INPUT/OUTPUT EQUATIONS AND REALIZABILITY

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Abstract

This paper establishes a precise correspondence between realizability and the existence of algebraic differential equations relating derivatives of inputs and outputs of a continuous time system. The only assumption needed is that the data be "well-posed" in a suitable sense. Our results serve to relate the notion of realizability proposed by Fliess in the context of differential algebra with the more standard concept used in nonlinear state-space systems.

1 Introduction

It is often useful to model system behavior through differential equations of the type

$$E\left(u(t), u'(t), u''(t), \ldots, u^{(r)}(t), y(t), y'(t), y''(t), \ldots, y^{(r)}(t)\right) \;=\; 0 \tag{1}$$

where $u(\cdot)$ and $y(\cdot)$ are the input and output signals respectively. The form of the functional relation E may be deduced experimentally, for instance thorugh least squares techniques if a certain form (e.g. polynomials) is chosen. For instance, in linear systems theory one often deals with "autoregressive moving average" representations such as

$$y^{(k)}(t) = a_1 y(t) + \ldots + a_k y^{(k-1)}(t) + b_1 u(t) + \ldots + b_k u^{(k-1)}(t) \;\; . \tag{2}$$

On the other hand, in most theoretical developments in nonlinear control, one uses a state-space formalism, where inputs and outputs are related by a system of first order differential equations

$$\dot{x} = f(x) + G(x)u \;, \quad y = h(x)$$

where the state x is multidimensional.

It is a classical (and easy) fact that equations (2) can be reduced, by adding state variables for enough derivatives of the output y, to systems of first order equations, linear finite-dimensional systems. (In frequency-domain terms, rationality of the transfer function is equivalent to realizability). For nonlinear systems this reduction, essentially the problem of realization, is a far more difficult problem, one that is to a great extent unsolved.

Work by one of the authors in discrete-time ([15]) provided one approach to relating these two types of representations –with difference equations appearing instead,– and this

[1]Short book chapter, appeared as E.D. Sontag and Y. Wang. Input/output equations and realizability. In Realization and modelling in system theory (Amsterdam, 1989), volume 3 of Progr. Systems Control Theory, pages 125-132. Birkhäuser Boston, Boston, MA, 1990. Placed online by request, but please refer to full papers such as: (1) Y. Wang and E.D. Sontag. Orders of input/output differential equations and state-space dimensions. SIAM J. Control Optim., 33(4):1102-1126, 1995, (2) Y. Wang and E.D. Sontag. Algebraic differential equations and rational control systems. SIAM J. Control Optim., 30(5):1126-1149, 1992, (3) Y. Wang and E.D. Sontag. Generating series and nonlinear systems: analytic aspects, local realizability, and i/o representations. Forum Math., 4(3):299-322, 1992.

was used as a basis of identification algorithms by other authors; see for instance [8] and [3]. (The former reference shows also how to include stochastic effects in the resulting approach.)

The present paper deals with continuous time. A number of partial results are available in that case; see for instance [13] or [2], and recently [4] showed that it is natural to add inequality constraints to (1). On the other hand, the discrete time approach mentioned above has recently been extended to continuous-time *bilinear* systems and a theorem showed that realizability by such systems is equivalent to the existence of an E of a special form, namely affine on y (see [16]). We show in this paper how to generalize the full power of the results in ([15]) to the continuous case.

The view proposed in [14], [15] is that one should attack the problem as follows. One should separate the issue of existence of a realization from the question of "well-posedness" of the equation. For example, the equation

$$u(t)y'(t) = 1$$

can never be satisfied by all the input/output pairs corresponding to a state space system, as remarked in [16], nor is this true for

$$y''(t) = u'(t)^2 \quad .$$

In both of these cases, not only cannot the equation be reduced to state-space form but –as one can easily prove– even more basically, it cannot be satisfied by any "input/output map" of the type that we shall consider. Indeed, we shall show that *if* the equation would have been well-posed, in the sense that it is an equation satisfied by all input/output pairs corresponding to a *Fliess operator* –i.e. one described by a convergent generating series– and if E is a polynomial, then it is always realizable in the sense to be explained below.

We view our results being used as follows. The idea is very similar to that employed in the discrete case, and explored in detail in [3]. If there is reason to believe that the system producing the observed data is well-posed, then an equation E may be fit to the data. We are assured that there is then a realization of the type to be considered, and we then try to find this realization by any method. Efficient techniques for obtaining the realization are an important topic for further research, but the following example illustrates the basically constructive character of the proofs.

Consider the input/output equation

$$u\ddot{y} = y^2u^2 + \dot{y}\dot{u} \tag{3}$$

and assume that it is "well-posed" in the sense mentioned above, that is, that there is a Fliess operator $y = F_c[u]$ so that every pair $(u, F_c[u])$ satisfies the equation. Then we know, because of our main result to be given later ("recursive equation" part), that F_c can be realized by some polynomial state space system

$$\dot{x} = f(x) + g(x)u \tag{4}$$
$$y = h(x) \; . \tag{5}$$

So we have

$$\begin{aligned} \dot{y} &= L_f h(x) + L_g h(x)u \\ \ddot{y} &= L_f^2 h(x) + (L_f L_g h(x) + L_g L_f h(x))\, u + L_g^2 h(x)u^2 + L_g h(x)\dot{u}. \end{aligned}$$

Substituting $y, \dot{y}, \ddot{y}$ into equation (3) we get the following formulas:

$$L_f h = 0, \tag{6}$$
$$L_f L_g h + L_g L_f h = h^2, \tag{7}$$
$$L_g^2 h = 0. \tag{8}$$

Formulas (6) and (7) suggest that $L_f^2 h = 0$ and $L_f L_g h = h^2$. Now let

$$z_1 = h(x), \;\; z_2(x) = L_g h(x).$$

Then along any trajectory $x(t)$ of (4),

$$\begin{aligned} \dot{z}_1(t) &= L_f h(x(t)) + L_g h(x(t))u(t) = z_2(t)u(t) \\ \dot{z}_2(t) &= L_f L_g h(x(t)) + L_g^2 h(x(t))u(t) = z_1(t)^2. \end{aligned}$$

Hence, F_c can be realized by the following polynomial system

$$\begin{aligned} \dot{z}_1 &= z_2 u \\ \dot{z}_2 &= z_1^2 \\ y &= z_1 \;. \end{aligned}$$

We close this introduction by pointing out that our results provide a link with the differential-algebraic work of Fliess, who in [5] defined realizability by the requirement that outputs be differentiably dependent on inputs, in other words, that an equation such as ((1) hold. We show then that this is basically the same as realizability in the more classical sense. Yet another link is with the recent work of Willems and his school. Consider the *behavior* $w(\cdot) = (u(\cdot), y(\cdot))$ associated to an input/output description. If we write the equation as

$$E\left(w(t), w'(t), w''(t), \ldots, w^{(r)}(t)\right) \;=\; 0$$

as preferred in some of the recent system-theoretic literature (see [18]), then what we do is to relate the fact that the behavior satisfies an algebraic differential equation to realizability.

In this conference paper, we only give definitions and statements, and sketches of proofs of results.

2 Statements of Main Results

By a *Fliess operator* (or i/o mapping) we mean an operator

$$y(\cdot) \;=\; F[u(\cdot)]$$

described by a convergent generating series. (An equivalent definition can be given in terms of Volterra series with analytic kernels). This is a very general kind of i/o causal operator, and it includes a large variety of nonlinear systems; the book [7] provides an introduction to such mappings, and the basic definitions, in the way needed here, are given in [16] too.

Definition 2.1 The Fliess mapping F is *realizable* (by a singular polynomial state-space system) if there exists an integer n, polynomial vector fields $f, g_1, \ldots, g_m$ on $\mathbb{R}^n$, some $x_0 \in \mathbb{R}^n$, and two polynomial functions $q, h : \mathbb{R}^n \to \mathbb{R}$ such that the following properties hold:

1. for each i/o pair, there is some solution $x(\cdot)$ of

$$q(x(t))\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} u_i(t) g_i(x(t))$$

   such that $x(0) = x_0, \; y(t) = h(x(t))$ for all t; and

2. there holds the following *regularity* condition: For a set of $\mathcal{C}^\omega$ pairs generic in the Whitney topology, there exists some $\mathcal{C}^\omega$ solution $x(\cdot)$ as above such that for almost all t, $q(x(t)) \neq 0$. ∎

Singular systems appear naturally in control theory, for instance in robotics; see [12] for many examples. The following is our main result:

Theorem 1 *The operator F is realizable if and only if it satisfies an algebraic i/o equation (E polynomial).* ∎

An algebraic equation will be called *recursive* if it has the form

$$a\left(u(t),\ldots,u^{(k)}(t)\right) y^{(k)}(t) = b\left(u(t),\ldots,u^{(k)}(t),y(t),\ldots,y^{(k-1)}(t)\right), \tag{9}$$

as for instance the example (3). It will be shown that existence of a recursive equation implies that there is a (nonsingular) polynomial system realizing F, that is, a system as above with $q = 1$.

Obviously nonsingular systems are to be preferred, but we do not know if there is always a realization of that type (for nonrecursive equations). In any case, we have been able to prove that about every point where q vanishes there is another system, locally defined in terms of analytic functions, that realizes (locally) the desired behavior. The picture that emerges then is that, at least, one can cover the possibly singular part with local analytic realizations. In a computer simulation, this would be achieved by passing to a subroutine to deal with trajectories near this set.

3 Comments

The proofs are based on a careful analysis of the concept of *observation space*, introduced in [10] and [6], and later rediscovered by many authors. The main technical result, given in [17], relates two different definitions of this space, one in terms of smooth controls and another in terms of piecewise constant ones; these two definitions are seen to coincide. One of them immediately relates to i/o equations, while the other is related to realizability through the notion of *observation algebras* and *observation fields*. These are the analogues of the corresponding discrete-time concepts studied in [15]; for differential equations they were first employed in [1].

Among future directions in this area, we are planning to concentrate on the study of the possibility of realizability by nonsingular polynomial systems as well as by "rational systems" (those for which q never vanishes). It is very possible that under extremely weak conditions every i/o operator might be realizable by some such systems, or at worst a system which can be naturally decomposed into a finite number of components of this type. Essentially this is the case for discrete-time in [15], where it is shown that realizability is equivalent to the existence of realizations whose state spaces are (Grothendieck) schemes which can be stratified in this fashion. In the present context, the treatment is complicated by the problem of understanding the meaning of differential equations over such spaces, as well the difficulties that arise when trying to prove existence and uniqueness results for differential equations on such spaces, i.e. integrability results for vector fields defined by formal derivation operators in the corresponding algebra of functions. The work [1] (and other papers by the same author) has already given preliminary results in that direction, however, and we plan to exploit these.

In addition to single i/o maps, it is also natural to study *families of i/o maps*, defined by a *family of convergent generating series*. To study a single i/o map may seem to be natural as a formal description of a "black box", but in general, a system may induce more than one i/o map. For example, a system described by an ordinary differential equation on a manifold may induce infinitely many i/o maps, each of them corresponding to some initial state. One should study all the i/o maps induced by the system simultaneously than to study them individually, unless a fixed initial state is of particular interest. This leads to the concept of a family of i/o maps. One question arises naturally: when can a family of i/o maps be realized by *one* state space system? i.e., when can all the members of the family be realized by some singular polynomial system, each of them is associated to some initial state of the system? The result we have so far obtained is that a family of i/o maps is realizable if and only if all the members of the i/o maps satisfy an i/o equation simultaneously. Details will be explained in a forthcoming paper.

4 Some Technical Details

Let m be a fixed integer, and consider noncommuting variables $\eta_0, \ldots, \eta_m$. A *power series* in these variables is a formal expression

$$c = \langle c, \phi \rangle + \sum \langle c, \eta_\iota \rangle \eta_\iota$$

where the sum is over all possible sequences of indices

$$\iota = (i_1, \ldots, i_l),\ l \geq 0$$

with each $i_r \in \{0, \ldots, m\}$, including the empty sequence ε ($l = 0$), and where we denote

$$\eta_\iota := \eta_{i_1} \ldots \eta_{i_l},$$

and $\eta_\varepsilon := 1$. The coefficients $\langle c, \eta_\iota \rangle$ are real numbers. The set of all formal power series on $\eta_0, \ldots, \eta_m$ forms a real vector space under the coefficientwise operations

$$\langle rc_1 + c_2, \eta_i \rangle = r\langle c_1, \eta_i \rangle + \langle c_2, \eta_i \rangle.$$

We can also define the *shuffle product* for the power series in the following way: for $c = \sum \langle c, \eta_\iota \rangle \eta_\iota$ and $d = \sum \langle d, \eta_\kappa \rangle \eta_\kappa$

$$c ⧢ d = \sum \langle c, \eta_\iota \rangle \langle d, \eta_\kappa \rangle \eta_\iota ⧢ \eta_\kappa.$$

For the definition of $\eta_\iota ⧢ \eta_\kappa$, we refer the readers to [11], [9], [17]. With the operations "+" and "⧢" defined as above, the set of all power series forms a commutative $\mathbb{R}$-algebra.

We shall say that c is *convergent* if there exist $M, K > 0$ such that, for each sequence ι as above,

$$|\langle c, \eta_\iota \rangle| \ \leq\ KM^l l!.$$

For any fixed T, we let $\mathcal{U}_T$ to denote the set of all essentially bounded functions, $u :\ [0,\ T] \to \mathbb{R}^m$ *endowed with the* L_1 *topology*.

Then for each T, each $u \in \mathcal{U}_T$, and each multiindex ι as above, we define inductively the functions $V_\iota = V_\iota[u] \in C[0, T]$ by $V_\varepsilon \equiv 1$ and

$$V_{i_1,\ldots,i_{k+1}}(t) = \int_0^t u_{i_1}(\tau) V_{i_2,\ldots,i_{k+1}}(\tau) d\tau,$$

where $u_i(\tau)$ is the i-th coordinate of $u(\tau)$ for $i = 1, \ldots, m$ and $u_0(\tau) \equiv 1$. It is easy to prove that each operator

$$\mathcal{U}_T \to C[0, T] :\ u \mapsto V_\iota[u]$$

is continuous. Further, if c is a convergent series and K, M are as above, then for $T < (Mm + M)^{-1}$, the series of functions

$$F_c[u](t) = F[u](t) = \sum \langle c, \eta_\iota \rangle V_\iota(t)$$

is absolutely and uniformly convergent for all $t \in [0,T]$ and all those $u \in \mathcal{U}_T$ such that $\sup|u_i(t)| \leq 1$ for all i; see [7], chapter III, for details. Thus the operator F_c is also continuous on the subset of $\mathcal{U}_T$ satisfying this magnitude constraint.

Furthermore, c is in turn determined by F_c, in the sense that if $F_c = F_d$ for small enough T then $c = d$, or equivalently, $F_c = 0$ implies that $c = 0$. We have not been able to find a complete proof of this fact –that generating series are well-defined– in the literature. However, it can readily be proved by the following argument: if $F_c[u] = 0$ for a *piecewise constant* control u, then the derivatives of $F_c[u]$ with respect to switching times and the (constant) values of u are all zero. It can be shown that every coefficient of c is one of those derivatives with some piecewise constant control. Therefore, $F_c[u] = 0$ for all piecewise constant controls implies that $c = 0$.

If T and u are like the above and u is of class $\mathcal{C}^{k-1}$, then $y := F[u]$ is of class $\mathcal{C}^k$; we call such a pair (u, y) a $\mathcal{C}^k$ *i/o pair* associated to c. It can also be proved that y is of class $\mathcal{C}^\omega$ if u is of class $\mathcal{C}^\omega$.

We shall say that the i/o map F_c satisfies an *algebraic i/o equation* if there exist an integer k and a nonzero polynomial E such that for any $\mathcal{C}^k$ i/o pair $w = (u,\ F[u])$,

$$E\left(w(t), w'(t), \ldots, w^{(k)}(t)\right) = 0 \text{ for any } t.$$

The i/o equation is called a *recursive equation* if it can be written as in (9). where a, b are polynomials and $a \not\equiv 0$.

In realization theory, *observation spaces* play an important role. One may define the observation space in two different ways. The following is the first approach. For a power series $c = \langle c, \phi\rangle + \sum\langle c, \eta_\iota\rangle\eta_\iota$ and monomial $\alpha = \eta_\iota$ we define $\alpha^{-1}c$ by

$$\langle \alpha^{-1}c, \eta_{i_1}\ldots\eta_{i_l}\rangle = \langle c, \eta_{j_1}\ldots\eta_{j_k}\eta_{i_1}\ldots\eta_{i_l}\rangle$$

for any $\alpha = \eta_{j_1}\ldots\eta_{j_k}$.

The *observation space* $\mathcal{F}_0$ is defined to be the space spanned by the $F_{\alpha^{-1}c}$'s over $\mathbb{R}$, i.e., $\mathcal{F}_0 = \text{span}_{\,\mathbb{R}}\{F_{\alpha^{-1}c}\}_\alpha$. This space is in fact isomorphic to the space $\text{span}_{\,\mathbb{R}}\{\alpha^{-1}c\}_\alpha$ and the latter was used earlier in [16] to study realizability by bilinear systems.

The *observation algebra* $\mathcal{A}_0$ is defined to be the $\mathbb{R}$-algebra generated by $\mathcal{F}_0$, and the *observation field* $\mathcal{Q}_0$ is defined to be the quotient field of $\mathcal{A}_0$. (It can be proved that $\mathcal{A}_0$ is an integral domain, thus $\mathcal{Q}_0$ is well defined.) In fact, A_0 is isomorphic to the $\mathbb{R}$-algebra generated by all the power series of the form $\alpha^{-1}c$.

Lemma 4.1 Let c be a convergent power series. Then

(a) F_c is realizable by a singular polynomial system if $\mathcal{Q}_0$ is a finitely generated field extension of $\mathbb{R}$.

(b) F_c is realizable by a non-singular polynomial system, (i.e., a singular polynomial system in which q is identically 1) if $\mathcal{A}_0$ is a finitely generated algebra. □

Now we discuss the second type of observation space. For the i/o map F_c, we define

$$G^{\mu_0\mu_1\cdots\mu_{k-1}}[u](t) := \frac{d^k}{d\tau^k}|_{\tau=0_+}\, F_c[(u,t)(\omega,\tau)](t+\tau).$$

where $\omega(\tau) = \mu_0 + \mu_1\tau + \cdots + \mu_{k-1}\tau^{k-1}$ and $v = (u,t)(\omega,\tau)$ is the concatenated control defined by

$$v(s) = \begin{cases} u(s) & \text{for } 0 \leq s \leq t, \\ \omega(s) & \text{for } t < s \leq t+\tau. \end{cases}$$

For $k = 0$ we define $G^{\phi} = F_c$.

The (second type of) *observation space* $\mathcal{F}_1$ is defined as follows:

$$\mathcal{F}_1 := \text{span}_{\mathbb{R}}\{G^{\mu_0\mu_1\cdots\mu_{k-1}} : \ k \geq 0, \ \mu_i \in \mathbb{R}, \}.$$

The corresponding *observation algebra* $\mathcal{A}_1$ is defined to be the $\mathbb{R}$-algebra generated by $\mathcal{F}_1$ and the *observation field* $\mathcal{Q}_1$ is the quotient field of $\mathcal{A}_1$. Again, $\mathcal{Q}_1$ is well defined since it can be proved that $\mathcal{A}_1$ is an integral domain.

Lemma 4.2 Suppose that c is a convergent power series. Then

(a) F_c satisfies an algebraic i/o equation $\Longrightarrow$ $\mathcal{Q}_1$ is a finitely generated field extension of $\mathbb{R}$.

(b) F_c satisfies a recursive i/o equation $\Longrightarrow$ A_1 is a finitely generated $\mathbb{R}$-algebra. □

Lemma 4.1 shows that $\mathcal{F}_1$ is closely related to realizability of the i/o map while lemma 4.2 shows that F_2 is closely related to existence of an i/o equation. Thus the relation between $\mathcal{F}_0$ and $\mathcal{F}_1$ should give the desired relation between the realizability and the i/o equations. The following important lemma shows that the two spaces are indeed the same. This is the main result of [17].

Lemma 4.3 The two observation spaces $\mathcal{F}_0$ and $\mathcal{F}_1$ are the same. □

Applying lemma 4.1, lemma 4.2 and lemma 4.3, we obtain the conclusion that the existence of an i/o equation implies realizability. For the other direction, one can first prove that for those analytic i/o pairs $(u, \ y)$ corresponding to $q(x(t)) \not\equiv 0$, there is an i/o equation $E = 0$. Now by using the *regularity* property and continuity, one can prove that the equation is satisfied by every $\mathcal{C}^k$ i/o pair.

Finally, using lemma 4.3 and part *(b)* in both lemma 4.1 and lemma 4.2, the following conclusion can also be proved:

Theorem 2 *F_c can be realized by a non-singular polynomial system if F_c has a recursive i/o equation.* ∎

Generally, an i/o map F_c may fail to satisfy a recursive equation even if it satisfies an algebraic i/o equation. By theorem 1, we know that F_c admits a realization by a singular system Σ. If the initial state x_0 in the definition of realization satisfies $q(x_0) \neq 0$, then Σ is a local analytic realization of F_c. In the general case, we know that there is some $\mathbb{C}^{\omega}$ input ω such that for the corresponding trajectory x_{ω}, $q(x_{\omega}(t)) \neq 0$ for $0 < t < \delta$ for some small δ. Now we construct a family of generating series in the following way:

$$c^{\tau} = \sum \langle c^{\tau}, \eta_{\iota} \rangle \eta_{\iota}; \quad \langle c^{\tau}, \eta_{\iota} \rangle = F_{\eta_{\iota}^{-1}c}[\omega](\tau).$$

It can be proved that

$$F_{c^{\tau}}[u](t) = F_c[\omega^{\tau} u](t + \tau),$$

where $\omega^{\tau} u$ is the concatenated input whose value is $\omega(s)$ for $s \in [0, \tau]$ and $u(s)$ for $s > \tau$. Thus $F_{c^{\tau}}$ is realized by Σ with initial state $x_{\omega}(\tau)$. Therefore, the *Lie rank* – in the sense of Fliess, see[7], of c^{τ} is bounded by n – the dimension of the system Σ for $\tau \neq 0$. Note that c^{τ} is continuous in the sense that $\langle c^{\tau}, \eta_{\iota} \rangle \to \langle c, \eta_{\iota} \rangle$ as $\tau \to 0$ for any $\eta_{\iota} \in P^*$. Since the Lie rank is lower semicontinuous, it follows that it is bounded by n, and hence finite, also at $\tau = 0$. By the main result in [7], we obtain:

Theorem 3 *If F_c satisfies an algebriac i/o equation, then it has a local analytic realization.*
∎

5 References

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