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REMARKS ON CONTINUOUS FEEDBACK

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Abstract

We show that, in general, it is impossible to stabilize a controllable system by means of a continuous feedback, even if memory is allowed. No optimality considerations are involved. All state spaces are Euclidean spaces, so no obstructions arising from the state space topology are involved either. For one dimensional state and input, we prove that continuous stabilization with memory is always possible.

1. Introduction

It is well known that optimal control problems often result in solutions that can naturally be implemented in terms of discontinuous feedback laws. It appears to be less widely appreciated, however, that it is in general not possible to control a ("controllable") system using continuous feedback, even without any optimality requirements. Of course, the state contains all the "information" needed for control, but this does not imply the existence of a continuous feedback for, say, a stabilization problem.

The (rather elementary) remarks given in this note will show that: (I) sometimes continuous "dynamic feedback" may be available for stabilization even if no continuous (constant) state feedback stabilizers, exist, and (II) in general not even dynamic stabilizers can be defined continuously. Our systems will be of the form

$$\dot{x} = f(x,u),\ x \in R^n,\ u \in R^m,$$

so that the state space is R^n. Of course, one could consider more complicated state spaces (e.g., manifolds) but, since our purpose is to show that certain things cannot be done in general, it is better to take the simplest possible spaces. In particular, this will make it clear that the impossibility of finding continuous feedback is not due to topological obstructions caused by the topology of the state space or the input space.

In §3, we give necessary and sufficient conditions for the existence of a constant feedback stabilizers that are either continuous or locally Lipschitz, if $n = m = 1$. These conditions make it easy to construct examples where no continuous feedback exists, as well as examples where a continuous feedback stabilizer exists but a locally Lipschitzian one does not. We then prove that, when $n = m = 1$, a time-varying feedback always exists. We show that this situation is atypical and that, when $n = 2$, it is possible to have a "completely controllable" system that admits no continuous dynamic feedback, even when allowing memory in possible regulators. Finally, in §4, we briefly mention some positive results on the existence of discontinuous feedback which is not too pathological.

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2. Definitions And Notations

(2.1) Definition. A control system is defined by giving (a) an open subset X of some Euclidean space R^n, (b) a metric space U, (c) a map $F: X\times U \to R^n$, and (d) a class $\mathcal{U}$ of admissible control functions $u(\cdot)$, defined on subintervals of the real line, and taking values in U.

We require that $(X,U,f,\mathcal{U})$ satisfy:

(I) $f: X\times U \to R^n$ is a locally Lipschitz map.

(II) Every piecewise continuous U-valued function is an admissible control.

(III) If I is an interval, and $u(\cdot): I \to U$ is admissible, then (a) $u(\cdot)$ is measurable and (b) the time-varying vector field $(x,t) \to f(x,u(t))$ satisfies Carathéodory-type conditions. (Precisely: let

$$A(K,u) = \sup\{\|f(x,u)\|: x\in K\}$$

$$B(K,u) = \sup\{\frac{\|f(x,u)-f(y,u)\|}{\|x-y\|}: x \in X,\ y \in K,\ x\neq y\}.$$

We assume that

$$\int_J A(K,u(t))dt < \infty \quad \text{and} \quad \int_J B(K,u(t))dt < \infty$$

for every compact $K\subseteq X$, and every compact subinterval $J \subseteq I$.)

It follows from the usual existence and uniqueness theorems that whenever $u(\cdot) \in \mathcal{U}$, $x_0 \in X$, $t_0 \in I$ (= domain $(u(\cdot))$), then there exists a unique maximal trajectory $x(\cdot)$ corresponding to the control $u(\cdot)$, and such that $x(t_0) = x_0$. The domain of $x(\cdot)$ is an interval, which is relatively open in I. (By a trajectory for an admissible control $u(\cdot): I \to U$ we mean an absolutely continuous curve $x(\cdot): J \to X$ $(J\subseteq I)$ such that

$$(*)\ \dot{x}(t) = f(x(t),u(t))$$

for almost all $t \in J$.)

Moreover, the following is an easy consequence of our technical assumptions:

(2.2) Piecewise Constant Approximation Lemma: If $u(\cdot): [a,b] \to U$ is an admissible control for the system (*), and $x(\cdot): [a,b] \to X$ is a trajectory for $u(\cdot)$, then there exists a sequence $\{u_n(\cdot)\}$ of piecewise constant controls defined on $[a,b]$, and trajectories $x_n(\cdot): [a,b] \to X$, such that $u_n(t) \to u(t)$ for almost all $t \in [a,b]$, and that $x_n(\cdot) \to x(\cdot)$ uniformly.

(2.3) Definition. An equilibrium state is an $\bar{x} \in X$ such that

$$f(\bar{x},\bar{u}) = 0 \text{ for some } \bar{u} \in U.$$

We shall assume that $0 \in X$, and that 0 is an equilibrium state.

(2.4) Definition. The system (*) is asymptotically controllable to a point $\bar{x}$ (a.c. to $\bar{x}$) iff for each $x_o \in X - \{\bar{x}\}$ there exists an admissible $u(\cdot): [0,\infty[\to U$ such that (i) the trajectory $x(\cdot)$ corresponding to $u(\cdot)$ and the initial condition $x(0) = x_o$ is defined for all $t \geq 0$, and (ii) $x(\cdot)$ satisfies

$$(\#) \quad \lim_{t\to+\infty} x(t) = \bar{x}$$

(2.5) Definition. A constant asymptotic feedback stabilizer (c.a.f.s.) for (*) is a pair $(K,\hat{u})$ where

(i) K is a map from $X-\{0\}$ to U,

(ii) $\hat{u}: X-\{0\} \to \mathcal{U}$ is a map which selects, for each $x_o \in X-\{0\}$, an admissible control

$$\hat{u}_{x_o}(\cdot): [0,\infty[\to U,$$

in such a way that, for each $x_o \in X-\{0\}$, the trajectory $x(\cdot)$ for $\hat{u}_{x_o}(\cdot)$ and the initial condition $x(0) = x_o$ satisfies:

(ii.a) $x(t)$ is defined for all $t \geq 0$

(ii.b) $\lim_{t\to+\infty} x(t) = 0$, and

(ii.c) $\hat{u}_{x(t)}(s) = K(x(t+s))$ for all $t \geq 0$ and almost all $s \geq 0$.

(2.6) Remark. The above definition says, approximately, that a c.a.f.s. is a map $K: X \to U$ such that, if $x_o \in X$, $x_o \neq 0$, then the solution $x(\cdot)$ of

$$(**) \quad \dot{x} = f(x,K(x))$$
$$x(0) = x_o$$

is defined for all $t \geq 0$ and converges to 0 as $t \to \infty$. The reason why we do not say it exactly that way is that, in general, $f(x,K(x))$ may fail to be locally Lipschitz, or even continuous. (In fact, the point of this paper is precisely to show that such K's are unavoidable!) When this happens, we cannot talk about "the" solution of (**), because there may be many solutions, or none. Hence we must add the assumption that (**) has a solution and, if more than one solution exists, the specification of K must be supplemented with the choice of one solution of (**) for each x_o. So we must specify both the map K and, for each x_o, a solution of (**). Equivalently, we may specify, for each x_o, an admissible control $\hat{u}_{x_o}$ whose trajectory $x(\cdot)$ is a solution of (**), with $\hat{u}_{x_o}(t) = K(x(t))$. Thus we arrive at Def. 2.5.

(2.7) Remark. One can give a totally analogous definition of what it means for (*) to be controllable (i.e., "controllable in finite time") to a point $\bar{x}$, by making the following changes in Def. 2.4:

(a) the domain of definition of $u(\cdot)$ is required to be an interval $[0,T]$, rather than $[0,\infty[$, and (b) condition (#) is replaced by: $x(T) = \bar{x}$. Similarly, one can define a constant finite-time feedback stabilizer (c.f.t.f.s.) exactly as in Def. 2.5, except that now the admissible control $\hat{u}_{x_o}(\cdot)$ is defined on some finite interval $[0,T]$ (with T depending on x_o), and that condition (ii.b) is replaced by $x(T) = 0$.

(2.8) Remark. If $K: X-\{0\} \to U$ is locally Lipschitz on $X-\{0\}$, then $x \to f(x,K(x))$ is locally Lipschitz, and (**) has a unique solution $x(\cdot)$ such that $x(0) = x_0$, for any $x_0 \in X-\{0\}$. If, for every x_o, $x(\cdot)$ is defined for all $t \geq 0$, and $x(t) \to 0$ as $t \to +\infty$, we will say that K is a locally Lipschitz constant asymptotic feedback stabilizer (l.L.c.a.f.s.). In this case, there exists a unique $\hat{u}: X-\{0\} \to \mathcal{U}$ such that the pair $(K,\hat{u})$ is a c.a.f.s. in the sense of Def. 2.5, so this new definition agrees with Def. 2.5.

(2.9) Definition. A dynamic asymptotic feedback stabilizer (d.a.f.s.) is a triple $(g,K,\hat{u})$, where $\dot{z} = g(z,x)$ defines a control system with state set Z and input value set X, and where $(K,\hat{u})$ is a c.a.f.s. for the system

$$(***) \quad \begin{aligned} \dot{x} &= f(x,u) \\ \dot{z} &= g(z,x) \end{aligned}$$

(2.10) Remarks. (a) One can define in a similar way the concept of a dynamic finite-time feedback stabilizer (d.f.t.f.s.). (b) when K is continuous on $X-\{0\}$, and the trajectories of $(K,\hat{u})$ depend continuously on the initial state, we say that $(K,\hat{u})$ is a continuous feedback. (c) Other definitions of d.a.f.s. might be reasonable as well. For instance, we may eliminate the requirement that the state of the regulator coverage to 0, or we may only ask convergence to 0 of the trajectories that start at some fixed $z(0)$. Although possibly of theoretical interest, such a controller design would not provide a basic feature of feedback: the stabilization even under sudden perturbations of the plant.

In any case, our positive result (Theorem 3.5) will prove existence of a l.L.d.a.f.s. in the "strong" sense of Def. 2.9, for $n = m = 1$, whereas our counterexample for $n = 2$ proves nonexistence of a continuous d.a.f.s. with the weakest possible definition.

(2.11) Definition. A "time-varying feedback stabilizer" for the system (*) is a pair $(K,\hat{u})$, where $K: X\times[0,\infty[\to U$ and, for each $x_0 \in X$, $t_0 \geq 0$, $\hat{u}_{x_0,t_0}: [t_0,\infty[\to U$ is an admissible control such that the trajectory $x(\cdot)$ for $\hat{u}_{x_0,t_0}$, which satisfies $x(t_0) = x_0$, is defined for all $t \geq t_0$, converges to 0 as $t \to \infty$, and satisfies $K(t,x(t)) = \hat{u}_{x_0,t_0}(t)$ for $t \geq t_0$. Equivalently, if we let $\tilde{u}_{x_0,t_0}(t) = \hat{u}_{x_0,t_0}(t+t_0)$, for $0 \leq t$, we see that $\tilde{x}$ is the trajectory for $\tilde{u}_{x_0,t_0}$ such that $\tilde{x}(0) = x_0$, and $\tilde{x}$ satisfies $K(t+t_0,\tilde{x}(t)) = \tilde{u}_{x_0,t_0}(t)$ for $t \geq 0$, so that $\tilde{x}$ is a solution of

$$(****)\qquad \dot{\tilde{x}} = f(\tilde{x}(t),K(t+t_0,\tilde{x}(t)))$$

Therefore, $(K,\tilde{u})$ is a c.a.f.s. for the system

$$\dot{x} = f(x,u),\ \dot{z} = 1,\ x \in X,\ z \in R_+,\ u \in U,$$

except for the fact that, as $t \to \infty$, $(x(t),z(t))$ converges to $(0,+\infty)$, rather than to $(0,0)$. By making the change of variable $w = \frac{1}{z}$, a time-varying feedback for (*) gives rise to a d.a.f.s., in the sense of Def. 2.9.

3. Continuous Feedback

We consider first the case of systems with $X = U = R$. Let

$$\mathcal{O} = \{(x,u): xf(x,u) < 0\}$$

$$\tilde{\mathcal{O}} = \{(x,u): x \neq 0,\ xf(x,u) \leq 0\}.$$

Let $\pi: R^2 \to R$ be the projection $(x,u) \to x$.

(3.1) Lemma. If (*) is a.c. to 0 then $\pi(\mathcal{O}) = R-\{0\}$.

Proof. Let $x_0 > 0$, and $u(\cdot) \in \mathcal{U}$ such that $x(t) \to 0$ as $t \to \infty$, where

$$x(0) = x_0,\ \dot{x}(t) = f(x(t),u(t))$$

for $t \geq 0$. Pick $T > 0$ such that $x(T) < \frac{x_0}{2}$. Then pick a piecewise constant $u'(\cdot): [0,T] \to R$ such that $x'(T) \leq \frac{x_0}{2}$, where $x'(0) = x_0$, and

$$\dot{x}'(t) = f(x'(t),u'(t))$$

for $0 \leq t \leq T$ (here we use 2.2). Let

$$t_0 = \sup\{t: x'(t) \geq x_0\}.$$

Then

$$0 \leq t_0 < T,\ x'(t_0) = x_0,$$

and $x'(t) < x_0$ for $t_0 < t < T$. If $\varepsilon > 0$ is small enough, then $u'(\cdot)$ has the constant value u_0 for $t_0 \leq t \leq t_0+\varepsilon$. So the restriction of $x'(\cdot)$ to $[t_0,t_0+\varepsilon]$ is an integral curve of the locally Lipschitzian vector field $f(\cdot,u_0)$. Clearly, this implies that $f(x_0,u_0) < 0$. Therefore, $(x_0,u_0) \in \mathcal{O}$, and so $x_0 \in \pi(\mathcal{O})$. A similar argument applies if $x_0 < 0$. Q.E.D.

(3.2) Proposition. Assume that $X = U = R$. Then
(a) the system (*) admits a locally Lipschitz c.a.f.s. if and only if there exists a continuous function $k: R-\{0\} \to R$ whose graph is included in $\mathcal{O}$.
(b) (*) admits an a.f.s. $(K,\hat{u})$ such that K is continuous on $R-\{0\}$ if and only if there exists a continuous function $k: R-\{0\} \to R$ such that (i) the graph of k is included in $\tilde{\mathcal{O}}$ and (ii) the function $x \to f(x,k(x))^{-1}$ is locally integrable on $R-\{0\}$.

Proof. Suppose that $k: R-\{0\} \to R$ is a locally Lipschitz c.a.f.s. Then $x \to f(x,k(x))$ is a locally Lipschitz vector field whose trajectories go to 0. Hence $f(x,k(x)) < 0$ if $x > 0$, and $f(x,k(x)) > 0$ if $x < 0$. So Graph $(k) \subseteq \mathcal{O}$. Conversely, if Graph $(k) \subseteq \mathcal{O}$, and k is locally Lipschitz, then k is a c.a.f.s. If k is merely continuous, then one can approximate k by a locally Lipschitz k' such that Graph $(k') \subseteq \mathcal{O}$. This completes the proof of (a).

Now let $k: R-\{0\} \to R$ satisfy (i) and (ii). Let $x_0 > 0$, and define $T:]0,x_0] \to R_+$ by

$$T(x) = -\int_x^{x_0} \frac{dy}{f(y,k(y))}. \tag{3.3}$$

Then T is well defined, finite, positive, and strictly decreasing. So $T(0) = \lim_{x\to 0} T(x)$ exists (and may be finite or infinite). Let

$$x(\cdot): [0,T(0)[\to R$$

be the unique continuous function such that $T(x(t)) = t$ for $0 \leq t < T(0)$. Then $x(\cdot)$ is strictly decreasing, and $\lim_{t \to T(0)} x(t) = 0$. If $0 < t_o < T(0)$, then the compact interval $I = [x(t_o), x_o]$ is contained in $\{x: x > 0\}$, and so $0 \leq f(x,k(x)) \leq C$ for $x \in I$, and some constant $C > 0$. So $\frac{dT}{dx} \geq \frac{1}{C}$ throughout I, and so $T(y) - T(x) \geq \frac{1}{C}(x-y)$ whenever $y < x$, $y \in I$, $x \in I$. This shows that $x(\cdot)$ is Lipschitzian on I. So $x(\cdot)$ is absolutely continuous, and satisfies $\dot{x}(t) = f(x(t),k(x(t)))$ for almost all $t \in [0,T(0)]$. Let $\hat{u}_{x_o}(t) = k(x(t))$ for $0 \leq t < T(0)$. Define $k(0) = \bar{u}$, where $\bar{u}$ is such that $f(0,\bar{u}) = 0$ (recall that 0 is an equilibrium point), and let $\hat{u}_{x_o}(t) = \bar{u}$ for $t \geq T(0)$. Construct $\hat{u}_{x_o}$ for $x_o < 0$ in a similar fashion. Then $(k,\hat{u})$ is a continuous a.f.s. (recall that our definition of continuity does not require that k be continuous at 0).

Conversely, suppose $(k,\hat{u})$ is a continuous c.a.f.s. Then the graph of k must be included in $\tilde{\mathcal{O}}$. (Otherwise, if there is an $x_o > 0$ such that $f(x_o,k(x_o)) > 0$, then there exists an open interval I containing x_o and such that $f(x,k(x)) > 0$ for $x \in I$. Hence no curve $x(\cdot)$ which satisfies $\dot{x}(t) = f(x(t),k(x(t)))$ can cross I from right to left, contradicting the fact that $(k,\hat{u})$ is a c.a.f.s.) Along a curve $x(\cdot)$ which is a trajectory of $\hat{u}_{x_o}$, for some $x_o > 0$, we can reparametrize using x, rather than t, as the parameter. Then we can write $t = T(x)$, and T satisfies (3.3). Since $T(x)$ remains finite for $x > 0$, the function $x \to T(x,k(x))^{-1}$ is locally integrable on $\{x: x > 0\}$. The proof that it is locally integrable on $\{x: x < 0\}$ is similar.

Q.E.D.

<u>(3.4) Example</u>. Consider the system $\dot{x} = f(x,u)$, where $f(x,u) = u(1+x+\alpha\arctan(u))$, and $\alpha \in R$ is a constant. Then $\pi(\tilde{\mathcal{O}}) = R-\{0\}$, for any α, but $\pi(\mathcal{O}) = R-\{0\}$ iff $\alpha > 0$ (and $\pi(\mathcal{O}) = R-\{0,1\}$ if $\alpha \leq 0$.) So the system is not a.c. to 0 if $\alpha \leq 0$. Now suppose that $0 < \alpha < \frac{2}{\pi}$. Then $\mathcal{O}$ is the union of three sets, namely,

$\mathcal{O}_1 = \{(x,u): x>0, u<0\}$, $\mathcal{O}_2 = \{(x,u): u>0, -1-\alpha \arctan u < x < 0\}$, and $\mathcal{O}_3 = \{(x,u): u < 0, x < -1-\alpha \arctan u\}$. We have $\text{clos}\, \mathcal{O}_3 \cap \text{clos}\, \mathcal{O}_1 = \emptyset$, $\text{clos}\, \mathcal{O}_3 \cap \text{clos}\, \mathcal{O}_2 = \{(-1,0)\}$ and $\text{clos}\, \mathcal{O}_2 \cap \text{clos}\, \mathcal{O}_1 = \{(0,0)\}$. It is easy to construct a continuous $k: R \to R$ such that $(x,k(x)) \in \mathcal{O}$ whenever $x \neq 0$, $x \neq -1$. Moreover, such a k can be taken to be of class C^∞. However, $(x,k(x))$ has to belong to $\mathcal{O}_3$ for $x < -1$, because $\pi(\mathcal{O}_2)$ does not extend all the way to $-\infty$. And $(x,k(x))$ must be in $\mathcal{O}_2$ for $-1 < x < 0$, because $\pi(\mathcal{O}_3)$ does not extend all the way to 0. Hence $k(-1)$ must equal 0, and so $f(-1,k(-1)) = 0$. Therefore, $(-1,k(-1)) \notin \mathcal{O}$. So there is no continuous $k: R-\{0\} \to R$ such that $(x,k(x)) \in \mathcal{O}$ for all $x \neq 0$. Hence, no locally Lipschitz c.a.f.s. exists. However, if we no longer insist that k be locally Lipschitz, it is clear that we can choose k such that

(i) k is C^∞ except at $x = 1$,

(ii) $|k(x)| \sim |1+x|^\beta$ as $x \to -1$, and

(iii) $(x,k(x)) \in \mathcal{O}$ whenever $x \neq 0$, $x \neq -1$.

Here $\beta > 0$ is arbitrary. If we choose β such that $\beta < \frac{1}{2}$, then $|f(x,k(x))| \sim \alpha|1+x|^{2\beta}$ as $x \to -1$, and so $x \to f(x,k(x))^{-1}$ is integrable near $x = -1$. Therefore, k satisfies conditions (i) and (ii) of part (b) of Prop. 3.2. So, we see that the system $\dot{x} = f(x,u)$ admits a <u>continuous</u> c.a.f.s., but doesn't admit a <u>locally Lipschitzian</u> c.a.f.s.

If $\alpha > \frac{2}{\pi}$, then there is a unique $u_o \in R$ that satisfies $1+\alpha \arctan u_o = 0$. Then u_o is necessarily negative, and so $f(x,u_o) \in \mathcal{O}$ for all $x \neq 0$. So in this case, the system admits a <u>constant</u> (i.e., independent of x) c.a.f.s. Finally, for an example where no continuous c.a.f.s. exists, take $\dot{x} = ((u-1)^2-(x-1))(x-2+(u+1)^2)$. Then $\mathcal{O} \cap \{(x,u): x > 0\}$ consists of two disjoint pieces, namely, the set $\mathcal{O}_1$ of points to the right of the parabola $x = 1+(u-1)^2$, and the set $\mathcal{O}_2$ of points (x,u) such that $x > 0$ and that (x,u) lies to the left of the parabola $x = 2-(u+1)^2$. $\tilde{\mathcal{O}}$ is simply the closure of $\mathcal{O}$. It is easy to see that there exists no continuous $k: R_+ \to R$ such that $(x,k(x)) \in \tilde{\mathcal{O}}$ for all $x > 0$. On the other hand, one sees easily that $\pi(\mathcal{O}) = R-\{0\}$, so the system is a.c. to 0, and that there exists a piecewise constant c.a.f.s.

We now show that, in dimension one, every system which is a.c. to 0 can be stabilized by a continuous <u>dynamic</u> feedback $u = K(t,x)$.

(3.5) Proposition. (Still $X = U = R$.) The system (*) is a.c. to 0 if and only if it admits a continuous d.a.f.s. (in particular, the converse of (3.1) is also valid.)

Proof. By (3.1), it is enough to prove that $\pi(0) = R-\{0\}$ implies the existence of a $K(t,x)$ as in (****). We first construct $K(t,x)$ for $t \geq 0$, $x > 0$. Pick first any $\bar{u}$ with $f(0,\bar{u}) = 0$ and define $K(t,0) := \bar{u}$ for all $t \geq 0$.

Since $\pi(0) \cap \{(x,u): x>0\}) =]0,+\infty[$, there exist 3 sets of reals $\{a_j, j\in Z\}$, $\{b_j, j\in Z\}$, and $\{u_j, j\in Z\}$ such that the following properties hold:

(i) all $a_j, b_j > 0$; $\{a_j\}$ $\{b_j\}$ have no nonzero limit points; $a_j \to \infty$ as $j \to +\infty$, $a_j \to 0$ as $j \to -\infty$;

(ii) $a_{j+1} < b_j < a_{j+2}$ for all j;

(iii) $f(x,u_j) < 0$ for all x in $I_j := [a_j, b_j]$.

For each j, let M_j and m_j be positive numbers such that

(iv) $|f(x,u)| < M_j$ for all x in $[a_{j+1}, b_j]$ and all u between u_j and u_{j+1}, and

(v) $f(x,u_j) < -m_j$ for all x in I_j.

We now define a map $\hat{K}$ for $t > 0$ and $x > 0$ (which can be analogously extended to $x < 0$, to give K.) First, for each j, let:

$$\hat{K}t,x) = u_{j+1} \quad \text{for } x \text{ in } [b_j, a_{j+2}] \tag{3.6}$$

and all $t \geq 0$. It remains to define $\hat{K}(t,x)$ for x in intervals of the type $[a_{j+1}, b_j]$ in such a way that $\hat{K}$ is continuous.

To simplify notations, take now a fixed j and let $c = a_{j+1}$, $d = b_j$, $m = \max(m_j, m_{j+1})$, $M = M_j$, $\delta = (d-c)/4$, $t_1 = 3\delta/m$, and $t_2 = t_1 + \delta/M$. Further, define a continuous piecewise linear map α_r on $[c,d]$, for each r between d and $d-3\delta = c+\delta$ by:

$$\alpha_r(x) = u_{j+1} \quad \text{for } r \leq x \leq d$$

$$\alpha_r(x) = u_j \quad \text{for } c \leq x \leq r-\delta$$

$$\alpha_r(x) = \text{linear on } [r-\delta, r].$$

Now let

$$\hat{K}(t,\cdot) = \alpha_{d-mt} \quad \text{for } 0 \leq t \leq t_1 \tag{3.7}$$

and

$$\hat{K}(t,\cdot) = \alpha_{3M(t-t_1)+c+\delta} \quad \text{for } t_1 \leq t \leq t_2. \tag{3.8}$$

Note that $\hat{K}$ is continuous in (x,t), and that $\hat{K}(t_2,\cdot) = \hat{K}(0,\cdot)$. Extend $\hat{K}$ to all pairs (t,x) with x in $[c,d]$ and $t \geq 0$, as a periodic function with period t_2. Note that for x in $[c,d]$ this gives that $f(x,\hat{K}(x,t))$ is locally Lipschitz on x, uniformly in t. This construction can be carried out for all j, yielding corresponding $t_1(j)$, $t_2(j)$'s. An analogous construction gives $\hat{K}(t,x)$ for $x < 0$. Thus $\hat{K}$ is defined for all (t,x) with $t \geq 0$.

We now prove:

Claim I. Let $x(\cdot)$ be a solution of $\dot{x}(t) = f(x(t),\hat{K}(t,x(t)))$, suppose that $0 < x(t_0) \leq b_j$ for some t_0 and some j. Then $x(t_0+3t_2) < a_{j+1}$ for $t_2 = t_2(j)$. For this fixed j use the previous notations. Let t_0' be an integer multiple of t_2 satisfying $t_0 \leq t_0' \leq t_0 + t_2$. Since $f(d,\hat{K}(t,d)) < 0$ for all t, also $x(t_0') \leq d$. Since $\hat{K}$ is periodic in t with period t_2, the curve $\xi: t \to x(t_0'+t)$ is also a solution of $\dot{x}(t) = f(t,\hat{K}(t,x(t)))$. We know that $\xi(0) \leq d$, and we want to prove that $\xi(t_0+3t_2-t_0') < c$. Since $\hat{K}(t,c) = u_j$ for all t, we have $f(t,\hat{K}(t,c)) < 0$ for all t, and therefore $\xi(t) < c$ for all $t \geq \bar{t}$, if $\xi(\bar{t}) < c$. In particular, the conclusion that $\xi(t_0+3t_2-t_0') < c$ will follow if we prove that $\xi(2t_2) < c$. So all we need is to prove that, whenever $x(\cdot)$ satisfies $x(0) \leq d$, it follows that $x(2t_2) < c$. Since trajectories do not cross at any given time, it will be enough to prove that $x(2t_2) < c$ whenever $x(0) = d$. Consider the map $\beta(t) = x(t) + mt-d$. Then $\beta(0) = 0$. If $\beta(\tau) = 0$ for some $0 \leq \tau \leq t_1$ then $\dot{x}(\tau) = f(x(\tau),u(\tau))$ with $u(\tau) = K(\tau,d-m\tau) = u_{j+1}$, so $\dot{x}(\tau) < -m$ (by (v) above). Thus $\dot{\beta}(t) < 0$ whenever $\beta(t) = 0$, if $0 \leq t \leq t_1$. It follows that $\beta(t) < 0$ for such t, and in particular that $x(t_1) < c+\delta$. Now consider $x(t)$ for $t_1 \leq t \leq t_2$. We may assume that $x(t_1)=c+\delta$. By the mean value theorem and (iv) above,

$$x(t_2) \leq x(t_1) + M(t_2-t_1) \leq c+2\delta.$$

With an agrument similar to the one used above, one shows that $x(t_2+2\delta/m) \leq c$, so also $x(2t_2) \leq c$, as wanted.

Pick now any $x_o > 0$ and consider the trajectory $x(t)$, corresponding to (****), and such that $x(0)=x_o$. We claim that $x(t) \to 0$. To prove this, it suffices to assume that $t_o = 0$. Moreover, it is enough to prove that for every $j \in Z$ the inequality $x(t) < a_j$ holds for sufficiently large t (because $a_j \to 0$ as $j \to -\infty$). Since $a_j \to \infty$ as $j \to +\infty$, there is a j such that $x(0) \leq a_j$. If we prove that $x(t) \leq a_j$ implies that $x(\tau) \leq a_{j-1}$ for sufficiently large τ, the conclusion will follow by a simple induction. Moreover, by definition of $\hat{K}$, we have $\hat{K}(t,a_{j-1})=u_{j-2}$ for all t, and so $f(a_{j-1},\hat{K}(t,a_{j-1})) < 0$ for all t. Therefore, $x(\tau) < a_{j-1}$ for all $\tau > \tau_0$ if $x(\tau_0) \leq a_{j-1}$. So, it suffices to prove that $x(t) \leq a_j$

implies that $x(\tau) \leq a_{j-1}$ for some $\tau > t$. Now $a_{j-1} < b_{j-2} < a_j$, and $f(x,\hat{K}(t,x)) = f(x,u_{j-1}) < 0$ for all t, and all x such that $b_{j-2} \leq x \leq a_j$. Hence $f(x,\hat{K}(t,x))$ is bounded above by a fixed strictly negative constant for $b_{j-2} \leq x \leq a_j$, $t \geq 0$. Therefore, $x(t) \leq a_j$ implies that $x(\tau) \leq b_{j-2}$ for some $\tau > t$. But then, by Claim I, $x(\eta) < a_{j-1}$ for some $\eta > \tau$. This concludes the proof that any solution of $\dot{x}(t) = f(t,\hat{K}(t+t_0,x(t)))$ that starts at an $x(0) > 0$ converges to 0 as $t \to \infty$. The proof for the case $x(0) < 0$ is similar.

Note that the above argument could be trivially modified to imply the existence of a K which is infinitely differentiable in x, for $x \neq 0$. Even a K which is continuous everywhere can be obtained, if one modifies the above K by a time-varying linear interpolant in a shrinking neighborhood of the origin.

For dimension greater than one the existence of even a dynamic f.s. is not insured by $(*)$ being a.c. An example of this is provided by the 2-dimensional system $(X = R^2, U = R^2)$:

$$\dot{x}(t) = (4-y^2(t))u^2(t)$$
$$\dot{y}(t) = e^{-x(t)}+y(t)-2e^{-x(t)}\sin^2 u(t).$$

This system is a.c. to 0. However, no continuous d.a.f.s. exists. This follows from the arguments in Sussmann [1979, Appendix]. In fact, those arguments show that it is impossible in this example to even choose continuously a path from states to the origin. (The obstructions here are in the choice of directions, while those in dimension one are only in the choice of particular input values.)

4. Discontinuous Regulation

We mention here some results from SUSSMANN [1979] and from SONTAG [1980].

These results indicate how to regulate nonlinear systems using various types of (not necessarily continuous) feedback. Since the technical details may be found in the above references, we only explain here the intuitive ideas of each approach.

In the case of SUSSMANN [1979] one proves that a "piecewise analytic" c.f.s. always exists provided that the original system (plant) be completely controllable (any state can be driven to any other state) and that the "f" in $(*)$ be real-analytic in x. The complete controllability assumption can be weakened considerably into a local condition around the origin, but the condition on f cannot since the results use facts from the theory of subanalytic sets.

In contrast, the approach in SONTAG [1980] applies to the construction of sampled feedback stabilizers under rather minimal conditions on f (continuity, and differentiable with controllable linearization at the origin). "Sampled" refers to the fact that control is assumed to be based on a constant-rate sampling; in a sense sampled-feedback lies somewhere in between "true" feedback and open-loop control, but is closer to the former than the latter due to the constant sampling rate. (In any digital implementation of a feedback law, a sampled feedback is necessarily used.)

References

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