SOME CONNECTIONS BETWEEN CHAOTIC DYNAMICAL SYSTEMS AND CONTROL SYSTEMS*

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Abstract. This paper shows how to extend recent results of Colonius and Kliemann, regarding connections between chaos and controllability, from continuous to discrete time. The extension is nontrivial because the results all rely on basic properties of the accessibility Lie algebra which fail to hold in discrete time. Thus, this paper first develops further results in nonlinear accessibility, and then shows how a theorem can be proved, which while analogous to the one given in the work by Colonius and Klieman, also exhibits some important differences. A counterexample is used to show that the theorem given in continuous time cannot be generalized in a straightforward manner.
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1 Introduction

In a recent paper, Colonius and Kliemann [3] provide an elegant connection between controlled and classical (no-input) dynamical systems. Essentially, they first associate to each finite-dimensional control system Σ an infinite-dimensional dynamical system Σ^* whose state space consists of all states and possible inputs for Σ, and then they show how to establish a one-to-one correspondence between "control subsets" of Σ and "chaotic subsets" of Σ^*. These concepts are defined more precisely below; the former are basically sets in which approximate controllability holds, and the latter are sets in which the dynamics is chaotic in the sense defined in [4].

The results in [3] are given in continuous-time, and the purpose of this note is to study their discrete-time analogues. In principle, the analogy is straightforward, and it is obvious how one should generalize the above correspondence. However, there are difficulties that have to do with controllability properties of discrete-time systems. A basic property that holds –under appropriate Lie-algebraic conditions– in the continuous case is that of (forward) accessibility: from any given state one can reach in positive time an open subset, and one can do so with respect to any fixed open set containing the state in question. In discrete-time, in contrast, this fails to be true in general, even under assumptions analogous to those made in continuous-time. As this basic property underlies most of the results in [3], it would appear at first sight that one cannot generalize those results unless one makes additional and very strong ad-hoc assumptions.

Fortunately, there is a way of applying the recent theory developed in [5] and [1] in such a manner that the above analogy can indeed be completed. In [3] a key fact employed, which follows from the above accessibility property, is that the interiors of control sets are exactly (rather than approximately) controllable. While not true here, we are able to replace his fact by the weaker property that inside each such set there exists an open dense subset, which we call its "core," in which controllability holds. The existence of such a subset is perhaps the most interesting part of our work, both because in order to establish it one must use some of the results given in the above cited papers, and also because it provides new insight into the structure of nonlinear discrete-time systems. Because of the need to work with this dense subset and various associated technical difficulties, the proofs become considerably more involved. But otherwise the arguments closely parallel those in [3], and we organized this note in such a manner that a direct comparison with that paper should be easy. A counterexample in the last section illustrates the differences.

It should be noted that our results are given only for systems that are invertible (each control induces a diffeomorphism) and analytic. The first restriction can probably be relaxed considerably, and is due to the method of proof. The second restriction is however much deeper, and it is doubtful that similar results would hold in a more general smooth situation –for instance, a central fact used is that from a dense set of points one has accessibility, but this may fail for smooth but nonanalytic systems, even under further Lie-algebraic assumptions.

2 Notations

We will consider a discrete-time system Σ as follows:

$$x(t+1) = f(x(t), u(t)) \tag{1}$$

where $x(t) \in M$ and $u(t) \in U$ for each $t = 0, 1, 2, \ldots$. We assume that:

(1) M is a Riemannian, paracompact, connected, and analytic manifold of dimension n.

(2) $U \subseteq \mathbb{R}^n$ is a compact, convex set with at least two points and so that $U \subseteq \operatorname{clos}\operatorname{int} U$.

(3) $f : M \times \tilde{U} \longrightarrow M$ (where $\tilde{U} \supseteq U$ and is open) is analytic, and denoting by $f_u : M \longrightarrow M$ the map $f(\cdot, u)$, this map is a global diffeomorphism of M for each $u \in U$ (i.e., the system Σ is invertible).

(4) The system Σ is controlled-transitive.

We will use f_u^- to denote the inverse of the map f_u. See [5] for basic background on such discrete-time systems. The property called here "controlled-transitivity" is often called simply "transitivity"; we use a different terminology because of the later use in

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this paper of the word transitivity in its dynamical systems sense. Another name for controlled-transitivity is "forward-backward accessibility". It means that every state can be reached from every other state by some finite iteration of maps of the type f_u or f_u^-. The property can be elegantly characterized in Lie-algebraic terms (see [5]). If the system Σ is not controlled-transitive, there is a decomposition of M into a disjoint union of invariant submanifolds, in such a manner that the system can be restricted to a controlled-transitive one on each of them. This reduces the analysis of general systems to the controlled-transitive case.

For each $x \in M$, each subset S of M, and each $k \geq 0$, we will use the following notations:

$R^k(x) = \{y \in M \mid \exists u_1, \ldots, u_k \in U$ such that $f_{u_k} \circ \ldots \circ f_{u_1}(x) = y\}$,

$R(x) = \bigcup_{k \geq 0} R^k(x)$, $R^k(S) = \bigcup_{s \in S} R^k(s)$,

$R(S) = \bigcup_{k \geq 0} R^k(S)$, $\bar{R}(x) = \operatorname{clos} R(x)$,

and $\bar{R}(S) = \operatorname{clos} R(S)$.

For any sequence $\omega = u_1, \ldots, u_k \in U^k$ we also denote $f_\omega^k = f_{u_k} \circ \ldots \circ f_{u_1}$. The following two facts are easy to establish:

(1) $\bar{R}(\bar{R}(S)) = \bar{R}(S) \quad \forall S \subset M$.

(2) $\bar{R}(S_1 \cup S_2) = \bar{R}(S_1) \cup \bar{R}(S_2)$.

Let $\Omega = U^{\mathbb{Z}} = \{\omega : \mathbb{Z} \longrightarrow U\}$, equipped with the pointwise convergence topology. Since U is compact, it can be shown that with this topology Ω is a compact, complete, separable metric space when the distance between two elements ω, ω' is given by:

$$d(\omega\ ,\omega') = \sum_{k \in \mathbb{Z}} 2^{-|k|} \mid \omega\ (k) - \omega'(k) \mid .$$

Depending on the context (control or dynamical systems, respectively), the letter ω will denote a finite sequence $\omega = u_1, \ldots, u_k \in U^* = \cup_{k \geq 0} U^k$ or a double infinite sequence $\omega \in \Omega$. The meaning will be clear from the context.

To the system Σ we associate a classical dynamical system Σ^* on $M \times \Omega$, where the flow Φ is given by the following: $\{\phi_k; k \in \mathbb{Z}\}$, where $\phi_k : M \times \Omega \longrightarrow M \times \Omega$, $\phi_k(x, \omega) = (f_\omega^k(x), \sigma^k \omega)$, with: $(\sigma^k \omega\ )(i) = \omega\ (i + k)$ i.e. σ represents the shift operator, and with:

$$f_\omega^k(x) = \begin{cases} f_{u_k} \circ f_{u_{k-1}} \circ \ldots \circ f_{u_1}(x) & \text{if } k > 0 \\ x & \text{if } k = 0 \\ f^-_{u_{k+1}} \circ f^-_{u_{k+2}} \circ \ldots \circ f^-_{u_0}(x) & \text{if } k < 0 \end{cases}$$

where $u_i = \omega(i)$. Observe that the notation f_ω^k is consistent with that used, in the context of control, when ω is just a finite sequence.

It can be shown that the flow Φ defines a continuous dynamical system on $M \times \Omega$, meaning that $\phi_{k+s} = \phi_k \circ \phi_s$ for all k, s, and each mapping ϕ_k is continuous on $M \times \Omega$.

Observe that $y \in R^k(x)$ if and only if $\exists\ \omega \in \Omega$ such that $f_\omega^k(x) = y$.

3 Properties of Control Sets

The next definition is a precise analogue of that in [3], except that we make the assumption of nonempty interior.

Definition 3.1 A set $D \subseteq M$ is called a *precontrol set* if $D \subseteq \bar{R}(x)$ for all $x \in D$ and $\operatorname{int} D \neq \emptyset$. A precontrol set which is maximal with respect to set inclusion is called a *control set*.

Note that if D is a precontrol set, then in D the system Σ is "almost" controllable, in the sense that if $x, y \in D$ then from x it is possible to reach any neighbourhood of y.

Next we state some properties of control sets that will be useful later. For the proofs of the following results we refer to [2].

Lemma 3.2 Let $D \subseteq M$ be a control set. Pick any two elements $\bar{x}$, $\bar{y}$ in $\operatorname{int} D$. Then, for each sequence $(u_0, \ldots, u_T) \in U^{T+1}$ such that $f_{u_T} \circ \ldots \circ f_{u_0}(\bar{x}) = \bar{y}$ we have that, necessarily, also

$f_{u_k} \circ \ldots \circ f_{u_0}(\bar{x}) \in \operatorname{int} D \quad$ for $k = 0, \ldots, T-1$. ∎

Lemma 3.3 Let $D \subseteq M$ be a precontrol set. Then we have: $D \subseteq \operatorname{clos} F_k(\operatorname{int} D)$ for all $k = 0, 1, 2, \ldots$ where $F_k(\operatorname{int} D) = \bigcup_{l \geq k} R^l(\operatorname{int} D)$ ∎

Lemma 3.4 Let $D \subseteq M$ be a control set. Then: $\operatorname{clos} D = \operatorname{clos} \operatorname{int} D$ ∎

Definition 3.5 Let $x \in M$ and $S \subseteq M$. We say that x is *forward accessible (f.a.) in* S (resp. *backward accessible (b.a.) in* S) if $\operatorname{int}(R(x) \cap S) \neq \emptyset$ (resp. $\operatorname{int}(C(x) \cap S) \neq \emptyset$). If we simply say that x is forward (backward) accessible, we mean forward (backward) accessible in M.

Lemma 3.6 Let $S \subseteq M$ and define:

$S_f = \{\ x \in M \mid x$ is forward accessible in $S\ \}$,

then S_f is open. ∎

Lemma 3.7 Let $D \subseteq M$ be a precontrol set and pick any $x \in D$. If x is forward accessible then x is forward accessible in D. ∎

We conclude the following important property of precontrol sets.

Theorem 1 *Let $D \subseteq M$ be a precontrol set. Then every point of D is forward accessible in D.* ∎

The definition of precontrol set is not reversible in time, so we cannot conclude backward accessibility from every point. However, the next result provides backward accessibility from a dense subset.

Proposition 3.8 Let $D \subseteq M$ be a control set. Then there exists some (necessarily nonempty) open subset $E \subseteq D$ such that:

(1) $\operatorname{clos} E = \operatorname{clos} D$,

(2) if $y \in E$ then y is backward accessible in D. ∎

Lemma 3.9 Let $D \subseteq M$ be a control set and let E be any set as in the conclusion of the previous Proposition. Then $E \subseteq R(x)$ for each $x \in D$. ∎

Definition 3.10 For any set $S \subseteq M$, define: $\mathrm{Core}\,(S) := \{\ x \in \mathrm{int}\, S \mid x$ is forward and backward accessible in $S\ \}$.

Using Lemma 3.6 twice (once for Σ and another time for the "inverse" system $x(t+1) = f_u^-(x(t))$), we can conclude the following.

Lemma 3.11 For any subset $S \subseteq M$, $\mathrm{Core}\,(S)$ is open. ∎

For a control set D, we proved (see results in Theorem 1 and Proposition 3.8) that $\mathrm{Core}\,(D) \supseteq E$ for some $E \subseteq D$ such that $\mathrm{clos}\, E = \mathrm{clos}\, D$. Thus we have:

$$\boxed{\mathrm{clos}\,\mathrm{Core}\,(D) = \mathrm{clos}\, D \quad \text{for a control set} \quad D.}$$

Moreover the result in Lemma 3.9 can be rephrased as follows.

Proposition 3.12 If D is a control set, and $E = \mathrm{Core}\,(D)$, then $E \subseteq R(x)$ for all $x \in D$. ∎

If D is a control set, then, by the previous results, $\mathrm{Core}\,(D)$ is a dense subset of D in which we have exact controllability. Note that (as it is implicitely used in [3]) if Σ was a continuous time system then $\mathrm{Core}\,(D)$ would have been equal to $\mathrm{int}\, D$. An example of a control set D for which $\mathrm{Core}\,(D)$ is strictly contained in $\mathrm{int}\, D$ will be given later. However the density property of $\mathrm{Core}\,(D)$ will allow us to derive analogous results of those in [3].

4 Main Results

Now we associate to each given control set D a subset D^* of the dynamical system Σ^* on the state space $M \times \Omega$. Let D be any control set; we define D^* as follows, in exact analogy to [3]:

Definition 4.1 $D^* = \mathrm{clos}\,\{(x, \omega) \in M \times \Omega \mid f_\omega^k(x) \in \mathrm{int}\, D \text{ for all } k \in \mathbb{Z}\}$

Note that this set is obviously invariant, so Φ can be restricted to D^*.

Remark 4.2 Note that, under our assumptions, D^* is certainly non-empty. In fact, choose any $\bar{x},\ \bar{y} \in \mathrm{Core}\,(D)$ with $\bar{x} \neq \bar{y}$. By Proposition 3.12 there exist input sequences $\omega^1,\ \omega^2 \in \Omega$ and integers $k_1,\ k_2$ (with $k_i \geq 1$ for $i = 1, 2$) such that: $f_{\omega^1}^{k_1}(\bar{x}) = \bar{y}$ and $f_{\omega^2}^{k_2}(\bar{y}) = \bar{x}$. Let:
$\bar{\omega} \equiv (\ldots \omega^2(1), \ldots, \omega^2(k_2), \omega^1(1), \ldots, \omega^1(k_1), \ldots)$ with $\bar{\omega}(0) = \omega^2(k_2)$.

Then the trajectory starting at $\bar{x}$ with input sequence $\bar{\omega}$ is periodic (in fact $f_{\bar{\omega}}^{k_1+k_2}(\bar{x}) = \bar{x}$), thus by Lemma 3.2 it lies entirely in $\mathrm{int}\, D$. So $(\bar{x}, \bar{\omega}) \in D^*$.

The next results will extablish a correspondence between control sets D and their "lifts" D^*. In order to make this correspondence clear, we first recall some definitions for dynamical systems; the definition of chaotic system is from [4].

Definition 4.3 Let (X, Ψ) be a discrete-time dynamical system. (X, Ψ) is called *topologically mixing* if for any two open sets V_1, V_2 of X there exist $k_1, k_2 \in \mathbb{Z}$ with $k_2 > 0$ such that, for all $l \in \mathbb{N}$,

$$\psi_{-lk_2+k_1}(V_1) \cap V_2 \neq \emptyset.$$

Definition 4.4 A discrete-time dynamical system (X, Ψ) on a metric space (X, d) is said to have *sensitive dependence on initial conditions* if there exists some $\delta > 0$ such that for each $x \in X$ and each neighbourhood W of x there are $y \in W$ and a positive integer l such that: $d(\psi_l(x), \psi_l(y)) \geq \delta$.

Definition 4.5 Let (X, Ψ) as before; (X, Ψ) is said to be *chaotic* if it is topologically mixing, has sensitive dependence on initial condition, and has a dense set of periodic points.

The result in the next Proposition and its proof are a discrete-time parallel of Proposition 3.5 in [3].

Proposition 4.6 Let D be a control set, and define $D^* \subseteq M \times \Omega$ according to Definition 4.1. Then D^* is a chaotic set, and $D^* = \mathrm{clos}\,\mathrm{int}\, D^*$.

Proof. We will show:
(1) Periodic points are dense in D^*,
(2) $\Phi|_{D^*}$ is topologically mixing,
(3) $\Phi|_{D^*}$ has sensitive dependence on initial conditions,
(4) $D^* = \mathrm{clos}\,\mathrm{int}\, D^*$.

(1) It is sufficient to show that each $(\bar{x}, \bar{\omega}) \in D^*$ with $f_{\bar{\omega}}^k(\bar{x}) \in \mathrm{int}\, D$ for all $k \in \mathbb{Z}$ can be approximated by periodic points.

Let $W = (N \times V) \cap D^*$ be a neighbourhood of $(\bar{x}, \bar{\omega})$. We may assume that N, V have the following forms: $N = \{y \in M \mid d_1(\bar{x}, y) < \epsilon\}$, $V = \{\omega \in \Omega \mid d_2(\bar{\omega}, \omega) < \epsilon\}$; where $d_1,\ d_2$ represent the distances respectively on M and on Ω. There exists $l > 0$ such that:

$$\Sigma_{k \in \mathbb{Z} \setminus \{-l, \ldots, 0, \ldots, l\}} 2^{-|k|} \leq \frac{\epsilon}{\mathrm{diam}\, U} \tag{2}$$

where $\mathrm{diam}\, U = \sup\{|u_1 - u_2|;\ u_1,\ u_2 \in U\}$. By assumption on $(\bar{x}, \bar{\omega})$ we know that $f_{\bar{\omega}}^l(\bar{x})$ and $f_{\bar{\omega}}^{-l}(\bar{x})$ are in $\mathrm{int}\, D$. Thus we can find $\tilde{\epsilon}$ ($\tilde{\epsilon} \leq \epsilon$), for which the $\tilde{\epsilon}$-neighbourhoods of $\bar{x}$, $f_{\bar{\omega}}^l(\bar{x})$ and $f_{\bar{\omega}}^{-l}(\bar{x})$ are contained in D. By continuity, there exists some δ so that:

> if $d_1(y, f_{\bar{\omega}}^{-l}(\bar{x})) < \delta$ then, denoting by $\tilde{\omega} = \sigma^{-(l+1)}(\bar{\omega})$, we have $d_1(f_{\tilde{\omega}}^{l+1}(y), \bar{x}) < \tilde{\epsilon}$ and $d_1(f_{\tilde{\omega}}^{2l+1}(y), f_{\bar{\omega}}^l(\bar{x})) < \tilde{\epsilon}$.

Let $E = \mathrm{Core}\,(D)$. Since $\mathrm{clos}\, E = \mathrm{clos}\, D$, we can choose a point y^{-l} in E whose distance from $f_{\bar{\omega}}^{-l}(\bar{x})$ is less than δ. Then, denoting $y^0 = f_{\tilde{\omega}}^{l+1}(y^{-l})$ and $y^l = f_{\tilde{\omega}}^{2l+1}(y^{-l})$, we know that y^0 and y^l lie in $\mathrm{int}\, D$. So, by Proposition 3.12, there exists $k > 0$ and $v \in \Omega$ such that $f_v^k(y^l) = y^{-l}$. Define

$$\omega_p(i) \ = \begin{cases} \bar{\omega}(i) & \text{for } i \in \{-l, \ldots, 0, \ldots, l\} \\ v(i-l) & \text{for } i \in \{l+1, \ldots, l+k\} \end{cases}$$

and extend $\omega_p(\cdot)$ by periodicity. Then (y^0, ω_p) is a periodic point of period $2l + 1 + k$. Moreover:

(i) $d_1(y^0, \bar{x}) \le \tilde{\epsilon}$, so $y^0 \in N$,

(ii) $d_2(\omega_p, \bar{\omega}) = \Sigma_{i\in\mathbb{Z}} 2^{-|i|} \mid \omega_p(i) - \omega(i) \mid =$
$= \Sigma_{i\in\mathbb{Z}\setminus\{-l,\ldots,0,\ldots,l\}} 2^{-|i|} \mid \omega_p(i) - \omega(i) \mid \le$
$\le (\operatorname{diam} U)\Sigma_{i\in\mathbb{Z}\setminus\{-l,\ldots,0,\ldots,l\}} 2^{-|i|} \le \epsilon$, so $\omega_p \in V$,

(iii) by Lemma 3.2, since y^0 and y^{-l} are in $\operatorname{int} D$, all the trajectories joining them lie interely in $\operatorname{int} D$, so $f^k_{\omega_p}(y^0) \in \operatorname{int} D$ for all $k \in \mathbb{Z}$.

Thus $(y^0, \omega_p) \in W$.

(2) We have to show that for every pair $W_1,\ W_2$ of open sets in D^* there exists $k_0 \in \mathbb{Z}$ and $k_1 > 0$ such that for all $n \in \mathbb{N}$ $\phi_{-nk_1+k_0}(W_2) \cap W_1 \neq \emptyset$.

Since, by (1), the periodic points are dense, we may assume that the sets W_i $i = 1,2$, are of the form: $W_i = (N_i \times V_i) \cap D^*$ with $V_i = \{\omega \in \Omega \mid d_2(\omega, \omega_i) < \epsilon\}$, $N_i = \{y \in M \mid d_1(y, x_i) < \epsilon\}$, where (x_i, ω_i) $i = 1,2$ are periodic points with periods l_i.

Choose $l > 0$ as in eqution 2; we may assume that $l_i > l$ for $i = 1,2$. Using the same argument as when proving (1), we can find periodic points $(y_i, v_i) \in W_i$, $i = 1,2$, with periods $L_i > l_i > l$, such that, if we denote $E = \operatorname{Core}(D) \subseteq D$, as before, then $y_i \in E$ and $v_i(k_i) = \omega_i(k_i)$ for $k_i \in \{-l_i, \ldots, 0, \ldots, l_i\}$.

Since $E \subseteq R(x)$ for all $x \in D$ (Proposition 3.12), there exist $v_0 \in \Omega$ and $L_0 \in \mathbb{N}$ such that $f^{L_0}_{v_0}(y_2) = y_1$. For any $n \ge 1$ we define $\omega_n(i)$ as follows:

$$\begin{array}{ll} v_1(i) & \text{if } i \ge -nL_1 \\ v_0(i+1+nL_1+L_0) & \text{if } -nL_1 > i \ge -nL_1 - L_0 \\ v_2(i+1+nL_1+L_0) & \text{if } -nL_1 - L_0 > i. \end{array}$$

Since $v_1(i) = \omega_1(i) = \omega_n(i)$ for $i \in \{-l, \ldots, 0, \ldots, l\}$, we have that $\omega_n \in V_1$ for all n. Futhermore, since $f^k_{\omega_n}(y_1) = f^k_{v_1}(y_1)$ for $k \ge -nL_1$ and $f^k_{\omega_n}(y_1) = f^{k+nL_1+L_0}_{v_2}(y_2)$ for $k < -nL_1 - L_0$, we can conclude $f^k_{\omega_n}(y_1) \in \operatorname{int} D$ for all $k \in \mathbb{Z}$. Thus $(y_1, \omega_n) \in W_1$. On the other hand:

(i) $f^{-nL_1+(-L_0-L_2)}_{\omega_n}(y_1) = y_2$,

(ii) $\omega_n(-nL_1 - L_0 - L_2 + k) = v_2(k - L_2 + 1) = v_2(k) = \omega_2(k)$ for $k \in \{-L_2, \ldots, 0, \ldots, L_2\}$ (thus in particular for $k \in \{-l, \ldots, 0, \ldots, l\}$).

So we can conclude $\phi_{-nL_1-L_0-L_2}(y_1, \omega_n) = (y_2, \sigma^{-nL_1-L_0-L_2}(\omega_n)) \in W_2$ since from what we have observed before, $v_2(k) = \sigma^{-nL_1-L_0-L_2}(\omega_n)(k)$ for $k \in \{-l, \ldots, 0, \ldots, l\}$, and so their distance is less than ϵ. Thus (2) holds with $k_1 = L_1$ and $k_0 = -L_0 - L_2$.

(3) If we consider the dynamical system (Ω, σ^k) for $k \in \mathbb{Z}$ it can be shown, in complete analogy to [3], proposition 2.8, that this system has sensitive dependence on initial conditions. Thus the sensitive dependence on initial conditions of $\Phi|_{D^*}$ is an immediate consequence of the same property of (Ω, σ^k), since if d is any metric on $M \times \Omega$, we have that if $d_1(\omega, v) \ge \delta$ then $d((x,\omega),(y,v)) \ge \delta$.

(4) Let $(x, \omega) \in D^*$. There exists a sequence $(x_n, \omega_n) \to (x, \omega)$ such that $f^k_{\omega_n}(x_n) \in \operatorname{int} D$ for all k. By (1) we know that the periodic points are dense in D^*. Thus for each n there exists (y_n, v_n) periodic of period l_n with $d((x_n, \omega_n),(y_n, v_n)) < 1/n$. So $(y_n, v_n) \to (x, \omega)$. Moreover we can assume that $y_n \in E$, where $E = \operatorname{Core}(D)$. From the facts that (y_n, v_n) is periodic and E is open, it is not difficult to prove that $(y_n, v_n) \in \operatorname{int} D^*$.

Thus we have $D^* \subseteq \operatorname{clos} \operatorname{int} D^*$. From this we can conclude $D^* = \operatorname{clos} \operatorname{int} D^*$, as desired. ∎

Proposition 4.7 Let $D^* \subseteq M \times \Omega$ be a chaotic set for Φ, such that $D^* = \operatorname{clos} \operatorname{int} D^*$. Define:

$$\Pi_M D^* = \{x \in M \mid \exists\ \omega \in \Omega \ \text{ with } \ (x, \omega) \in D^*\}.$$

Then there exists some precontrol set $D \subseteq M$ such that $D \subseteq \Pi_M D^*$ and $\operatorname{clos} D = \Pi_M D^*$.

Proof. Since the system Σ is analytic and controlled-transitive we know that there exist two open sets $A_1,\ A_2$ from which we have backward and forward accessibility respectively (see [1]).

Let $A = A_1 \cap A_2$, then A is still open and dense. Let $P = \{x \in \Pi_M D^* \mid \exists\ \omega \in \Omega$ for which (x, ω) is a periodic point of $D^*\}$.

Since by assumption periodic points are dense in D^*, P is dense in $\Pi_M D^*$.

Now let $E = A \cap P \cap \operatorname{int} \Pi_M D^*$. Then $E \neq \emptyset$ since $D^* = \operatorname{clos} \operatorname{int} D^*$ implies $\Pi_M D^* = \operatorname{clos} \operatorname{int} \Pi_M D^*$, so in particular $\operatorname{int} \Pi_M D^* \neq \emptyset$. Moreover, since A is open and dense and P is dense in $\Pi_M D^*$, we can conclude that: $\operatorname{clos} E = \operatorname{clos} \operatorname{int} \Pi_M D^* = \Pi_M D^*$.

Claim 1 $E \subseteq \operatorname{Core}(\Pi_M D^*)$.

Pick any $e \in E$; in particular e is forward accessible. By analyticity we know that, for all k sufficiently large, $\operatorname{clos} R^k(e) = \operatorname{clos} \operatorname{int} R^k(e)$. Futhermore, $e \in P$, thus there exist $\omega \in \Omega$ and $l > 0$ such that $f^l_\omega(e) = e$. Since l can be chosen arbitrarily large (in particular $l > k$), then $e \in \operatorname{clos} \operatorname{int} R^l(e)$. So from the fact that $e \in \operatorname{int} \Pi_M D^*$ we have that e is forward accessible in $\Pi_M D^*$. A similar argument shows that e is also backward accessible in $\Pi_M D^*$. Thus $e \in \operatorname{Core}(\Pi_M D^*)$.

Claim 2 $E \subseteq R(x)$ for all $x \in E$.

Pick $x_1, x_2 \in E$. By Claim 1, there exist two open sets $W^{x_1}_+,\ W^{x_2}_-$ contained in $\Pi_M D^*$ such that $W^{x_1}_+ \subseteq R(x_1)$ and $W^{x_2}_- \subseteq C(x_2)$. Let:

$$V_1 = (W^{x_1}_+ \times \Omega) \cap D^* \qquad V_2 = (W^{x_2}_- \times \Omega) \cap D^*$$

By the mixing property, there exists some $L > 0$ such that $\phi_{-L}(V_2) \cap V_1 \neq \emptyset$. Thus we can find $\omega \in \Omega$, $y_1 \in W^{x_1}_+$, and $y_2 \in W^{x_2}_-$ such that $f^k_\omega(y_1) = y_2$ for some $k \in \mathbb{N}$; this implies $x_1 \in R(x_2)$, as desired.

By Claim 2, in E we have exact controllability. Next we extend E to a precontrol set. Define:

$$D = \{x \in \Pi_M D^* \mid x \ \text{ is f.a. in } \ \Pi_M D^*\}.$$

Note that $E \subseteq D \subseteq \Pi_M D^*$. In order to prove that D is a precontrol set we need to show: (i) $D \subseteq \bar{R}(x)$ for all $x \in D$, and (ii) $\operatorname{int} D \neq \emptyset$.

To prove (i), pick any $x,\ y \in D$. Since from x one can reach an open subset of $\Pi_M D^*$ and E is dense in $\Pi_M D^*$, we have that from x one can reach a state $e \in E$. Moreover, using again the density of E, there exists a sequence of elements $e_n \in E$ so that $e_n \to y$. By Claim 2, we know that $e_n \in R(e)$ for all n. Therefore also $e_n \in R(x)$, which in turn implies $y \in \bar{R}(x)$, as wanted.

We now show (ii). Choose an element $e \in E$. By Claim 1, there exists an open subset W of $\Pi_M D^*$ so that $W \subseteq C(e)$. Since e is forward accessible in $\Pi_M D^*$, any element $y \in W$ is forward accessible in $\Pi_M D^*$; thus $W \subseteq D$, which implies $\operatorname{int} D \neq \emptyset$. ∎

Definition 4.8 If (X, Ψ) is a dynamical system, we say that $W \subseteq X$ is a *maximal chaotic set* if:

1. W is closed and Ψ-invariant.
2. $\Psi|_W$ is chaotic.
3. For all $W' \supset W$ with $\Psi|_{W'}$ topologically mixing we have $W' = W$.

The next theorem will establish a one-to-one correspondence between control sets of the control system Σ on M and maximal chaotic sets of the dynamical system Σ^* on $M \times \Omega$.

Theorem 2 *Let $\mathcal{C}$ be the class of all control sets $D \subseteq M$, and let $\mathcal{C}^*$ be the class of all maximal chaotic sets D^* for Φ such that $D^* = \operatorname{clos} \operatorname{int} D^*$. Define $\alpha : \mathcal{C} \to \mathcal{C}^*$ and $\beta : \mathcal{C}^* \to \mathcal{C}$ as follows:*

$$\alpha(D) = \operatorname{clos} \{(x, \omega) \in M \times \Omega \mid f^k_\omega(x) \in \operatorname{int} D \ \forall k \in \mathbb{Z}\}$$

$$\beta(D^*) = \{x \in \Pi_M D^* \mid x \text{ is f. a. in } \Pi_M D^*\}.$$

Then α and β give a one-to-one correspondence between $\mathcal{C}$ and $\mathcal{C}^*$ with $\alpha \circ \beta = id_{\mathcal{C}^*}$ and $\beta \circ \alpha = id_{\mathcal{C}}$.

Furthermore if either $D^* = \alpha(D)$ or $D = \beta(D^*)$, we have:

1. $\operatorname{int} D \subseteq \operatorname{int} \Pi_M D^*$,
2. $\operatorname{clos} D = \Pi_M D^*$.

Proof. We divide the proof in 3 steps. In part (a) we prove that α is well defined and satisfies (1) and (2), in (b) we show the same properties for β, and in (c) we prove that α and β are inverse maps.

Recall that if (X, Ψ) is a discrete-time dynamical system, and $x \in X$, the *ω-limit set* of x is

$$\omega(x) = \{y \in X \mid \exists\, n_k \to \infty \text{ for which } \ \psi_{n_k}(x) \to y\}.$$

Moreover, the system (X, Ψ) is called *transitive* if there exists some $x \in X$ for which $\omega(x) = X$. It is a consequence of Proposition I.11.4 of [6] that, for the system Σ^*, topologically mixing implies transitivity. Since chaotic sets are topologically mixing, we are allowed, in what follows, to use the fact that chaotic sets are transitive.

(a) In proposition 4.6 we have already proved that $\alpha(D)$ is a chaotic set such that $\alpha(D) = \operatorname{clos} \operatorname{int} \alpha(D)$. Thus in order to see that α is well defined (i.e. $\alpha(D) \in \mathcal{C}^*$), we need to establish maximality of $\alpha(D)$.

Suppose that there exists some $D' \supseteq \alpha(D)$ with D' topologically mixing. Then since D' is also transitive there exists $(x, \omega) \in D'$ such that the ω-limit set of (x, ω) is all of D'. Let $E = \operatorname{Core}(D)$ and choose $e \in E$; then there exists a sequence k_n, with $k_n \to \infty$, such that $f^{k_n}_\omega(x) \to e$. So, since E is open, for $\bar{n}$ sufficiently large $f^{k_{\bar{n}}}_\omega(x) = \tilde{e} \in E$.

By maximality of D and the fact that $f^{k_n}_\omega(x)$ belongs to $\operatorname{int} D$ for infinitely many $n \geq \bar{n}$, it follows that $f^k_\omega(x) \in \operatorname{int} D$ for all $k \geq k_{\bar{n}}$. Since $\tilde{e} \in E$, we can also find $\tilde{\omega}$ so that $f^k_{\tilde{\omega}}(\tilde{e}) \in \operatorname{int} D$ for all $k \leq k_{\bar{n}}$. Define:

$$\bar{\omega}(i) = \begin{cases} \tilde{\omega}(i) & \text{for } i < k_{\bar{n}} \\ \omega(i + k_{\bar{n}}) & \text{for } i \geq k_{\bar{n}}. \end{cases}$$

Then $(\tilde{e}, \bar{\omega}) \in \alpha(D)$, since $f^k_{\bar{\omega}}(\tilde{e}) \in \operatorname{int} D$ for all $k \in \mathbb{Z}$.

Moreover, the ω-limit set of $(\tilde{e}, \bar{\omega})$ is equal to the ω-limit set of (x, ω), which is D'. Thus, since $\alpha(D)$ is closed, $D' \subseteq \alpha(D)$. So $D' = \alpha(D)$, from which we have maximality.

Next we prove that (1) and (2) hold for D and $\alpha(D)$.

(1) Suppose that $x \in \operatorname{int} D$, and take $\epsilon > 0$ so that the open ball B_ϵ centered at x is contained in D. Let $y \in B_\epsilon \cap E$. Then, as shown in the proof of part (4) of proposition 4.6, there exists $\omega \in \Omega$ so that $(y, \omega) \in \operatorname{int} \alpha(D)$.

So we have $B_\epsilon \cap E \subseteq \Pi_M \alpha(D)$. Since $\Pi_M \alpha(D)$ is closed and $\operatorname{clos} E = \operatorname{clos} D$, we have: $\operatorname{clos} B_\epsilon \subseteq \Pi_M \alpha(D)$, which implies $x \in \operatorname{int} \Pi_M \alpha(D)$.

(2) Since $E \subseteq \Pi_M \alpha(D)$, $\Pi_M \alpha(D)$ is closed, and $\operatorname{clos} E = \operatorname{clos} D$, we have $\operatorname{clos} D \subseteq \Pi_M \alpha(D)$.

Conversely, take $x \in \Pi_M \alpha(D)$. Then there exists ω such that $(x, \omega) \in \Pi_M \alpha(D)$. This in turn implies that there exists $(x_n, \omega_n) \to (x, \omega)$ with $x_n \in \operatorname{int} D$, by definition of $\alpha(D)$. Thus $x \in \operatorname{clos} D$.

(b) By proposition 4.7 we know that $\beta(D^*)$ is a precontrol set. So, also in this case, to see that β is well defined we need to prove maximality of $\beta(D^*)$. Suppose that $\beta(D^*)$ is not maximal, and let: $\tilde{D} = \cup_\lambda D_\lambda$ where D_λ is a precontrol set such that $D_\lambda \supset \beta(D^*)$.

Since $\operatorname{int} \beta(D^*) \neq \emptyset$, it is easy to see that $\tilde{D}$ is a control set. If we show that $\tilde{D} \subseteq \Pi_M D^*$, then we can conclude that $\tilde{D} = \beta(D^*)$ since, by Theorem 1, we know that all $x \in \tilde{D}$ must be forward accessible in $\tilde{D}$. Since $\tilde{D}$ is a control set, we can apply α to $\tilde{D}$.

Let $E^* = \{(x, \omega) \in \operatorname{int} D^* \mid (x, \omega)$ is a periodic point and x is forward accessible $\}$; then E^* is dense in D^*, since the set of periodic points is dense. Moreover if $(x, \omega) \in E^*$ then $x \in \beta(D^*)$; thus $x \in \tilde{D}$ and, futhermore, $(x, \omega) \in \alpha(\tilde{D})$.

So $E^* \subseteq \alpha(\tilde{D})$, since $\alpha(\tilde{D})$ is closed, $D^* = \operatorname{clos} E^* \subseteq \alpha(\tilde{D})$. By the maximality of D^*, we have $D^* = \alpha(\tilde{D})$.

By part (a) we know that: $\tilde{D} \subseteq \Pi_M \alpha(\tilde{D}) = \Pi_M D^*$. Thus $\beta(D^*)$ is maximal, and so it is a control set.

Next we prove (1) and (2) for D^* and $\beta(D^*)$. Using proposition 4.7 we know that $\operatorname{clos} \beta(D^*) = \Pi_M D^*$. Thus (2) holds. To prove (1), it is sufficient to notice that $\Pi_M E^* \subseteq \operatorname{int} \beta(D^*)$, and now, using the same arguments as in the proof of (1) of part (a), we can conclude $\operatorname{int} \beta(D^*) \subseteq \operatorname{int} \Pi_M D^*$.

(c) Let D be a control set. We need to prove $\beta(\alpha(D)) = D$. By property (1) we know that $\operatorname{clos} D = \Pi_M(\alpha(D)) = \operatorname{clos} \beta(\alpha(D))$, thus the equality follows by the maximality of both D and $\beta(\alpha(D))$.

Conversely, let D^* be a maximal chaotic set. Again choosing E^* as in part (b), E^* is a dense subset of D^* which is contained in $\alpha(\beta(D^*))$. Thus, by closeness and maximality of D^* and $\alpha(\beta(D^*))$, we can conclude $D^* = \alpha(\beta(D^*))$, as needed. ∎

The result of the previous Theorem is the discrete-time version of the result given in Theorem 3.9 of [3]. The main difference is that in the continuous time case the inclusion (1) is always an equality, while in the discrete time case it can be a proper inclusion, as it is shown in the example given in the next section.

5 An Example

Here we provide an example of a system for which $\operatorname{int} D \neq \operatorname{int} \Pi_M(\alpha(D))$, and D is a control set so that $\operatorname{Core}(D) \neq \operatorname{int} D$.

Example 5.1 Consider the function $g(x) = \frac{\sin(\pi x)}{\pi x}$. It is easy to verify that $|g'(x)| \leq 1$ for all $x \in \mathbb{R}$. Moreover, $g(x) = 0$ if and only if $x \in \mathbb{Z} \setminus \{0\}$. Now let us consider the following discrete-time, analytic system: $M = \mathbb{R}^2$, $U = [-1, 1]^2$, and equations

$$\begin{aligned} x^+ &= x + 1 + uy \\ y^+ &= y + \frac{v}{2} g(x) \end{aligned}$$

where $g(x)$ is the above function.

This system is invertible. In fact the determinant of the Jacobian matrix of the map $f_{u,v}(x, y)$ is given by: $1 - \frac{uv}{2} g'(x)$, which is never zero since $u, v \in [-1, 1]$, and $|g'(x)| \leq 1$. Moreover it is easy to verify that for each $(u, v) \in U$, the map $f_{u,v}(\cdot, \cdot)$ is bijective. It is also easy to prove that this system is transitive.

For this system we can see that for all $k \in \mathbb{N}$ with $k \geq 1$ the following hold:

1. the points of the type $(-k, 0)$ are not backward accessible,
2. the points of the type $(k, 0)$ are not forward accessible.

Letting $B = \{(k, 0) \mid k \in \mathbb{N},\ k \geq 1\}$, we want to show that $D = \mathbb{R}^2 \setminus B$ is a control set.

Notice that D is certainly maximal; in fact, no points in B could belong to a control set, since they are not forward accessible. To prove that D satisfies:

$$D \subseteq \bar{R}(\xi) \quad \text{for all } \ \xi \in D \tag{3}$$

we will prove the following:

$$\mathbb{R}^2 \setminus \{\ (k, y) \mid k \in \mathbb{Z},\ y \in \mathbb{R}\ \} \subseteq R(\xi) \tag{4}$$

which, by taking the closure in both sides, implies 3. Let $F = \{\ (k, y) \mid k \in \mathbb{Z},\ y \in \mathbb{R}\ \}$.

First we notice that, since $|\sin(\pi(x+1))| = |\sin(\pi x)|$, if we apply to any (x, y) a control sequence of the form:

$$u_l = 0, \quad v_l = \ \text{sign}\ (g(x + l - 1)), \tag{5}$$

then, after k steps, we will reach the point:

$$\begin{aligned} x_k &= x + k \\ y_k &= y + \frac{|\sin(\pi x)|}{2\pi} \sum_{l=0}^{k-1} \frac{1}{|x + l|}. \end{aligned}$$

Using this fact and the divergence of the series $\sum_n 1/n$ we will prove 4.

Fix $(\bar{x}, \bar{y}) \in D$ and $(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 \setminus F$. Notice that, since $(\bar{x}, \bar{y}) \notin B$, it is not restrictive to assume:

$$g(\bar{x}) \neq 0 \ \text{ and } \ \bar{y} \neq 0.$$

First we choose u_l, v_l as in 5. Since $g(\bar{x}) \neq 0$ there exists k such that $y_k > 1$. Next we apply a control sequence with all $v_l = 0$ so as to reach a state (x', y') of the type:

$$x' = \tilde{x} - n \quad \text{and} \quad y' = y_k$$

where n is a positive integer that will be chosen later. Notice that we can assume $\tilde{y} < y'$.

Now we want to find a sequence of controls $(0, v_l)$ such that we get the state $(\tilde{x}, \tilde{y})$ in exactly n steps. It is clear that this is possible if and only if:

$$y' - \frac{|\sin(\pi \tilde{x})|}{2\pi} \sum_{l=0}^{n-1} \frac{1}{|\tilde{x} - n + l|} \leq \tilde{y} \tag{6}$$

So we just have to choose n large enough such that 6 is satisfied. This is possible since $\sin(\pi \tilde{x}) \neq 0$ and:

$$\sum_{l=0}^{n-1} \frac{1}{|\tilde{x} - n + l|} = \sum_{m=1}^{n} \frac{1}{|\tilde{x} - m|}$$

is divergent. Thus D is a control set.

Notice that, for this control set D, $\operatorname{Core}(D)$ is strictly contained in $D = \operatorname{int} D$. In fact, none of the points of the type $(-k, 0)$ with k a strictly positive integer, belongs to $\operatorname{Core}(D)$.

Now if we consider the corresponding maximal chaotic set $\alpha(D)$, it is clear that $\alpha(D) = M \times \Omega$. Thus, in this case, the inclusion (1) in Theorem 2 is proper (i.e. $\operatorname{int} D \neq \operatorname{int} \Pi_M \alpha(D)$).

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