SOME CONNECTIONS BETWEEN STABILIZATION AND FACTORIZATION

Eduardo D. Sontag*
SYCON - Rutgers Center for Systems and Control
Department of Mathematics, Rutgers University, New Brunswick, NJ 08903
e-mail: sontag@fermat.rutgers.edu

Appeared as *Proc. IEEE Conf. Decision and Control, Tampa, Dec. 1989*, IEEE Publications, 1989, pp. 990-995.

ABSTRACT

Coprime right fraction representations are obtained for nonlinear systems defined by differential equations, under assumptions of stabilizability and detectability. A result is also given on left (not necessarily coprime) factorizations.

1 Introduction

There has been some interest in problems related to parameterizations of controllers for nonlinear systems, and this has motivated the search for coprime factorization conditions; see e.g. [6], [8], [2], [21], [20], [1], as well as our paper [18]. In the latter reference, we showed how the existence of right coprime factorizations for the input to state mapping of systems linear in controls

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) \tag{1}$$

is very closely related to the problem of smooth stabilization for such systems. This relation is of course not surprising, since the "classical" way to obtain such factorizations for linear systems is indeed through the use of state feedback stabilization (see [12], as well as [7] and [2] for related and previous work in the nonlinear case.)

In trying to extend these results to more general systems

$$\dot{x} = f(x, u) \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, f is a differentiable function from $\mathbb{R}^{n+m}$ into $\mathbb{R}^n$ not necessarily affine in u as in (1), and 0 is an equilibrium point for the system, $f(0,0) = 0$, it was noted in [16] and [17] that a weaker notion of coprimeness, which we called "weak coprimeness", seems to be needed, analogous to the various notions of coprimeness used e.g. in [14], [20], [21]. (Weak coprimeness is not necessarily equivalent to the stronger notion from [18], which we shall call here a "Bezout" factorization.) When adding an output map

$$h : \mathbb{R}^n \to \mathbb{R}^p,\ x(t) \mapsto y(t) = h(x(t)), \tag{3}$$

*This research was supported in part by US Air Force Grant AFOSR-88-0235.

to (2), the problems become even harder, since observers must be used. (The independent work [9] does not employ observers, but the relation to our work is not very clear, since the definitions of stability are very different, in addition to the fact that [9] works in discrete time.)

In this paper, we show how to employ a state feedback combined with an observer construction in order to obtain factorizations. Again, this is not a particularly original idea, since it is standard for linear systems. Our contribution here in that regard is at most to clarify what type of observer is needed and to state explicitely some "theorems" which are basically tautological once the definitions have been given and the results in our previous papers are used. Nonetheless, we hope that this will serve to motivate further work into the construction of observers of the type needed here, as well as into the smooth state stabilization problem, which has attracted considerable attention lately on its own.

Part of what follows could be stated more abstractly, in terms of axiomatic notions of stability, as for instance done in [21]. However, the interesting results will have to do with stability in the sense of differential equations and "bounded-input bounded-output" behavior. From now on, we use the notations and terminology from [18]. For the reader's convenience, these are summarized in an appendix.

2 Coprime Fractions

Consider an operator $P : \mathcal{D}(P) \to L^p_{\infty,e}$, where $\mathcal{D}(P) \subseteq L^m_{\infty,e}$. A pair of IOS operators

$$N : L^m_{\infty,e} \to L^p_{\infty,e}, \text{ and } D : L^m_{\infty,e} \to L^m_{\infty,e}$$

such that D is causally invertible, $\mathcal{D}(D^{-1}) = \mathcal{D}(P)$, and

$$P = ND^{-1} \tag{4}$$

will be called a *right (fractional) factorization* of P. If there exists some IOS operator

$$Q : L^{m+p}_{\infty,e} \to L^m_{\infty,e}$$

so that

$$Q \begin{bmatrix} D \\ N \end{bmatrix} = I \tag{5}$$

where the second term in the left-hand side of this equation indicates the map into $L^{m+p}_{\infty,e}$ that sends $v \mapsto (Dv, Nv)$ and I denotes the identity in $L^m_{\infty,e}$, the factorization (N, D) will be said to be a *weakly coprime right factorization.*

The Appendix compares this definition with that used in [18]; we use the term "weak" coprimeness to distinguish it from that reference, though a better terminology would probably be simply "coprime" for that used here and "Bezout" in [18].

A *left* factorization for P is a pair $(\widetilde{N}, \widetilde{D})$ of IOS operators with D causally invertible and so that

$$\widetilde{D}P = \widetilde{N} . \tag{6}$$

Note that this equality forces P to be defined everywhere, since $\widetilde{N}$ is. So when studying left factorizations we shall assume that $\mathcal{D}(P) = L^m_{\infty,e}$.

Coprime right factorizations, if they exist, are unique in the following sense. Given two fractional representations $N_1 D_1^{-1}$ and $N_2 D_2^{-1}$ of P, assume that Q_1 and Q_2 are the respective one sided inverses as in (5). If $v = D_2^{-1} D_1 u$ then $D_2 v = D_1 u$ and therefore also

$$N_2 v = P D_2 v = P D_1 u = N_1 u$$

which implies that

$$D_2^{-1} D_1 u = v = Q_2(D_1(u), N_1(u))$$

and thus that $M := D_2^{-1} D_1 = Q_2(D_1(\cdot), N_1(\cdot))$ is an IOS operator. Similarly, $M^{-1} = D_1^{-1} D_2$ is also IOS, so one may write

$$D_1 = D_2 M$$

with M causal, stable, and with an inverse that is also causal and stable. This uniqueness is entirely analogous to e.g. Theorem 3.11 in [8] or the corresponding results in [21].

3 Stabilizability and Observers

The system (2) is *(smoothly) stabilizable* if there exists a smooth map $K : \mathbb{R}^n \to \mathbb{R}^m$ with $K(0) = 0$ such that the zero state for $\dot{x} = f(x, K(x))$ is GAS. In [18] we defined the system (2) to be "smoothly input to state stabilizable" if there is such a K so that the system

$$\dot{x} = f(x, K(x) + u)$$

becomes ISS. As pointed out in [16] and [17], this last notion cannot be expected to give a satisfactory theory for systems (2) that are not affine in controls as in (1). Instead we suggested there a definition more in accordance with the standard practice in nonlinear control, namely control laws of the type

$$u = K(x) + G(x)v \tag{7}$$

where G is an $n \times n$ matrix of smooth functions invertible for all x, not necessarily the identity, and K is as above. One of the main results from [16] is as follows:

Theorem 1 from [16]: If (2) is stabilizable, then there exist K, G as above so that the new system

$$\dot{x} = f(x, K(x) + G(x)v) =: f_{cl}(x, v) \tag{8}$$

is ISS. ∎

The proof in fact shows more, namely that if K makes $\dot{x} = f(x, K(x))$ GAS then the same K works for the ISS property, and only G needs to be obtained.

Analogous definitions and results could be given that do not require K to be everywhere smooth; the tutorial paper [19] discusses that point and provides extensive references to the problem of smooth stabilizability.

From now on, we fix a pair (f,h) specifying a system with outputs consisting of (2) together with a K-bounded mapping (3), and we let

$$\mathcal{P}_s : L^m_{\infty,e} \to L^n_{\infty,e}$$

be its (partially defined) input to state operator (initial state zero) and

$$\mathcal{P} : L^m_{\infty,e} \to L^p_{\infty,e} : u \mapsto h(\mathcal{P}_s(u))$$

be its (partially defined) input/output operator. We wish to construct factorizations for $\mathcal{P}$.

We now turn to observers. In this paper, a *state observer* for the given system (f,h) will be an i/o operator

$$\widetilde{\mathcal{P}}_s : L^{m+p}_{\infty,e} \to L^n_{\infty,e}$$

such that

$$\widetilde{\mathcal{P}}_s(u, \mathcal{P}(u)) = \mathcal{P}_s(u) \tag{9}$$

for each $u \in \mathcal{D}(\mathcal{P}_s)$. That is, $\widetilde{\mathcal{P}}_s$ causally reconstructs the state of the system $\dot{x} = f(x,u), x(0) = 0$, using input and output observations. Of course, unless more is assumed, the zero initial state on the plant makes the observer concept trivial (and useless), since one could *define* $\widetilde{\mathcal{P}}_s(u,y) := \mathcal{P}_s(u)$. What makes the concept interesting is the following assumption:

(O1) $\widetilde{\mathcal{P}}_s$ is an IOS operator.

For example, if $\mathcal{P}$ is in fact the input to state map, i.e. $h(x) \equiv x$, then we can take $\widetilde{\mathcal{P}}_s(y) \equiv y$, which is stable. More interestingly, for linear detectable systems $\dot{x} = Ax + Bu, y = Cx$, the input to state map of any Luenberger observer

$$\dot{z} = (A + LC)z + Bu - Ly,\ z(0) = 0 \tag{10}$$

is an observer and, having chosen L so that $A+LC$ is Hurwitz, is stable. (A similar argument can be used for systems that can be linearized by output injection and coordinate changes.) The fact that the Luenberger observer functions correctly (but only asymptotically) even for nonzero initial states will not appear in our nonlinear generalizations, though it is essential in understanding abstract issues of input/output stabilization for nonlinear systems ([15]).

Motivated by the linear situation, we shall say that (f,h) is *detectable* if it admits an observer satisfying (O1). It means essentially that small observed inputs and outputs should result in small state estimates.

We will prove:

Theorem 1 *If (f,h) is stabilizable and detectable then $\mathcal{P}$ admits a coprime right factorization.*

In order to obtain left fractional representations, two more assumptions are needed; we let $\widetilde{\mathcal{P}} := h\widetilde{\mathcal{P}}_s$.

(O2) $y \mapsto \widetilde{\mathcal{P}}(0,y) - y$ is causally invertible.

(O3) $u \mapsto \widetilde{\mathcal{P}}(u,y) - \widetilde{\mathcal{P}}(0,y)$ is stable uniformly on y.

By (O3) we mean precisely the following. The i/o operator

$$\Delta(u,y) := \widetilde{\mathcal{P}}(u,y) - \widetilde{\mathcal{P}}(0,y)$$

is so that $\mathcal{D}(\Delta) = L_{\infty,e}^{m+p}$ and there exist a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that, for each pair of times $0 \leq T \leq t$,

$$|\Delta(u,y)(t)| \leq \beta(\|u_T\|, t-T) + \gamma(\|u^T\|) \tag{11}$$

for each $(u,y) \in L_{\infty,e}^{m+p}$. For instance, if it were the case, as it happens for linear systems, that there is a decomposition

$$\widetilde{\mathcal{P}}(u,y) = \widetilde{\mathcal{P}}_1(u) + \widetilde{\mathcal{P}}_2(y),\ \widetilde{\mathcal{P}}_1(0) = \widetilde{\mathcal{P}}_2(0) = 0, \tag{12}$$

then (O3) follows from (O1), since stability of $\widetilde{\mathcal{P}}$ implies stability of each of $\widetilde{\mathcal{P}}_1$ and $\widetilde{\mathcal{P}}_2$, and $\Delta = \widetilde{\mathcal{P}}_1$. Property (O2) is satisfied for instance if $\widetilde{\mathcal{P}}_s$ is "strictly causal" in any of the usual senses, as discussed in a later section on dynamic observers. In particular, it is satisfied for Luenberger observers.

For example, the input to state map $\mathcal{P}_s$ admits an observer satisfying (O1-O2); now we cannot take $\widetilde{\mathcal{P}}_s(y) \equiv y$, but

$$\dot{z} = -z + y + f(y,u),\ z(0) = 0 \tag{13}$$

does satisfy all assumptions. Note that stability follows from the fact that $\widetilde{\mathcal{P}}_s$ is in this example the composition of the memoryless map $(u,y) \mapsto y + f(y,u)$ with an i/s map of a linear stable system. That this is really an observer follows from the fact that $\dot{x} = f(x,u)$ implies that x satisfies also (13) when $y \equiv x$, and property (O2) follows from strict causality. Unfortunately, obtaining property (O3) seems harder even in this case. The only obvious example is that in which one can write

$$\dot{x} = f_0(x) + f_1(x,u),\ f_1(x,0) \equiv 0,$$

and $|f_1(x,u)| \leq \gamma(u)$ for some function of class $\mathcal{K}$. In that case, the operator $\Delta(u,y)$ is the input to state operator for

$$\dot{z} = -z + f_1(x,u)$$

and is therefore stable uniformly on $y = x$.

The main fact about left factorizations is

Theorem 2 *If (f,h) admits an observer satisfying (O1) to (O3) then $\mathcal{P}$ has a left factorization.*

4 Proofs of Results

As we said earlier, the proofs of the two theorems are essentially trivial, given the definitions and previous results.

For the first theorem, we first apply Theorem 1 from [16] to get K, G as in equation (8). If μ is the memoryless operator induced by the mapping

$$M : \mathbb{R}^{m+n} \to \mathbb{R}^m : \ (v, \xi) \mapsto G^{-1}(\xi) \ [v - K(\xi)]$$

then we define

$$Q(u, y) := \mu(u, \widetilde{\mathcal{P}}_s(u, y)) \tag{14}$$

which is IOS since it is a composition of stable operators. The operators N and D are chosen basically in the same manner as in [18], as follows. First let N_s be the i/s mapping of the closed-loop system (8), and let

$$N = hN_s \tag{15}$$

be the i/o map of (8)-(3), stable by construction. Finally, let μ^* be the memoryless operator induced by

$$M^* : \mathbb{R}^{m+n} \to \mathbb{R}^m : \ (\nu, \xi) \mapsto K(\xi) + G(\xi)\nu$$

and define

$$D := \mu^*(\cdot, N_s(\cdot))$$

which is IOS since it is a composition of IOS operators.

The closed-loop solution $N_s(v)$ of (8) with initial condition $x(0) = 0$ and input v is the same as the solution of (2) with the same initial condition and with $u = \mu^*(v, N_s(v))$, that is,

$$N_s = \mathcal{P}_s D$$

which will imply (4) once that D is shown to be invertible. (The same argument also shows that the range of D coincides with the domain of $\mathcal{P}_s$, and hence of $\mathcal{P}$.) On the other hand, from the definitions of μ and μ^* it follows that

$$v = \mu(Dv, N_s v) = \mu(Dv, \mathcal{P}_s Dv) \tag{16}$$

for all v, so D is indeed causally invertible (since $\mu, \mathcal{P}_s, D$ are causal). Property (5) follows from:

$$\begin{aligned} Q(Dv, Nv) &= \mu(Dv, \widetilde{\mathcal{P}}_s(Dv, Nv)) \\ &= \mu(Dv, \widetilde{\mathcal{P}}_s(Dv, PDv)) \\ &= \mu(Dv, P_s Dv) \\ &= v \end{aligned}$$

where the last equality is a consequence of (16). This completes the proof of Theorem 1. ∎

The mappings in the proof are interpreted as follows in terms of the system connection in which v is fed as an external output and $u = K(x) + v$: $N : v \mapsto y$, $D : v \mapsto u$, and Q is obtained using the maps $(u, x) \mapsto v$ and the observer output $(u, y) \mapsto x$.

The proof of the second Theorem is even easier. It is only necessary to take

$$\widetilde{D} : y \mapsto \widetilde{\mathcal{P}}(0, y) - y \tag{17}$$

which is IOS by assumption (O1) and causally invertible by (O2). The composition

$$\widetilde{N} := \widetilde{D}P$$

is then equal to $\Delta(\cdot, \mathcal{P}(\cdot))$ and is therefore stable too. (Note that uniform stability independently of the last factor is needed because $\mathcal{P}$ will in general not be stable.) ∎

5 Comparison With the Linear Case

In this section we show how the coprime factorization derived here reduces, for linear systems, to the one well-known in the literature. We first recall the standard formulas for coprime right factorizations of linear systems (left factorizations are simply obtained from these by duality). These formulas first appeared in the paper [12], and have since been rediscovered by other authors. (Unfortunately, the fact that the formulas were given in far more generality, in the context of factorizations of "systems over rings", originally obscured the fact that they were significant even for linear finite-dimensional systems.)

Given a strictly proper transfer function $W(s)$, then, one wishes to obtain a factorization

$$W = ND^{-1}$$

where N and D are stable and proper, D is square and invertible as a rational matrix, as well as stable proper transfer matrices S and T so that the Bezout equation

$$SN + TD = I$$

holds. This is done as follows. We first find a minimal state space realization

$$W(s) = C(sI - A)^{-1}B$$

and a feedback law F so that $A + BF$ is a Hurwitz matrix and an L such that $A + LC$ is Hurwitz. In [12], equation (3.12), we choose the Luenberger observer solution (over arbitrary rings, the context of that reference, Luenberger observers do not always exist, as discussed in [10], but it is the obvious choice over fields), so that

$$\left[(sI - A - LC)^{-1} \quad -(sI - A - LC)^{-1}L\right] \begin{bmatrix} sI - A \\ B \end{bmatrix} = I$$

(there is a missprint in equation (3.12), in that the roles of "M" and "N" are interchanged there) and from this

$$S = F(sI - A - LC)^{-1}L, \quad T = I - F(sI - A - LC)^{-1}B$$

using [12], line after equation 5.12. Finally, using [12], equation (5.9), we pick:

$$D := \left(I - F(sI-A)^{-1}B\right)^{-1} \quad = \quad I + F(sI-A-BF)^{-1}B$$

and therefore $N = WD$ is also determined; explicitely,

$$\begin{aligned} N &= C(sI-A)^{-1}B\left(I - F(sI-A)^{-1}B\right)^{-1} \\ &= C(sI-A-BF)^{-1}B \; . \end{aligned}$$

The matrices S, T, N, D given above are the same as those called respectively $-\widetilde{Y}, \widetilde{X}, M, N$ in the book [4], which should be consulted for more details.

We now turn to showing why our formulas indeed reduce to these for the linear case. Using a Luenberger observer (10) together with a linear feedback stabilizer $u := Fx + v$ in (7), N in formula (15) is the i/o map of $\dot{x} = (A+BF)x + Bv, y = Cx$, and hence its transfer matrix is $C(sI-A-BF)^{-1}B$ as in the standard linear case. From here it follows that D must also be the same. On the other hand, equation (14) says that

$$Q(u,y) = u - Fz$$

where z is the solution of (10), and thus it is of the form $Q_1(u) + Q_2(y)$, where Q_1 and Q_2 have transfer matrices T and S respectively. For left factorizations, the operator $\widetilde{D}$ in (17) has transfer matrix

$$I + C(sI-A-LC)^{-1}L$$

which is precisely the transfer matrix obtained by duality in the linear case (last paragraph before section VI in [12], and the matrix "$\widetilde{M}$" in the book [4]).

6 Some Remarks

Often one obtains observers which are themselves given by differential equations (see e.g. [5], [13], [11]). In this case, there is a system

$$\dot{z} = F(z,u,y), \; z(0) = 0 \tag{18}$$

together with an output map $H(z)$, so that the i/o operator of this system is the desired observer $\widetilde{\mathcal{P}}_s$. For linear systems, for instance, Luenberger observers (10) correspond to models like this with $H(z) = z$ and F linear.

Assumption (O2) is automatically satisfied for such observers. Indeed, more generally, a map of the type $u \mapsto y := H(z) - u$, where z satisfies

$$\dot{z} = f(z,u) \; , z(0) = 0$$

is always causally invertible, since its inverse can be computed using a negative feedback loop. That is, $y = H(z) - u$ if and only if u is the output of

$$\dot{\tilde{z}} = f(\tilde{z}, H(\tilde{z}) - y), \; \tilde{z}(0) = 0, \; u = H(\tilde{z}) - y \; .$$

Techniques for obtaining observers satisfying all the desired properties should be the object of much further research. In some simple cases, however, such as linear systems with bounded controls or some classes of bilinear systems, observers can be easily constructed, and only the smooth stabilizability problem presents an obstruction.

The emphasis on *smooth* stabilization was only for simplicity. Various types of continuous or even discontinuous feedback laws could be used, provided that at least local existence and uniqueness can be guaranteed for the various differential equations involved. However, using discontinuous feedback means that the notion of stability may have to be relaxed, since "K-bounded" maps are used in the construction of the various memoryless operators that appeared in the proofs.

In our definition of stability (IOS) we require that the operator be everywhere defined. Possibly it is better to relax this requirement, and to ask for instance that Q in the definition of coprimeness only need to be defined on all the pairs (Dv, Nv). This might give a more interesting theory, in that more operators may be factorizable, but it will affect parameterization questions.

7 Appendix: Stability Notions

In this appendix we recall some of the terminology and basic results from [18], as well as some of the notations used.

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{K}$ if it is continuous strictly increasing and satisfies $\gamma(0) = 0$; it is *of class* $\mathcal{K}_\infty$ if in addition $\gamma(s) \to \infty$ as $s \to \infty$. Note that if γ is of class $\mathcal{K}_\infty$ then the inverse function γ^{-1} is well defined and is again of class $\mathcal{K}_\infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{KL}$ if for each fixed t the mapping $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed s it is decreasing to zero on t as $t \to \infty$.

For any vector ξ in Euclidean space, $|\xi|$ is its Euclidean norm. For measurable functions u taking values in such a space, $\|u\|$ is the sup norm

$$\|u\| := \text{ess.sup.}\, \{|u(t)|, t \geq 0\}.$$

This may be infinite; it is finite when u is essentially bounded.

The system (2) is *input to state stable (ISS)* if there is a function β of class $\mathcal{KL}$ and there exists a function γ of class $\mathcal{K}$ such that for each measurable essentially bounded control $u(\cdot)$ and each initial state ξ_0, the solution exists for each $t \geq 0$ and furthermore it satisfies

$$|x(t)| \leq \beta(|\xi_0|, t) + \gamma(\|u\|). \tag{19}$$

In particular, when $u \equiv 0$, ISS is equivalent to global asymptotic stability (GAS).

For each integer m, $L^m_{\infty,e}$ denotes the set of all measurable maps

$$u : [0, \infty) \to \mathbb{R}^m$$

which are locally essentially bounded, that is, such that the restriction of u to each finite subinterval of $[0, \infty)$ is essentially bounded, and L^m_∞ is the set of all essentially bounded

u, that is the set of all u with $\|u\| < \infty$, thought of as a Banach space with this norm. Given any $u \in L^m_{\infty,e}$ and any $T \geq 0$, the truncations u_T and u^T are defined as follows:

$$u_T(t) := \begin{cases} u(t), & \text{if } t \in [0,T], \\ 0, & \text{if } t \in (T,\infty) \end{cases}$$

and

$$u^T(t) := \begin{cases} 0, & \text{if } t \in [0,T], \\ u(t), & \text{if } t \in (T,\infty). \end{cases}$$

Note that $u_T \in L^m_\infty$ for each T.

An *i/o operator* is a partially defined mapping

$$F : \mathcal{D}(F) \to L^p_{\infty,e}$$

with $\mathcal{D}(F) \subseteq L^m_{\infty,e}$, which is *causal*, i.e. it is such that

$$[F(u_T)]_T = F(u)_T$$

for each $T \geq 0$ and each $u \in \mathcal{D}(F)$. Implicit in this definition is the requirement that $u_T \in \mathcal{D}(F)$ for each $T \geq 0$ whenever u is in $\mathcal{D}(F)$.

For each state space system (2) and any fixed initial state $\xi_0 \in \mathbb{R}^n$, – which for simplicity we always take to be $\xi_0 = 0$,– let $\mathcal{D}$ be the set of controls $u \in L^m_{\infty,e}$ for which the solution $x(\cdot)$ of (2) with $x(0) = \xi_0$ is defined for all t. Then the map

$$F(u)(t) := x(t),\ \mathcal{D}(F) = \mathcal{D},$$

is an i/o operator, the *input to state mapping* of the system.

Memoryless i/o operators are everywhere defined i/o maps of the form

$$F(u)(t) := h(u(t))$$

where $h : \mathbb{R}^m \to \mathbb{R}^p$. In order for F to be well defined as a map into $L^p_{\infty,e}$, one needs that it be a compact operator, in the sense that the following property should hold for the mapping h:

$$\sup\{|h(\mu)|, |\mu| \leq a\} < \infty \quad \text{for all} \quad a > 0. \tag{20}$$

If in addition to (20) it holds that $h(0) = 0$ and h is continuous at the origin, then h is *K-bounded*. The supremum in (20) is a nondecreasing function of a; if it vanishes at $a = 0$ and is continuous at 0, then it can be majorized by a function of class $\mathcal{K}$. Thus an equivalent definition of K-bounded function h is that there must exist a function α of class $\mathcal{K}$ such that

$$|h(\mu)| \leq \alpha(|\mu|)$$

for each $\mu \in \mathbb{R}^m$, and hence the terminology. Observe that any continuous map h such that $h(0) = 0$ is K-bounded. In particular, the feedback laws K in the definition of smooth stabilizability are automatically K-bounded.

More generally, consider systems with output. These are given by an equation such as (2) together with a K-bounded mapping (3) with some integer p. Taking the initial state $\xi_0 = 0$, the assignment $F(u)(t) := h(x(t))$ gives the *i/o operator of the system*. In the particular case when h is the identity, this is the same as the input to state map.

The i/o operator F is *input/output stable (IOS)* if $\mathcal{D}(F) = L^m_{\infty,e}$ and there exist a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that, for each pair of times $0 \leq T \leq t$,

$$|F(u)(t)| \leq \beta(\|u_T\|, t - T) + \gamma(\|u^T\|) \tag{21}$$

for each $u \in L^m_{\infty,e}$.

It is easy to see that if the system (2) is ISS, then the system with output (2)-(3) is IOS. Also, if $F : L^m_{\infty,e} \to L^q_{\infty,e}$ and $G : L^q_{\infty,e} \to L^p_{\infty,e}$ are both IOS i/o operators then the composition $G \circ F$ is also IOS.

If the i/o operator $F : \mathcal{D}(F) \to L^p_{\infty,e}$ is one-to-one then there exists a well-defined left inverse

$$F^{-1} : \mathcal{D}(F^{-1}) \to \mathcal{D}(F) \subseteq L^m_{\infty,e},\; F^{-1}F = \text{ identity on } \mathcal{D}(F)$$

whose domain $\mathcal{D}(F^{-1})$ is the image im F of F. (Using simply juxtaposition FG to denote functional composition $F \circ G$.) The operator F is *causally invertible* if it is one-to-one and its inverse F^{-1} is an i/o operator.

The operator $P : \mathcal{D}(P) \to L^p_{\infty,e}$ admits a *Bezout right factorization* if and only if there exist IOS operators

$$A : L^p_{\infty,e} \to L^m_{\infty,e}, N : L^m_{\infty,e} \to L^p_{\infty,e}, \quad \text{and} \quad B, D : L^m_{\infty,e} \to L^m_{\infty,e}$$

such that B and D are causally invertible, $\mathcal{D}(D^{-1}) = \mathcal{D}(P)$,

$$P = ND^{-1} \tag{22}$$

and, if I denotes the identity in $L^m_{\infty,e}$,

$$AN + BD = I. \tag{23}$$

In [18] this was called a *coprime* rather than a *Bezout* factorization. We now think that the terminology “Bezout” is more appropriate. In [9] it is shown that in a certain abstract sense, and with different definitions of stability, coprimeness in the sense of the present paper, i.e., weak coprimeness, in which $Q(u, y)$ cannot necessarily be written as a sum

$$Q_1(u) + Q_2(y),$$

is equivalent to the Bezout property. But in terms of our definitions of stability the equivalence is not at all clear, and in any case there is often a need for explicit formulas for Q, such as are provided by the observer constructions.

8 References

1. Chen, G., and R.J.P. de Figereido, “A constructive approach to coprime factorizations of nonlinear feedback control systems,” Rice University Report EE8925, August 1989. Submitted for publication.

2. Desoer, C.A., and M.G. Kabuli, "Right factorization of a class of time varying nonlinear systems," *Memo UCB ERL M87/55*, Berkeley, 1987.

3. Desoer, C.A., and M.G. Kabuli, "Nonlinear plants, factorizations, and stable feedback systems," *Proc. IEEE Conf. Decision and Control*, Los Angeles, Dec.87, pp. 155-156.

4. Francis, B.A., *A Course in H_∞ Control Theory*, Springer, NY, 1987.

5. Grasselli, O.M., and A. Isidori, "An existence theorem for observers of bilinear systems," *IEEE Trans. Autom. Control* **26**(1981): 1299-1301.

6. Hammer, J., "Nonlinear systems: stability and rationality," *Int.J. Control*, **40**(1984): 1-35.

7. Hammer, J., "On nonlinear systems, additive feedback, and rationality," *Int.J. Control*, **40**(1984): 953-969.

8. Hammer, J., "Fraction representations of nonlinear systems: a simplified approach," *Int.J. Control*, **46**(1987): 455-472.

9. Hammer, J., "Fraction representations of nonlinear systems and non-additive state feedback," preprint, University of Florida, January 1989.

10. Hautus, M.L.J. and E.D. Sontag, "An approach to detectability and observers," in *AMS-SIAM Symp. Applied Math., Harvard, 1979* AMS-SIAM Pbl., 1980, pp. 99-136.

11. James, M.R., "Asymptotic nonlinear filtering and large deviations with application to observer design," Systems Research Center Report TR-88-28, University of Maryland, 1988. (Ph. D. Dissertation.)

12. Khargonekar, P.P., and E.D. Sontag, "On the relation between stable matrix fraction decompositions and regulable realizations of systems over rings," *IEEE Trans.Autom. Control* **27**(1982): 627-638. (Summarized version in *Proc.IEEE Conf.Dec. and Control, San Diego*, Dec. 1981, pp. 1006-1011.)

13. Kou, S.R., D.L. Elliott, and T.J. Tarn, "Exponential observers for nonlinear dynamical systems," *Inform.and Control* **29**(1975): 204-216.

14. Krener, A.J., "Nonlinear controller design via approximate normal forms," in *Proc. IMA Conf.on Signal Processing, Minneapolis, June-Aug 1988*, Institute for Mathematics and its Applications, 1989.

15. Sontag, E.D., "Conditions for abstract nonlinear regulation," *Information and Control*, **51**(1981):105-127.

16. Sontag, E.D., "Further facts about input to state stabilization", Report 88-15, *SYCON - Rutgers Center for Systems and Control*, Dec.1988. To appear in *IEEE Trans. Automatic Control*.

17. Sontag, E.D., "Some recent results on nonlinear feedback," *Proc. Conf. Info. Sciences and Systems*, Baltimore, Mar.1989, Johns Hopkins University Press, 1989, pp. 151-156.

18. Sontag, E.D., "Smooth stabilization implies coprime factorization," *IEEE Trans. Automatic Control*, **34**(1989): 435-443.

19. Sontag, E.D., "An introduction to the feedback stabilization problem," in *Mathematical Theory of Networks and Systems*, (M.A. Kaashoek et al, eds,.), Birkhäuser, Boston, 1990, to appear.

20. Tay, T.T., and J.B. Moore, "Left coprime factorizations and a class of stabilizing controllers for nonlinear systems," *Proc. IEEE Conf. Decision and Control*, Austin, Dec.1988, pp. 449-454.

21. Verma, M.S., "Coprime fractional representations and stability of nonlinear feedback systems," *Int.J. Control* **48**(1988): 897-918.