AN APPROXIMATION THEOREM IN NONLINEAR SAMPLING

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Abstract

We continue here our investigation into the preservation of structural properties under the sampling of nonlinear systems. The main new result is that, under minimal hypothesis, a controllable system always satisfies a strong type of approximate sampled controllability.

Introduction

The papers [4],[3] began the study of the following type of question. Let

(1) $\dot{x}(t) = f(x(t),u(t)$

describe a nonlinear system which satisfies a certain system-theoretic property (e.g., weak controllability, controllability); one then asks if then same property is still satisfied by (1) when one restricts the controls to be *sampled* at a constant rate δ. Recall that a δ-sampled control is one which is constant in intervals $[k\delta,(k+1)\delta)$; this is the type of control usually available under digital regulation. For the results that follow, system (1) has to satisfy certain technical conditions: states x(t) evolve in a smooth n-dimensional manifold **M**, controls u(t) take values in a subset **U** of a smooth manifold **P**, and $f:\mathbf{M}\times\mathbf{P}\to T\mathbf{M}$ is a smooth map such that each vector field $X_u=f(\cdot,u)$ is complete; further, int(**U**) must be connected, with **U** included in the closure of its interior (relative to **P**) -see [3] for details, in particular, any convex subset of $\mathbf{R}^m$ satisfies the requirements.

Call (1) *controllable* if for each pair of states x,y there is some admissible control $u(\cdot)$ driving x to y; for definiteness, take here "admissible" to mean piecewise constant (but, of course, not necessarily sampled). A *sampled controllable* system (1) is one which satisfies the following property: given any compact subset K of **M**, there exists some $\delta>0$ such that, for each pair x,y of states in K, there is a δ-sampled control sending x to y.

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We shall need a few technical concepts and notations. The *T-time orbit* $\mathbf{O}^T(x)$ is the set of states of the form

$(X_1^{t_1}...X_r^{t_r})(x)$,

with the t_i real (possibly negative), and with $\sigma t_i=T$, and each X_i a controlled vector field of the form X_{u_i}. (Here X^t denotes the flow exp(tX).) For any $\delta>0$, the subset $\mathbf{O}_k^\delta(x)$ of $\mathbf{O}^{\delta k}(x)$ is the set of time-k sampled orbits: as above, but with all t_i now integer multiples of δ, adding to δk. The $\mathbf{O}_k^\delta$ [resp., $\mathbf{O}^T$] are connected submanifolds of $\mathbf{M}$, diffeomorphic to each other for different k,T. Let d(x) be the dimension of $\mathbf{O}^0(x)$. If $\mathbf{M}$ is controllable, d(x) is independent of x (the flow induced by any control sending x into y gives a diffeomorphism between $\mathbf{O}^0(x)$ and $\mathbf{O}^0(y)$); we denote in that case by d the common value of the d(x). Introduce also the notation $A_\delta(x)$ for the set of states reachable from x using sampled controls, i.e. the set of all $X_1^{t_1}...X_r^{t_r}(x)$ with the t_i *positive* multiples of δ.

One of the main results of [3] (theorem 4.5) states that controllability of (1) plus a certain Lie-algebraic condition ("d*=n" in the notation used there) implies sampled controllability. In fact, the technical condition can be weakened considerably: d*=n was used only to insure that a suitable "normal 0-time controllability" property be satisfied. But a recent result of Grasse (see [2],[5]), applied to the system obtained by extending (1) with a trivial equation dz/dt=1, implies that the needed condition is automatically satisfied if the system (1) is (weakly) controllable and d=n. Now, the latter condition is *necessary* for sampled controllability, since for any x the δ-sampled orbit is the union of the (countably many) $\mathbf{O}_k^\delta(x)$. Thus, the former having nonempty interior implies that every $\mathbf{O}_k^\delta(x)$, hence $\mathbf{O}^0(x)$, has dimension n. In conclusion, the result in [3] can be strenghtened to: controllability plus d=n is *equivalent* to sampled controllability. Further, it is easy to see that the result in [1], for analytic systems, establishing that controllability implies d*=n when the fundamental group of $\mathbf{M}$ has no elements of infinite order, can be extended to the general smooth case (the proof in [6], for example, extends directly). Thus, for instance for simply connected manifolds one concludes that sampled controllability is equivalent to controllability. Similar generalizations are immediate for weak controllability.

The case of weak controllability of *one*-dimensional analytic systems was considered in some detail in [4]. It is easy to see that analiticity was not used in the proof of the existence of global sampling periods in [4]. The same proof works in the smooth case. But it is false in general that the set of "good" δ is a discrete set: with the notations of [4], it is only necessary to choose a nonzero smooth $g:\mathbf{R}\to\mathbf{R}$ with g(t)=0 if 0<t<1. Then, for sampling times $\delta>1$ there are states whose orbits have dimension 0. The controllability case, incidentally, is easier than transitivity (weak controllability): *any* $\delta>0$ provides sampled control if the system is controllable. This is proved as follows. Let x be any state, and pick $\delta>0$. Let y be the sup of the set of states δ-reachable from x, assumed finite. By controllability, there is some $X=X_u$ such that $X(y)>0$. Thus, $X(z)>a>0$ in some ngbd I of y. Let z in I be δ-reachable from x. Then the sequence $\{X^{k\delta}\}$, $k>0$, is δ-reachable from x and is eventually to the right of y, contradicting its choice. So the above sup=$+\infty$, and similarly for the inf=$-\infty$. The desired result now follows from the fact that (because **U** is connected) the sets of states δ-reachable from x in k steps are intervals, increasing with k.

The simplest case in which sampled controllability is not a consequence of controllability is that of the system $\dot{x}=1$ in the unit circle. No δ provides sampled controllability in this example. But if δ/π is irrational, then at least the controllable states from any x form a *dense* subset of **M**. This motivates the following definition and result.

Definition. The system (1) is *approximately sampled controllable* (a.s.c.) if the following property holds: For each compact subset K of **M** there exists a $\delta>0$ such that, for every x in K, K is included in $\mathrm{clos}(A_\delta)$.

Theorem. If (1) is controllable then it is approximately sampled controllable.

A proof is outlined below. Only the case d=n-1 has to be considered, since for d=n one has (exact) sampled controllability. Note that the result is of course not a simple consequence of the continuity on control values of solutions of (1): such an argument would only provide, for each fixed tolerance ε, a δ such that for each x, any other y is at distance (choose any metric) less than ε from $A_\delta(x)$; the result claims that δ can be chosen independently of ε. The above example on the circle is a good illustration of this.

The proof will show that, in fact, there is for each K a Δ such that a "random" $\delta<\Delta$ will satisfy the desired property. (More precisely, it will hold for any $\delta<\Delta$ not in a fixed countable set.)

Details

We assume from now on that (1) is a given controllable system with d=n-1, X is an arbitrary vector field X_υ, $\upsilon \in$ **U**, K is a compact in **M**, and T any fixed real number. Further, we let K' be any compact which contains $\{X^t(x), x \in K, |t| <= T\}$, and let e:=T/4. Note that X has no singular points X(x)=0, since that would imply d=n. The following easy fact is needed later.

Lemma 1. For any state y there exist a ngbd U_y of y, a real c>0, and a diffeomorphism $g:Q(c) \rightarrow U_y$, $Q(c) = \{(s_1,...,s_r) \in \mathbf{R}^n$ s.t. $|s_i|<c\}$, g(0)=y, such that:

(a) V_y:=g(s_1=0) is an open ngbd of y in $\mathbf{O}^0$(y) (endowed with the usual manifold structure),

(b) $X^t(V_y)$=g(s_1=t) is an open ngbd of X^t in $\mathbf{O}^t$, and

(c) $X^t(g(0,s_2,...,s_r)) = g(t,s_2,...,s_r)$.

Proof. Let x be any other state. By the results in [5],[2], y is *normally* reachable from x, i.e., there is an integer r, controls $u_1,...u_r$, and positive t_i^0, such that the map

(2) $h:t=(t_1,...,t_r) \rightarrow (X_1^{t_1}...X_r^{t_r})(x)$

has $h(t_0)$=y at $t_0=(t_1^0,...,t_r^0)$ and has rank n differential at t_0. We may assume that X_1=X: if this were not the case, extend h to $\mathbf{R}^{n+1}$, with $h(t_o,t):=X^{t_o}(h(t))$; this still satisfies all properties, at $(0,t_0)$. The lemma is now just the implicit function theorem: pick integers i_j, j=1,...,n such that, in local coordinates, the partials $\delta/\delta t_i$ are linearly independent. Since X(x) =/= 0, we may assume that i_1=1. Now map $(s_1,...,s_r)$ into (t_i), where

$t_1 := s_1 - \sigma_{i=/=1} s_i + t_1^0$,

$t_{i_j} := s_j + t_{i_j}$, j=2,...,n,

and all other $t_i=t_i^0$. Take g as the composition of this map with the above h, restricted to an appropiate cubical ngbd Q(c). For constant s_1=t, g(s) is in $\mathbf{O}^t$(y) by construction. Further, g restricted to (s_1=t) must have rank at least n-1, so dim$\mathbf{O}^t$(y)=n-1 implies that the image of this restriction is open in the latter. (In fact, g restricted to (s_1=t) provides a typical coordinate chart.)

For further reference, we note that t_1 above could be assumed to be as large as wanted, say $t_1>\alpha$. This can be accomplished by letting $y':=X_\alpha$, and obtaining next a normal control from x to y'; the concatenation of the latter with υ for length α is as desired.

Let G(x) be the set of times t for which $\mathbf{O}^t(x)=\mathbf{O}^0(x)$. By controllability, G=G(x) is independent of x. It is easy to verify that G is a subgroup of **R** and that $\mathbf{O}^s(x)$ and $\mathbf{O}^t(x)$ can intersect only if they coincide, which happens iff $s-t\in G$. Observe that G is nontrivial: given any t>0, there is some control sending x to $X_{-t}(x)$ in (positive) time s. Thus s=-t mod G, or t+s =/= 0 is in G. In fact, G is *countable*: $A=g(s_1=t)$ is disjoint from $B=g(s_1=t)$ for t,s small and distinct, and A,B are open subsets of $\mathbf{O}^t(y)$ and $\mathbf{O}^s(y)$ respectively. If G were uncountable, there would be an uncountable set of t's with -c<t<c and $\mathbf{O}^0(y)=\mathbf{O}^t(y)$, so the correponding A's would constitute an uncountable set of pairwise disjoint opens in $\mathbf{O}^0(y)$, contradicting second countability of the latter submanifold.

The proof of the theorem will be a consequence of the following lemma.

Lemma 2. There exists a real Δ, $0<\Delta<e$, such that the following property holds for each $0<\delta<\Delta$. For any z,y in K' there is a real h, 0 <= h <= δ, such that $X_{-t}(X^h)(y)$ is in $A_\delta(X^{-t}(z))$ for each 0 <= $t<\delta$.

We shall show first how the theorem follows from this result. Let **Q**(G) be the rational vector space generated by G; this is again countable. Pick any $\delta<\Delta$ which is not in **Q**G. Fix any x,y in K. We want to show that y is in the closure of $A_\delta(x)$. Let $z:=X^T(y)$. Apply lemma 2 to this pair (z,y). Let S be the set

$\{X^t(y),\ h-\delta<t<T-\delta\}$.

It follows from another application of lemma 2 that there is some $\alpha<\delta$ such that $x^*:=X^\alpha(y)$ is in $A_\delta(x)$; thus we may assume without loss of generality that x=x* is in S.

Note that $z=X^T(y)$ and y is in $\mathbf{O}^{k\delta-h}(z)$, for some positive integer k (by lemma 2). Thus -T is congruent to $k\delta$-h mod G, or T-h = $-k\delta$ mod G, with $-k\delta<0<T-h$. It follows that T-h and δ cannot be rational multiples of each other. Let $t_k := [k\delta+\alpha]$, k=0,1,2..., where [s] denotes the residue of s mod (T-h) in I := $(h-\delta,T-\delta]$. Then $\{t_k\}$ is dense in I. Let $x_k := X^{t_k}(y)$. So $x_0=x$, and the set of x_k is dense in S. The result will be proved once we establish that each x_k is in $A_\delta(x)$.

But this is easy to see by induction on k: if $k\delta+\alpha$ is not larger than $T-2\delta$ then $x_{k+1}=X^{\delta}(x_k)$. Otherwise, $x_{k+1} = X^{-t}X^{h}(y)$, for $t=T-t_k-\delta$, which by lemma 2 is δ-reachable from $X^{-t}X^{T}(y) = X^{\delta}(x_k)$, so that x_{k+1} is again reachable from x_k.

Proof of lemma 2.

Let x,y be in K'. Obtain as in lemma 1 U_y, V_y, and $t_1^0,\ldots,t_r^0$, with $t_1^0>3e$, with the state "x" in the proof being the chosen x. Apply now lemma 1 to x, obtaining in turn cubical ngbds U_x and V_x of x in **M** and $\mathbf{O}^0(x)$ respectively. Let

$Q'(c):=\{t \text{ s.t. } s(t)\in Q(c)\}$.

Without loss, assume $c_x=c_y=c$, and that c is small enough so that $|t_1^0-t_1|<2e$ for $t\in Q'(c)$ and $c<e$. Consider the map

$\phi(z,t):=X^{t_1-\varepsilon}X_2^{t_2}\cdots X_r^{t_r}(z)$,

defined for t in Q'(c) and z in U_x, where $\varepsilon=\varepsilon(z)$ is that real for which z is in $X^\varepsilon(V_x)$. For any $t=(t_1,\ldots,t_r)$, denote $\#(t):= t_1+\ldots t_r$. Then, $\phi(z,t)\in X^{\#(t)}$. Consider the restriction of ϕ' of $\phi(z,t)$ to the set of t with $\#(t)=T=\#(t^0)$. Since ϕ' is continuous in z,t, its image is in U_y for t close to t^0 and z close to x. Redefining U_x if necessary, we may assume that $\phi'(z,t)\in U_y$ when $t\in Q'(c)$ and $z\in U_x$. Since $\mathrm{Im}\phi'$ is connected, it contains $y=\phi(x,t^0)$, and $\mathrm{Im}\phi'\subseteq\mathbf{O}^0(y)$, it follows that $\phi'(z,t)\in V_y$ for the above (z,t), and in fact ϕ' is continuous into V_y (embedded submanifold of U_y). Let U_y', V_y' be cubical ngbds of y for which $|s_i| <= c'$.

Pick any $\delta>0$ and t with $\#(t)=T$. Let $u[t,\varepsilon,\delta]$ be the δ-sampled control obtained from

$X_1^{t_1-\varepsilon}X_2\ldots X_r^{t_r}$

as in [4], theorem 2.2 (this was denoted "$v[t,\delta]$" there). Note that its length is $d[T-\varepsilon,\delta]$, where $d[r,\delta]$:= largest multiple of δ not larger than r. Let $v[t,\varepsilon,\delta]$ be the above control followed by the constant control υ applied for time $T-\varepsilon-d[T-\varepsilon,\delta]$. (This is "$u_r$*" in [3].) Thus $v[t,\varepsilon,\delta]$ has lenght $T-\varepsilon$ (independent of δ).

Let $H_{\delta,z}:V_y'\to V_y$ be defined as follows: $H_{\delta,z}(y')$ is the state reached when applying the control $u[t,\varepsilon,\delta]$ to z, where $\varepsilon=\varepsilon(z)$ is as above, and $g(t')=y'$. By an argument as before, this is well defined into V_y, for δ small enough. Further, each $H_{\delta,z}$ is a continuous map, and $H_{\delta,z}$ converges uniformly to the identity of V_y' as $\delta\to0$ and $z\to x$. So, by a reasoning as in [4],[3], there are new cubical ngbds U_y'', V_y'' such that

$V_y'' \subseteq H_{\delta,z}(V_y')$

for $z\in U_x$ (redefine U_x again, if necessary) and $0<\delta<\Delta$. Redefine now U_y, V_y (and c) so that these are the U_y'',V_y''; further, assume without loss that $\Delta<c/2$.

Pick now any $y' \in U_y$, and any $0<\delta<\Delta$. Then $y'=X^{\alpha}(y'')$, for some α and some $y'' \in V_y$. Let $h=h(\alpha,\delta)$ be 0 if $T+\alpha$ divides δ, and

$$d[T+\alpha,\delta]+\delta-T-\alpha$$

otherwise. So $0 <= h<\delta$.

Now pick any $z \in U_x$, and $\varepsilon=\varepsilon(z)$ as before. Since $y'' \in \mathrm{Im} H_{\delta,z}$, there is some control $v[t,\varepsilon,\delta]$ which applied to z results in y". Let $y^*:=X^{\alpha+\varepsilon+h}(y'')$. This is in $A_\delta(z)$, because $v[t,\varepsilon,\delta]$ is the concatenation of $u[t,\varepsilon,\delta]$ (δ-sampled) and υ with length $T-\varepsilon-d[T-\varepsilon,\delta]$, so y^* is obtained by concatenating $u[t,\varepsilon,\delta]$, with total length

$$\delta+d[T+\alpha,\delta]-d[T-\varepsilon,\delta].$$

(Note that, since $t_1>3e$ and each of α, ε, δ have magnitude <e, this is always a true, positive time, control). Further, $y^*=X^{\varepsilon+h}(y')$.

Now cover $K' \times K'$ by sets $(U_x/2)\times(U_w/2)$, where the notations indicate the use of c/2 in place of c. Pick a finite subcover, and a Δ smaller than all the corresponding $\Delta_{x,w}$. Pick now any z,y and any $0<\delta<\Delta$. Say that (z,y) is in an element $(U_x/2)\times(U_w/2)$ of the subcover, and $z=X^r(x)$. Let $y':=X_{-r}(y)$. Since $|r|<c/2$, y' is in U_w. Let $h:=h(\alpha,\delta)$ correspond to this y'. Let $z':=X^s(z)$, for any given $|s|<\delta$. So $z' \in U_x$, and $\varepsilon(z')=r+s$. Apply now the above argument to these (z',y'). It follows that $X^{s+h}(y) = X^{r+s+h}(y')$ is in $A_\delta(z')$, for any such z', as wanted.

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