–120

Karafyllis I, Krstić M (2019) Input-to-state stability for PDEs. Springer, London

Khalil HK (1996) Nonlinear systems, 2nd edn. Prentice-Hall, Upper Saddle River

Krstić M, Deng H (1998) Stabilization of uncertain nonlinear systems. Springer, London

Krstić M, Kanellakopoulos I, Kokotović PV (1995) Nonlinear and adaptive control design. John Wiley & Sons, New York

Lin Y, Sontag ED, Wang Y (1996) A smooth converse Lyapunov theorem for robust stability. SIAM J Control Optim 34(1):124–160

Massera JL (1956) Contributions to stability theory. Ann Math 64:182–206

Praly L, Wang Y (1996) Stabilization in spite of matched unmodelled dynamics and an equivalent definition of input-to-state stability. Math Control Signals Syst 9:1–33

Sepulchre R, Jankovic M, Kokotović PV (1997) Constructive nonlinear control. Springer, New York

Sontag ED (1989) Smooth stabilization implies coprime factorization. IEEE Trans Autom Control 34(4):435–443

Sontag ED (1998) Comments on integral variants of ISS. Syst Control Lett 34(1–2):93–100

Sontag ED (1999) Stability and stabilization: discontinuities and the effect of disturbances. In: Nonlinear analysis, differential equations and control (Montreal, QC, 1998). NATO Science Series C, Mathematical and physical sciences, vol 528. Kluwer Academic Publishers, Dordrecht, pp 551–598

Sontag ED (2006) Input to state stability: basic concepts and results. In: Nistri P, Stefani G (eds) Nonlinear and optimal control theory. Springer, Berlin, pp 163–220

Sontag ED, Wang Y (1995) On characterizations of the input-to-state stability property. Syst Control Lett 24(5):351–359

Sontag ED, Wang Y (1996) New characterizations of input-to-state stability. IEEE Trans Autom Control 41(9):1283–1294

Willems JC (1976) Mechanisms for the stability and instability in feedback systems. Proc IEEE 64:24–35