REMARKS ON THE PRESERVATION OF VARIOUS CONTROLLABILITY PROPERTIES UNDER SAMPLING

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Abstract: This note studies the preservation of controllability (and other properties) under sampling of a nonlinear system. More detailed results are obtained in the cases of analytic systems and of systems with finite dimensional Lie algebras.

1. Preliminaries. When a system is regulated by a digital computer, control decisions are often restricted to be taken at fixed times $0,\lambda,2\lambda,\ldots$; one calls $\lambda>0$ the *sampling time*. For a (continuous time) plant, the resulting situation can be modelled through the constraint that the inputs applied be constant on intervals of length λ. It is thus of interest to characterize the various controllability properties when the controls are so restricted. This problem motivated the results in [KHN], which studied the case of linear systems; more recent references are [BL], [GH]. For nonlinear systems, it appears that the problem has not been studied systematically. The recent paper [SS] began such a study, using tools of geometric control theory. We continue that study here. Reasons of space prevent us from repeating the material in [SS], which will be needed in a few places. The definitions and statements of results, however, will be self-contained.

The systems σ to be considered are those described by differential equations

(1.1) $(dx/dt)(t) = f(x(t),u(t)),\ x(t)\in \mathbf{M},\ u(t)\in \mathbf{U}$,

where $\mathbf{M}$ is a smooth (Hausdorff, second countable) n-dimensional manifold, $\mathbf{U}$ is a subset (see below) of a smooth manifold $\mathbf{P}$, $f:\mathbf{M}\times\mathbf{P}\to T\mathbf{M}$ is smooth, and $X_u:=f(\cdot,u)$ is a complete vector field on $\mathbf{M}$ for each u. "Smooth" means either infinitely differentiable or analytic; in the latter case, σ is an *analytic system*. The control set $\mathbf{U}$ can be very general; we shall only assume that $\mathbf{V}:=\mathrm{int}(\mathbf{U})$ is connected and that the following (local) condition is satisfied at each $u\in\mathbf{U}$: there exists a smooth path $g:[0,1]\to\mathbf{P}$ with $g(0)=u$, $g([0,1])\subseteq\mathbf{U}$, and $g(t)\in\mathbf{V}$ for almost all t. ("Smooth" meaning: defined and smooth in a ngbd of [0,1].) In particular, then, $\mathbf{U}$ must be path connected and it must satisfy $\mathbf{U}\subseteq\mathrm{clos}(\mathbf{V})$; the former is the essential property for most results.

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We shall use the notations T_pN or N_p for the tangent space at p to the manifold N. For a smooth map $f:A\times B\to C$, $[\mathbf{d}_x f(x,y)](v)$ denotes the differential of $f(\cdot,y)$, evaluated at x and applied to $v\in A_x$. If f depends only on x, we omit the subscript. *Submanifold* means throughout immersed second countable submanifold. For any (complete) vector field X, $X^t := \exp(tX)(x)$ denotes the flow generated by X.

We introduce now certain groups and semigroups of diffeomorphisms of M associated to the system σ. Let *S* [resp., S_λ] be the semigroup generated by all the diffeomorphisms X_u^t, t >= 0, $u\in\mathbf{U}$ [resp., X_u^λ, $u\in\mathbf{U}$], and let *G* [resp., G_λ] be the group generated by these. Consider also, for each $t\in\mathbf{R}$ [resp., t >= 0] the set G^t [resp., S^t] consisting of all compositions $X_{u_1}^{t_1}o...oX_{u_r}^{t_r}$ such that $\sigma t_i = t$ [resp., such that, further, all $t_i >= 0$]. Similarly, for each $k\in\mathbf{Z}$ (=integers) [resp., $k\in\mathbf{N}$ (=nonnegative integers)] the subset G_λ^k of $G^{k\lambda}$ [resp., S_λ^k of $S^{k\lambda}$] is obtained from the compositions $X_{u_1}^{k_1\lambda}o...oX_{u_r}^{k_r\lambda}$ with each k_i= +-1 [resp., all =1] and $\sigma k_i = k$. Note that G^0 [resp., G_λ^0] is the normal subgroup of *G* [resp., G_λ] generated by all the compositions $X_u^t o X_v^{-t}$ [resp., all $X_u^\lambda o X_v^{-\lambda}$], and its cosets are the sets G^t [resp., G_λ^k]. The identity of *G* is e = identity map. To emphasize the system, we may write *S*(σ), etc. Compositions goh in *G* will be denoted simply as gh, and evaluations g(h), for $x\in\mathbf{M}$, as g.x; these notations will be extended to sets: H.N:= {h.n, h∈H, n∈N}. In particular, we shall be interested in the following sets: O(x):= *G*.x (often called the "weakly controllable set from x"), $O^0(x) := G^0.x$ (the "zero time orbit of x", A(x):= *S*.x (the "accessible" or "reachable" set from x), $O_\lambda(x) := G_\lambda.x$, $O_\lambda^0(x) := G_\lambda^0.x$, $A_\lambda(x) := S_\lambda.x$, $A_\lambda^r(x) := S_\lambda^r.x$, $O_\lambda^r(x) := G_\lambda^r.x$, $A^t(x) := S^t.x$, and $O^t(x) := G^t.x$. We use primes: A_λ^r', etc., to indicate the corresponding concept obtained when controls are restricted to **V** = int(**U**). It is well-known (see for instance [SU1], [LO]) that O(x) and $O^0(x)$ are, in a natural way, connected submanifolds of **M**, for each x. It is possible to also give natural submanifold structures to $O_\lambda(x)$ and $O_\lambda^0(x)$. With these structures, $O_\lambda^0(x)$ becomes the connected component of $O_\lambda(x)$ at x, and $O_\lambda(x)$ [resp., $O_\lambda^0(x)$] is a submanifold of O(x) [resp., $O^0(x)$]; see the appendix to [SS] for details (see also [JA]). In fact, $O_\lambda(x) = O_\lambda'(x)$, so that it is only necessary to define a submanifold structure for the latter. This equality is proved as follows. We must show that for each $x\in\mathbf{M}$ and each $u\in\mathbf{U}$ there is an h in G_λ' with $X_u^\lambda(x)$ = h.x, and similarly for $X_u^{-\lambda}$. Let g be a path as in the first paragraph, and let D:= g([0,1)), E:= $D\cap\mathbf{V}$. Since $u\in D$, it will be enough to prove that $O_\lambda^E = O_\lambda^D$, the orbits which result when the control space is restricted to E or D respectively. Now let **Q** be the manifold $(-1,1)\subseteq\mathbf{R}$, and consider the map $\alpha:\mathbf{Q}\to\mathbf{P}$, $\alpha(a) := g(a^2)$. This is smooth and has image D. Consider the following two systems: $\sigma_{\mathbf{Q}}$ has control space **Q**, state space **M**, and system map f(x,α(a)), and $\sigma'_{\mathbf{Q}}$ with the same equations but with control space **Q'**:= $\alpha^{-1}(E)$ (open, hence seen as a submanifold of **Q**). It will be then enough to prove the equality of $O_\lambda(x)$ and $O'_\lambda(x)$, where these denote now the orbits of x under the systems $\sigma_{\mathbf{Q}}$ and $\sigma'_{\mathbf{Q}}$, respectively. By the assumptions on g, **Q'** is open dense in **Q**. The equality then follows as in [SS], remark A.12, with **Q'** = "**U**" there. The analiticity assumption needed in that argument is replaced by the density of **Q'**. The same proof also implies that O_λ^k coincides for both input sets, for each k: this follows easily by induction, once we observe that the above "h" can be in fact taken in $(G_\lambda^1)'$ (see proof in [SS]). Thus, we also conclude that O_λ^0 is the same, whether using **U** or **V**.

We shall denote by d(x), $d^0(x)$, and $d_\lambda(x)$ the dimensions of O(x), $O^0(x)$, and $O_\lambda(x)$ (or $O_\lambda^0(x)$) respectively.

We also consider certain Lie algebras of smooth vector fields on **M** which are associated to σ. These are: L:= Lie algebra generated by $\{X_u, u\in \mathbf{U}\}$, L^0:= ideal of L generated by $\{X_u - X_v,\ u,v\in \mathbf{U}\}$, and L_λ, defined as follows for any $\lambda>0$. Pick any $g\in G_\lambda$, $u\in \mathbf{V}$, $a\in T_u\mathbf{P}$, and r= +- 1. Associate to these the smooth map $\beta:\mathbf{M}\times\mathbf{V}\to\mathbf{M}$ given by $\beta(x,v):= (g^{-1}X_v^{-\lambda}X_u^{\lambda}g)^r(x)$. Then L_λ is the Lie algebra generated by $\{\phi(x):= [\mathbf{d}_u\beta(x,u)](a),\ (g,u,r,a) \text{ as above}\}$. The algebras L and L^0 are well known; the algebra L_λ coincides (at least for bilinear systems) with the algebra introduced in [CF] for "invertible" discrete time systems. Call σ *finite* if dim L (equivalently, dim L^0,) is finite. Denote by $L(x)$ the subspace $\{Y(x), Y\in L\}$ of $T_x\mathbf{M}$, and analogously for L^0 and L_λ. Then, $L(x)\subseteq T_xO(x)$ and $L^0(x)\subseteq T_xO^0(x)$ (well-known), and $L_\lambda(x)=T_xO_\lambda^0(x)$ (see [SS]). Let $d^*(x):= \dim L^0(x) <= d^0(x)$. We call $x\in\mathbf{M}$ *nice* if $d^*(x) = d^0(x)$. If σ is either analytic or finite, *every* x is nice --the proof is analogous to that of the similar statement for O(x), for which see for instance [LO]. In those cases, $\mathbf{d}^0$ can be calculated algebraically, by computing L^0. Note that the condition $d^*(x)=n$ always implies that x is nice. Regarding L_λ, note that $d_\lambda(x)=n$ is equivalent to $O_\lambda(x)$ being a ngbd of x.

The main properties which we wish to study are the following. A subset K of **M** is λ-ST (*sampled transitive*) if $K\subseteq O_\lambda(x)$ for each (equivalently, for *some*) $x\in K$; it is ST if it is λ-ST for some λ. The *system* σ is ST if every compact $K\subseteq\mathbf{M}$ is. If $K=\mathbf{M}$ is λ-ST for some λ, then σ is GST (*globally* ST). The set K is SC (*sampled controllable*) if $K\subseteq A_\lambda(x)$ for all $x\in K$; it is SC if this happens for some λ. Again, σ is SC if every compact is, and GSC if **M** is λ-SC for some λ. The purpose of this paper is to relate the above to the more standard notions: σ is *controllable* if **M**=A(x) for all x, *transitive* if **M**=O(x) for all x, and *strongly transitive* if $\mathbf{M}=O^0(x)$ for all x. (The last two are often refered to as the *accessibility* and *strong accessibility* properties.) Finally, we shall say that a property $P(\lambda)$ holds *for* *almost all* λ when $P(\lambda)$ is true for each $\lambda>0$ in the complement of a discrete set C of $[0,\infty)$ --i.e., C is discrete in $\mathbf{R}_+$ *and* C includes every λ small enough.

2. Some Technical Facts. We collect here some results to be used later. The following is a fixed-time version of the *normal controllability* concept in [SU2], relativized to zero-time orbits. Let σ be a given system. For any $r\in\mathbf{N}$ and any $T>0$, let $\mathbf{R}_r(T):=\{t=(t_1,...,t_r)\mid \text{all } t_i>0 \text{ and } \sigma t_i=T\}$, and for any $u=(u_1,...,u_r)\in\mathbf{U}^r$, $T>0$, and $x\in\mathbf{M}$, consider $F:\mathbf{R}_r(T)\to\mathbf{M}$ given by $F(t):=(X_{u_1}^{t_1}\circ...\circ X_{u_r}^{t_r})(x)$. For any y in the range of F, F can be seen as a (smooth) map into the submanifold $O^0(y)$, and $d^0(y)=d^0(x)$. Assume that $d^0(x)=k$. Let $F^T(x):=\{y\mid \exists u, \exists r, \exists t\in\mathbf{R}(T) \text{ with } F(t)=y \text{ and rank } \mathbf{d}F(t)=k\}$; as before, $(F^T)'(x)$ is the corresponding set when $\mathbf{V}$ is used as the control set. Note that (1) $F^T(x)$ is open in $O^0(y)$, (2) $F^T(x)\subseteq A^T(x)$, (3) $F^T(x)\subseteq\text{clos}[(F^T)'(x)]$, (4) $F^T(A^S(x))\subseteq F^{T+S}(x)$, and (5) $A^S(F^T(x))\subseteq F^{T+S}(x)$. Furthermore, if $y\in F^T(x)$, then for the *reversed* system σ^- [with dynamics $f^-(\cdot,u):=-f(\cdot,u)$] it holds that $x\in F^T(y)$.

(2.1) LEMMA. Assume that x is nice (i.e., $d^*(x)=d^0(x)$). Let B be any ngbd of x. Then $(F^T)'(x)\cap B$ is nonempty, for some T.

PROOF. This is basically (the "positive form" of) Chow's theorem (for which see e.g. [KR]). Consider the system $\sigma^\#$ on $\mathbf{M}^\#=\mathbf{M}\times\mathbf{R}$, which consists of (1.1) plus a scalar equation $dz/dt=1$. Let $x^\#=(x,0)$, and pick a ngbd $B^\#$ of $x^\#$ of the form $B\times B'$. If $d^0(x)=k$, then $d(x^\#)=k+1$, because $\alpha(x,t):=(X_v^t(x),t)$ gives a diffeomorphism between $O^0(x)\times\mathbf{R}$ and $O(x^\#)$ (orbit in $\sigma^\#$), for any fixed v. As in [SJ, p.103], one calculates directly that $\dim L(x^\#)=k+1$. Thus there is a finite set $\{u_1,...,u_r\}$ such that the subalgebra generated by the corresponding X_{u_i} is still $L(x^\#)$ at $x^\#$; since $\mathbf{U}\subseteq\text{clos}(\mathbf{V})$, these u_i may be taken in $\mathbf{V}$. Viewing $\sigma^\#$ as a system on the manifold $O(x^\#)$, there is by Chow's theorem a set C, open in $O(x^\#)$, contained in $A(x^\#)\cap B^\#$, and $A(x^\#)$ is now the set reachable using controls in $\{u_1,...,u_r\}$. Note that $(a,t)\in A(x^\#)$ iff $a\in A^t(x)$. Pick v in $\mathbf{V}$, and consider the corresponding map α as above. Then, $\alpha^{-1}(C)$ is open in $O^0(x)\times\mathbf{R}$, and hence contains a set $D\times\{T\}$, for some D open in $O^0(x)$ and some $T>0$. Note that then $X_v^T(D)\subseteq A^T(x)\cap B$. Consider now all the possible maps F as above (for this fixed T), corresponding to the possible sequences out of the controls u_i. The (countable) union of their images contains $A^T(x)$, and hence contains the open subset $X_v^T(D)$ of $O^0(X_v^T(x))$; thus one of these images intersects $X_v^T(D)$ with nonzero measure. The result then follows from Sard's theorem.

(2.2) LEMMA. Let $d^*(y)=n$. The following statements are equivalent: (a) $y\in F^T(x)$ for some T, (b) $y\in\text{int}[A^T(x)]$ for some T, and (3) $y\in\text{int}[A(x)]$.

PROOF. [a⇒b] If $d^*(y)=n$, then $O^0(y)$ is a ngbd of y. Thus $y \in \mathrm{int}[F^T(x)] \subseteq \mathrm{int}[A^T(x)]$. [b⇒c] Trivial. [c⇒a] Let $y \in B \subseteq A(x)$, B open. Applying 2.1 to the *reverse* system σ^-, we conclude that there is a $z \in B$ such that $y \in F^T(z)$. Say that $z \in A^S(x)$. Then $y \in F^{T+S}(x)$.

Note that, for analytic or finite systems, (b) already implies $d^*(y)=n$.

(2.3) PROPOSITION. Let $y \in F^T(x)$. There are then open ngbds A and B of x in $O^0(x)$, C of y in $O^0(y)$, a $\Lambda>0$, and for each $\lambda \in (0,\Lambda]$ a ngbd B_λ of x in $O^0(x)$, such that the following properties hold for each such λ:

(a) $(\exists s \in \mathbf{N})(\exists g \in G^{s\lambda})[B \subseteq g^{-1} S^s_\lambda.z$ for each $z \in A]$;

(b) $(\exists s \in \mathbf{N})(\exists g \in S^s_\lambda)[B_\lambda \subseteq g^{-1} S^s_\lambda.z$ for each $z \in A]$;

(c) $[\exists s \in \mathbf{N}$ s.t. $\lambda s=T] \Rightarrow [C \subseteq S^s_\lambda.z$ for each $z \in A]$;

(d) $d^0(x)=n \Rightarrow [\exists s \in \mathbf{N}$ s.t. $C \subseteq S^s_\lambda.z$ for each $z \in A]$.

[Note that (a) and (b) do not involve y, and hence depend only on $F^T(x)$ being nonempty; in particular, they must also hold for $(F^T)'(x)$.]

PROOF. The proof is a refinement of that of theorem 2.2 in [SS]. Let W, B, B', F, G, q, λ_0, u_i, T, $\underline{t}$, and all the $k_i[t,\lambda]$ and $v[t,\lambda]$ be as in that reference, except that B, B' are now ngbds relative to $O^0(y)$. Let $g[t,\lambda]$ be the element of S^s_λ corresponding to the control $v[t,\lambda]$, where $s = s(\lambda) = k_r[t,\lambda]$. Consider the constant control u_r^*, of length $h = h(\lambda) = T-\lambda k_r[t,\lambda]$, equal to u_r, and let $g_1 = X^h_{u_r}$ be the transformation associated to u_r^*. Finally, let $v'[t,\lambda]$ be the concatenation of $v[t,\lambda]$ and u_r^*. Thus, all the controls $v'[t,\lambda]$ have the same length T, and as in [SS], $v'[t,\lambda]$ differs from $v[t,0]$ on a set of measure less than $r\lambda$ (independently of $t \in \mathbf{R}_r(T)$). Further, $v'[t,\lambda]$ depends continuously on t, for any fixed λ. For such t,λ, let $p'[t,\lambda,z] := (g_1 g[t,\lambda]).z$ (=state reached using $v'[t,\lambda]$). We consider p' as a function on $W \times [0,\lambda_0) \times D$, with values in $O^0(y)$, where D is a ngbd of x in $O^0(x)$ chosen small enough so that the image of p' is included in B'. Consider the maps $H_{\lambda,z} := q(p'[G(.),\lambda,z])$: $B \to B$. These are all continuous, and they converge uniformly to the identity of B as $(\lambda,z) \to (0,x)$. Thus there are a $\Lambda>0$, a ngbd A of x in $O^0(x)$, and a ngbd C of y in the interior of B, such that C is included in $H_{\lambda,z}(B)$ for each $\lambda \in (0,\Lambda]$ and each $z \in A$. Thus $C \subseteq g_1 S^s_\lambda.z$ for all $z \in A$. Note that $s=s(\lambda)$ is by construction the integer part of T/λ, so (c) follows from this (since then $h(\lambda)=0$). Let $g_2 \in S^T$ be the transformation corresponding to $v[t,0]$, and let $B := g_2^{-1}.C$. Since g_2 gives a diffeomorphism between $O^0(x)$ and $O^0(y)$ sending x into y, B is a ngbd of x in $O^0(x)$; then (a) follows by taking $s=s(\lambda)$ and $g := g_1^{-1} g_2$. For any given λ, pick any $g\in S_\lambda^s$ such that $y=g_1g.x$; this g gives (b) with $B_\lambda := (g_1g)^{-1}.C$. Finally, (d) is proved using $v[t,\lambda]$ instead of $v'[t,\lambda]$; the proof is in that case analogous to that in [SS], except for the dependence of H on z, which is treated as above.

We shall also use the following remark; it is stated directly for the case $d^0=n$, since this is the only instance where we shall need it. Fix a sequence $a=(a_1,...,a_s)$, each a_i= +- 1, and a $\lambda>0$. Denote by $J_\lambda^a:\mathbf{M}\times\mathbf{U}^s\to\mathbf{M}$ the map $J_\lambda^a(x,w):= g_s...g_1.x$, where $w=(u_1,...,u_s)$ and $g_i:= X_{u_i}^{a_i\lambda}$.

(2.4) PROPOSITION. Assume that $\mathbf{d}_w J_\lambda^a(x,\underline{w})$ has rank n for some $\underline{w}\in\mathbf{V}^s$, for some λ and a as above. There are then ngbds C of $y=J_\lambda^a(x,\underline{w})$, A of x, and E of λ, such that $C\subseteq J_\alpha^a(\{z\}\times\mathbf{V}^s)$ for all $z\in A$ and $\alpha\in E$.

PROOF. There are $B\subseteq B'$ diffeomorphic to closed balls centered at y, a ngbd W of $\underline{w}$ in $\mathbf{V}^s$, and a $G:B\to W$ with $G(y)=\underline{w}$ and $J(x,\cdot)oG$ = identity. Let $p:(0,\infty)\times\mathbf{M}\times W\to\mathbf{M}$, $p(\alpha,z,w):= J_\alpha^a(z,w)$. Restricting to a compact ngbd of $(\lambda,x,\underline{w})$, we may assume that p is uniformly continuous; in particular, the maps $p(\alpha,z,\cdot)$ are all continuous and converge uniformly to $p(\lambda,x,\cdot)$ as $(\alpha,z)\to(\lambda,x)$. Further, we may assume that each $p(\lambda,z,\cdot)$ maps into B'. Let $q':B'\to B$ be a retraction mapping $B'\backslash B$ to the boundary of B. Let $H_{\alpha,z}:= q[p(\alpha,z,G(\cdot))]$. These are all continuous and converge uniformly to the identity of B. The proof is completed as before.

3. Sampled Transitivity. In this section we include some general results on ST; section 5 will include further refinements for the cases of group and finite systems.

(3.1) LEMMA. If x is nice, there is a ngbd A of x in $O^0(x)$ and a $\Lambda>0$ such that $A\subseteq O_\lambda^0(x)$ for all $\lambda\in(0,\Lambda]$.

PROOF. By lemma 2.1, there is a y as required in 2.3. Pick A and Λ as there. By part (b), $x\in O_\lambda^0(z)$ for all z in A and $\lambda\in(0,\Lambda]$; equivalently, $z\in O_\lambda^0(x)$.

(3.2) LEMMA. Assume that σ is analytic. Pick $x\in\mathbf{M}$. Then for almost all λ there exists an $s\in\mathbf{N}$ and a $g\in S_\lambda^s$ such that $g^{-1}S_\lambda^s.x$ is a ngbd of x in $O^0(x)$. In particular, $O_\lambda^0(x)$ is a ngbd of x in $O^0(x)$, for almost all λ.

PROOF. Apply 2.3, with control set $\mathbf{V}$ (c.f. remark following statement). There are then λ,s,g with $S_\lambda^s.x$ being a ngbd of g.x in $O^0(g.x)$, and this holds for λ in a ngbd of 0. Pick any such λ,s,g. With all

$a_i=1$, $J^a_\lambda(x,\cdot)$: $\mathbf{V}^s \to O^0(y)$ has rank $k=d^0(x)$ at some $\underline{w}\in\mathbf{V}$ (by Sard's theorem, because $\dim O^0(y)=k$). Now fix $\underline{w}$ but let λ be a variable. Let E be the set of α for which $\mathbf{d}_w J^a_\alpha(x,\underline{w})$ has rank >= k as a map into $\mathbf{M}$; E is the complement of an analytic set, and it is nonempty because $\lambda\in E$. So its complement is discrete in $(0,\infty)$. If $\alpha\in E$, then $g:=J^a_\alpha(\cdot,\underline{w})$ is as desired. Thus the property holds for $\alpha\in E$ as well as for α near 0.

Note that, while we know from 3.1 that there is for λ small a *common* ngbd A of x as desired, 3.2 only adds that for almost all λ there is *some* such ngbd. We turn now to the ST property itself. The following deals with the behavior of this property under perturbations; we provide a proof which generalizes readily to the case of controllability (4.3 below).

(3.3) PROPOSITION. If K is compact with nonempty interior, the set $N(K):= \{\lambda \mid K \text{ is } \lambda\text{-ST}\}$ is open.

PROOF. Pick any $\lambda\in N(K)$. Consider any $(x,y)\in K\times K$. By assumption, $O_\lambda(x)$ includes K, and hence contains a nonempty open subset B of K. Consider, for the given λ and x, the maps H_a:$\mathbf{P}^{s(a)}\to\mathbf{M}$, $H_a(w):=J^a_\lambda(x,w)$. Let D_a be the critical set of H_a, and F_a its complement. Finally, let F be the union of the (countably many) sets $B\cap H_a(\mathbf{U}^{s(a)}\cap F_a)$, and D the union of the $B\cap H_a(\mathbf{U}^{s(a)}\cap D(a))$. By Sard's theorem, D has measure 0. Since $B=F\cup D$, F is nonempty. Pick a,s, and $z\in B$ such that $z=H_a(v)$ and $\mathbf{d}H_a(v)$ has rank n. By continuity, we may assume that $v\in\mathbf{V}^s$. Since $K\subseteq O_\lambda(z)$ $(=O^{0\prime}_\lambda(z))$, there are a sequence b and a $v'\in V^{s(b)}$ such that $J^b_\lambda(z,v')=y$. Thus $J^{ab}_\lambda(x,\underline{w})=y$, with $\underline{w}=vv'$, and rank $J^{ab}_\lambda(x,\underline{w})$ is again n. We are thus in the situation of 2.4, and there are open ngbds A_{xy} of x, C_{xy} of y, and E_{xy} of λ such that $C_{xy}\subseteq O_\alpha(z)$ for all $\alpha\in E_{xy}$ and $z\in A_{xy}$. Pick E:= intersection of the E_{xy} corresponding to a finite subcover of $K\times K$ by the $A_{xy}\times C_{xy}$; then E is a ngbd of λ in N(K).

(3.4) THEOREM. Assume that $d^*(x)=n$ for every $x\in\mathbf{M}$. Then σ is sampled transitive.

PROOF. Note first that the hypothesis implies that $O^0(x)=\mathbf{M}$ for each x (strong transitivity). Now pick a compact K, as in the definition of ST. We may assume that K is connected (since $\mathbf{M}$ is). For each $x\in K$ there is an open ngbd A_x of x in $O^0(x)=\mathbf{M}$ and a Λ_x as in (3.1). Cover K by these A_x, and pick a finite subcover. Let Λ be smaller than all the Λ_x in this subcover. Take any $\lambda\in(0,\Lambda]$. Each z in K is in some A_x, hence in the interior of $O^0_\lambda(x)=O^0_\lambda(z)$. Thus the disjoint sets $O^0_\lambda(x)\cap K$ cover K; connectedness implies there is only one.

(3.5) COROLLARY. Assume that σ is either analytic or finite. Then, d*(x)=n for all x iff σ is ST.

PROOF. it is only necessary to prove the "if" part. Since σ is sampled transitive, there is for each x a λ such that $O^0_\lambda(x)$ is a ngbd of x. Since this is included in $O^0(x)$, we conclude that d*(x) = $d^0(x)$ = n.

4. Sampled controllability. The study of controllability is of course known to be very difficult in comparison with that of transitivity (weak controllability). Sampled controllability implies controllability, so the most that one can hope for is to state results which conclude ST from controllability (or from conditions known to imply it) *plus* other conditions. Locally, we can state:

(4.1) PROPOSITION. Assume that d*(x)=n and A(x) is a ngbd of x. Then there are a ngbd *B* of x and a Λ>0 such that *B* is λ-SC for all $\lambda \in (0,\Lambda]$.

PROOF. Since $x \in$ int(A(x)), it follows from 2.2 that $x \in F^T(x)$ for some T. Take then *A*, *C*, Λ as in 2.3(d), and *B*:= $A \cap C$.

(4.2) COROLLARY. Let σ be analytic of finite. Then, the following statements are equivalent: (a) d*(x)=n and $x \in$ int(A(x)), (b) $x \in$ int(A^T(x)) for some T, (c) $x \in$ int(A_λ(x)) for some λ, and (d) there is an SC ngbd of x.

This last result is of interest especially in view of the recent results in [SU3], which show that a certain algebraic condition (the "Hermes controllability condition") is sufficient, for a large class of systems, for (b) to hold. (More precisely, this condition implies that $x \in$ int(A(x)), and that x is an equilibrium point, so (b) indeed holds.) Some global statements follow; see section 5 for others. First note the following "stability" result on λ; the proof is analogous to that of 3.3 and it is therefore omitted. (We assume that **U** is a manifold because the equality $O_\lambda = O_\lambda$', used in the proof of 3.3, does not apply to this case.)

(4.3) PROPOSITION. Assume that **U**=**P**. If K is a compact with nonempty interior, {λ | K is λ-SC} is open.

(4.4) LEMMA. Assume that $d^0(x)$=n for all $x \in$ **M** and that, for each x,$y \in$ **M** there is a T with $y \in F^T(x)$. Then σ is SC.

PROOF. Take K compact. By 2.3(d), there are for each $(x,y)\in K\times K$ an open ngbd $A\times C$, and a $\Lambda>0$, with $b\in A_\lambda(a)$ whenever $a\in A$, $b\in B$, $\lambda\in(0,\Lambda]$. Pick any positive Λ less than all the Λ_{xy} corresponding to a finite subcover. For each $\lambda\in(0,\Lambda]$, K is SC.

(4.5) THEOREM. If $d^*(x)=n$ for all $x\in \mathbf{M}$, then σ is controllable iff it is sampled controllable.

PROOF. We apply 4.4. Note that $d^0(x)=n$ for all x. Pick $x,y\in \mathbf{M}$. By 2.1, there is a z in some $F^T(x)$. But controllability gives that $y\in A(z)$. Thus $y\in F^S(x)$, for some s. Alternatively, we could have used 4.1 on a (possibly larger) connected K.

Since SC implies $d^0(x)=n$ for all x, it also implies that $d^*(x)=n$ if x is nice. This gives corollary 4.6, while 4.7 is an immediate consequence of theorems 4.9 and 4.10 of [SJ], which conclude that under those hypothesis, $d^*(x)=n$ for all x if σ is controllable.

(4.6) COROLLARY. Let σ be analytic or finite. Then σ is controllable with $d^*(x)=n$ for all x iff σ is SC.

(4.7) COROLLARY. Let σ be analytic. Assume that either **M** has a compact covering space or that the fundamental group of **M** has no elements of infinite order (e.g., $\mathbf{M}=\mathbf{R}^n$). Then σ is controllable iff it is sampled controllable.

(4.8) REMARK. Two cases often singled-out in the controllability literature are those of "symmetric" systems (for each $u\in\mathbf{U}$ there is some $v\in\mathbf{U}$ with $X_v=-X_u$) and of "homogeneous" systems (where $f(x,u) = u_1X^{(1)}+...+u_sX^{(s)}$ and **U** is a ngbd of 0 in $\mathbf{R}^s$). As in the usual (non-sampled) case, the theory is considerably simplified in these cases. For homogeneous systems, $d^*(x)=n$ for all x is *equivalent* to σ being GSC; in fact, **M** is λ-SC *for all* λ. To establish this, it is enough to prove that, for all λ,u,t, there exist a $r\in\mathbf{N}$ and a $v\in\mathbf{U}$ with $X_u^t = X_v^{r\lambda}$. For these systems, $X_{u/a}^{at} = X_u^t$ for all a =/= 0, so it is only necessary to pick r and v with $(r\lambda)v=tu$. This is always possible because **U** is a ngbd of 0. Regarding symmetric systems, it is clear that $O_\lambda(x) = A_\lambda(x)$ for all x, so λ-ST and λ-SC become the same notion.

5. Group and finite systems. A *group* system will be by definition an analytic system for which **M**=G is a (connected) Lie group and all the vector fields X_u are right invariant. In order to simplify notations, we shall assume as part of the definition of group system that *the Lie algebra* L(G) *of the group* G coincides with the Lie algebra of the system $L(\sigma)$. (If this were not to hold, we may always restrict attention to the connected Lie subgroup of G with algebra *L*.) Let σ be a group system. The evaluation g→g(1) (1=identity of G) allows us to identify *G*(σ) with G (and, in particular, 1 with e). Under this identification, the action g.x becomes just the product gx in G, and $G^0:=O^0(e)$ equals G^0, $G_\lambda:=O_\lambda(e) = G_\lambda$, $G^0_\lambda:=O^0_\lambda(e) = G^0_\lambda$, $S_\lambda:=A_\lambda(e) = S_\lambda$, and S:=A(e) = *S*. Further, G^0 is then the (connected) normal Lie subgroup associated to L^0. Since G_λ and G^0_λ are submanifolds, they are also Lie subgroups. The vector fields in L_λ are right invariant: let R^x be the right translation by x; then $\beta(x,v) = R^x(\beta(e,v))$, for the β in the definition of L_λ, so $\phi(x) = [\mathbf{d}R^x(\beta(e,u))\mathrm{o}\,\mathbf{d}_u\beta(e,u)](a) = \mathbf{d}R^x(\phi(e))$. Since $L_\lambda(e) = T_eG^0_\lambda$, it follows that L_λ is here a *subalgebra* of *L* (in fact, the one corresponding to the Lie subgroups G^0_λ and G_λ).

Many control problems can be usefully modelled using group systems; see for instance the examples dealing with control system design, rigid body control, and electrical networks, in [BR1]. Note that group systems are in particular finite (to be treated below). We must still study them separately, both because some of the results do not generalize and, more importantly, because the results for finite systems will be obtained employing those for group systems. Let σ be a fixed group system, wih **M**=G.

(5.1) LEMMA. For almost all λ, (a) there exists a $g\in S_\lambda$ such that $g^{-1}S_\lambda$ is a ngbd of e in G^0, and (b) $G^0=G^0_\lambda$.

PROOF. part (a) is just lemma 3.2, applied at x=e. Since $g^{-1}S_\lambda\subseteq G^0_\lambda$, it follows that G^0_λ is a subgroup of the connected group G^0 which contains a ngbd of the identity. Thus (b) follows from topological group theory.

(5.2) THEOREM. The following statements are equivalent: (a) $L^0=L$, (b) σ is ST, (c) σ is GST, and (d) G is λ-ST for almost all λ.

PROOF. Since (a) is equivalent to $G=G^0$, (a)⇒(d) follows from 5.1(b); the rest is easy.

(5.3) LEMMA. Assume that σ is controllable. Then there is a λ>0 such that $G^0\subseteq S_\lambda$.

PROOF. By 2.1, $F^T(e)$ is nonempty. Say $z\in F^T(e)$. By controllability, $e\in A(z)$, so we conclude that $e\in F^T(e)$ for some (maybe different) T. By 2.3(c), there is a ngbd *C* of e in G^0 such that $C\subseteq S_\lambda$. Since

G^0 is connected, $G^0 = \cup\{C^k, k >= 1\} \subseteq S_\lambda$.

(5.4) THEOREM. σ is GSC $\Leftrightarrow$ σ is controllable and $L^0 = L$.

PROOF. [$\Rightarrow$] Trivial. [$\Leftarrow$] Since $G^0=G$ (by the Lie algebra assumption), we may apply 5.3 to conclude that $G=S_\lambda$. We deduce that $A_\lambda(g) = S_\lambda g = G$ for each $g \in G$, as wanted. Alternatively, we could apply (a)$\Rightarrow$(d) from 4.2 (at x=e) to again conclude that S_λ is a ngbd of e and hence that $S_\lambda=G$ (for all λ small enough: see 4.1).

When G is compact, the assumption that $L^0=L$ is *sufficient* to conclude GSC. This is because $L^0=L$ implies controllability for compact groups; see [JS]. Although not a corollary of that result, one may still apply basically the same *argument* as in that reference in order to prove that the closure of the semigroup S_λ includes the subgroup G_λ generated by S_λ; this establishes the following result, which will be useful later.

(5.5) LEMMA. Let G be compact and let λ be such that $G^0=G^0_\lambda$ (c.f. 5.1). Then $G^0 \subseteq \mathrm{clos}(S_\lambda)$.

Our study of finite systems ($\dim L<\infty$) relies upon the theory of Palais (see [PA], esp. chapter IV; for other applications of this theory to control problems, see for instance [HI]). We summarize those results of Palais that we need, using our notations. There is in this case a group system σ_G whose state space is $G=G(\sigma)$, input space is **U** (=input space for the original σ), and having equations dx/dt = $f^\$(x,u)$ with the property that each flow $X^{\$t}_u$ --as an element of G, via the identification with $X^{\$t}_u(e)$,-- is equal to X^t_u--as diffeomorphism on **M**. Thus $G(\sigma_G)$, and its subgroups and subsemigroups introduced earlier, identify naturally with the corresponding objects in $G(\sigma)$. Finally, one also identifies L(G) and $L(\sigma)$ in the following way. Let $ev_x:G\to M$ be the evaluation $ev_x(g):=g.x$. Define α_x: $T_eG=L(G) \to T_x\mathbf{M}$ by $\alpha_x(L):= \mathbf{d}ev_x(e)(L)$ and $\alpha:L(G)\to TM$ via $\alpha(L):=$ vector field $x\to\alpha_x(L)$. Then α is an isomorphism between L(G) and $L(\sigma)$. We shall then identify $L(\sigma)$, $L(\sigma_G)$, and L(G) where there is no danger of confusion.

When $L^0=L$, results for a finite system σ are immediate from the corresponding results for σ_G, since $G^0=G$ in that case. But the property $L=L^0$ is too restrictive in practice (for instance, not even *linear* systems satisfy this, in general). Thus we must work with G^0 itself. That is the reason for our having presented previous results relativized to $O^0(x)$. *For the rest of this section,* σ *is a finite system and* G *its associated Lie group*.

(5.6) THEOREM. σ is GST $\Leftrightarrow$ d*(x)=n for all x$\in$**M**.

PROOF. [$\Rightarrow$] Trivial. [$\Leftarrow$] Apply 5.1 to σ_G. Then $G^0=G^0_\lambda$ for almost all λ. Pick any such λ. Take any x$\in$**M**. The assumption d*(x)=n for all x implies that G^0 acts transitively on **M**. Thus $G^0_\lambda.x = G^0.x =$ **M**, as desired.

(5.7) LEMMA. Assume that d*(x)=n for all x. Then for almost all λ, int[$A_\lambda(x)$] is nonempty for all x.

PROOF. Pick any λ and g as in 5.1(a), for the system σ_G. The action of G^0 is again transitive. Thus we may apply Theorem I.2.5 of [HO] to conclude that ev_x is an open map, for each x. Since $g^{-1}S_\lambda$ has interior, $ev_x(g^{-1}S_\lambda) = g^{-1}A_\lambda(x)$ also does. So $A_\lambda(x)$ contains an open set itself.

(5.8) THEOREM. Assume that G is compact. Then σ is GSC $\Leftrightarrow$ d*(x)=n for all x$\in$**M**.

PROOF. [$\Rightarrow$] Trivial. [$\Leftarrow$] Apply 5.7 to the reversed system σ^-, and recall 5.1 and 5.5. We conclude that, for almost all λ, the following properties hold: (a) $G^0\subseteq$clos(S_λ), and (b) for each y$\in$**M** there is an open set *B* such that y$\in A_\lambda$(z) for each z$\in$*B*. Pick any x,y$\in$**M**. Since G^0=**M**, it follows that $A_\lambda(x)$ is dense. Pick *B* as above. Then *B* intersects $A_\lambda(x)$; thus y$\in A_\lambda(x)$.

We end this section with the following lemma, and an example of how it may be applied.

(5.9) LEMMA. L_λ is a subalgebra of *L*.

PROOF. Let (g,u,r,a) be as in the definition of *L*. Consider the induced vector fields ϕ and $\phi^\$$ on **M** and *G* respectively, as well as the corresponding maps β and $\beta^\$$. Since $\beta(x,v) = ev_x(\beta^\$(e,v))$, calculating differentials (and using that $\beta^\$(e,u)=e$) results in $\phi = \alpha(\phi^\$)$. Thus $\alpha(L_\lambda(\sigma_G)) = L_\lambda(\sigma)$. Since α establishes an isomorphism between $L(\sigma_G)$ and $L(\sigma)$, the result is a consequence of the group case treated earlier.

(5.10) REMARK. Consider the case of (continuous time) *polynomial systems* ([BA]). Since *L* consists then of polynomial vector fields, the same is true of L_λ, by the above result. Finite polynomial systems result in particular as the canonical realizations of a wide class of input/output behaviors, namely, those described by finite Volterra series (see [CR]). Bilinear systems are also a class of finite polynomial systems. The fact that L_λ consists of polynomial vector fields allows for the use of algebraic-geometric techniques. As a simple illustration, consider the following proof of the fact that ST implies GST (c.f. 5.6 above). Let $B(\lambda) := \{x \mid d_\lambda(x)<n\}$; this is then an algebraic set, for each fixed λ. Pick any $\lambda>0$, and consider the family of sets $B_n := B(\lambda/2^n)$. Then B_n is a descending sequence of algebraic sets, and hence is finite. Let B_N be a minimal element of this chain. For any given $x \in B_N$, there is by ST a $\Lambda>0$ such that x is not in $B(\lambda)$ for any $\lambda \in (0,\Lambda]$. Pick $n>N$ with $\lambda/2^n<\Lambda$. Then B_n is a proper subset of B_N, contradicting minimality. So B_N must be empty; equivalently, **M** is $\lambda/2^N$-ST.

6. Remarks and examples. Since many of the sampled notions have been proved to be equivalent to the corresponding non-sampled versions plus an algebraic condition ($d^*(x)=n$ for all x), most examples in the literature can be used to illustrate the above results, provided that one calculates d^*. Thus we shall give only one example, and concentrate after that on counterexamples to a few conjectures which would seem a priori to be true.

(6.1) EXAMPLE. Euler's equations for the angular momentum vector of a rotating rigid body, subject to a controlled torque along a non-necessarily principal direction, are given by $dx/dt = X(x)+uY(x)$, where states belong to $\mathbf{M}=\mathbf{R}^3$, $u \in \mathbf{U}$=(say)$\mathbf{R}$, and $X(x) = (a_1x_2x_3)\,\delta/\delta x_1 + (a_2x_1x_3)\,\delta/\delta x_2 + (a_3x_1x_2)\,\delta/\delta x_3$, and with $Y(x)$ = constant vector field $(b_1,b_2,b_3)'$. Such a system provides a model of a satellite steered by a pair of opposing jets; see for instance [BR2]. The quantities a_i can be expressed in terms of the moments of inertia with respect to the principal axes. Since X is Poisson stable, $d(x)=n$ for all $x \in \mathbf{M}$ is a necessary and sufficient condition for controllability. But since $\mathbf{M}=\mathbf{R}^3$, corollary 4.7 applies. Thus σ is in fact *SC* iff this condition holds. This, in turn, happens precisely when the above coefficients satisfy: $a_ib_j^2$ =/= $a_jb_i^2$ for i =/= j; see [BA] or [BO] for the corresponding calculations. Thus one has a simple algebraic condition for the sampled controllability of the satellite model.

(6.2) REMARK. The orbits O_λ are integral manifolds for the Lie algebras L_λ of vector fields on **M**. However, the situation is very different from that in the "classical" (non-sampled) case. For instance, $O_\lambda(x)$ may (and often does) fail to be connected, and may even be dense but proper (e.g., $dx/dt=1$ on $\mathbf{M}=\mathbf{S}^1$, with $\lambda/2\pi$ irrational). Further, the forward reachable set $A_\lambda(x)$ may not be connected even if $O_\lambda(x)$ is (e.g., $dx/dt = 4x+ux$ on $\mathbf{M}=\mathbf{R}_+$ and $|u| \le 1$; here $O_\lambda(x)$ is always connected and $A_\lambda(x)$ has 3 components, including one, {x}, without interior). Thus many of the arguments in the classical case, based on connectedness, have no analogue under sampling.

(6.3) EXAMPLE. A system for which L_λ fails to be a subalgebra of L. Take **M**:=**R**×**R**$_+$, **U**=**R**, and f(x,u):= (-x+uh(y)) δ/δx + y δ/δy. Let λ=1. For h take for example the function $h(y):= -(1+y)^{-2}$. Then L consists of linear combinations of -x δ/δx + y δ/δy and of vector fields of the form k(y) δ/δx, where each k(y) is rational with denominator $(1+y)^{-s}$, s >= 2. Consider the element φ of L_λ corresponding to g=e, u=0, r=1, a=1. Then φ(x,y) = q(y) δ/δx, where q(y) is rational with denominator (1+y)(1+ey). Thus φ is not in L.

(6.4) EXAMPLE. One of the most useful tools in the usual theory is (the positive form of) Chow's theorem, which implies for analytic systems that A(x) has nonempty interior whenever O(x) does. Here, however, it may happen that $O_\lambda(x)$ has interior (i.e., is a submanifold of dimension n,) but $A_\lambda(x)$ does not. We construct an example with **M**=**R**, using the method in [SS, lemma 3.6]. We first obtain an analytic function g:**R**→**R** whose derivative is bounded below, and for which a pair (x,λ) satisfies the condition [g(x+kλ) = g(x) for all k∈**Z**] iff it satisfies [x=2rπ, λ=2sπ, r,s∈**Z** and s does not divide r]. As in the above reference, this gives rise to a system for which $d_\lambda(x)=0$ iff (x,λ) is of this form. Further, assume that this g is such that, with $x_0=2\pi$, $g(x_0+2k\pi) = g(x)$ for all <u>positive</u> integers k. In that case we can conclude both that $O_{2\pi}(x_0)$ has interior and that $A_{2\pi}(x_0)$ = {2kπ, k >= 1}. An example of a g like this is g(x):=(sin x)/x. This example can be modified to obtain one where even $O_\lambda(x)$=**M** *for all* (x,λ) but such that still $A_\lambda(x_0)$ has empty interior for some x_0. For this, take the above g and introduce a $g_1(x):= \sigma 2^{-n} g^2(x+2\pi n)$, the sum over n >= 0. Again $x_0=\lambda=2\pi$ serves as a counterexample.

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