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Gradient Techniques for Systems With no Drift: A Classical Idea Revisited*

Eduardo Sontag
Dept. of Mathematics
Rutgers University
New Brunswick, NJ 08903

Abstract

This paper proposes a technique for the control of analytic systems with no drift. It is based on the generation of "nonsingular loops" which allow linearized controllability. Once such loops are available, it is possible to employ standard Newton or steepest descent methods. The theoretical justification of the approach relies on recent results on genericity of nonsingular controls as well as a simple convergence lemma.

1. Introduction

This paper deals with the problem of numerically finding controls that achieve a desired state transfer. That is, for any given initial and target states ξ_0 and ξ_F in $\mathbb{R}^n$, one wishes to find a time $T > 0$ and a control u defined on the interval $[0, T]$, so that u steers ξ_0 to ξ_F, for the system

$$\dot{x} = f(x, u) . \tag{1}$$

More precisely, the question of approximate controllability (for any $\varepsilon > 0$, find a control that brings the state to within ε distance of ξ_F) will be considered.

A number of preliminary results will be developed for general analytic systems of the type (1), but the controllability application is restricted to the case of systems *without drift*:

$$\dot{x} = G(x)u , \tag{2}$$

i.e., the right-hand side $f(x, u)$ is linear in u. For such systems it is relatively straightforward to decide controllability, but the design of explicit control strategies has attracted considerable attention lately.

Problems of steering systems without drift are in part motivated by the study of nonholonomic mechanical systems. Many sophisticated control strategies have been proposed, based on a nontrivial analysis of the structure of the Lie algebra of vector fields generated by the columns of G; see the various papers in this session. The approach presented in this paper is of an entirely different nature. It represents a simple-minded algorithm, in the style of classical numerical approaches, and it requires no symbolic computation to implement. In fact, a small piece of code in any numerical package such as MATLAB is all that is needed in order to obtain solutions. Obviously, as with any general procedure, it can be expected to be extremely inefficient, and to result in poor performance, when compared with techniques that use nontrivial information about the system being controlled. Perhaps it will be useful mainly in conjunction with other techniques, to provide a first step of global control, to be followed by finer local control.

Mathematically, the main contribution of this paper is in the formulation of the "generic loop" approach and the justification of the algorithm. The latter relies on a new result proving the existence of such loops with good controllability properties. This approach was motivated to a great extent by related work on time varying feedback laws by Coron and others; the last section of the extended version of this paper, [8], makes some remarks regarding connections with that work.

1.1. Classical Iterative Techniques

It is assumed from now on that in (1) the states $x(t)$ evolve in $\mathbb{R}^n$. (Systems on manifolds can also be considered, but doing so only complicates notations and adds in this case little insight.) Controls $u(t)$ take values in $\mathbb{R}^m$. Further, f is continuously differentiable (later results will impose analyticity). Given a state $\xi_0 \in \mathbb{R}^n$ and a measurable and locally essentially bounded control $u : [0, T] \to \mathbb{R}^m$ so that the solution $x : [0, T] \to \mathbb{R}^n$ of the equation (1) with this control and the initial condition $x(0) = \xi_0$ is defined on the entire interval $[0, T]$ —that is, u is *admissible for* x,— the state $x(t)$ at time $t \in [0, T]$ is denoted by $\phi(t, \xi_0, u)$. As discussed above, the objective, for any given initial and target states ξ_0 and ξ_F in $\mathbb{R}^n$, is to find a time $T > 0$ and a control u defined on the interval $[0, T]$, so that u steers ξ_0 to ξ_F, that is, so that $\phi(T, \xi_0, u) = \xi_F$, at least in an approximate sense. After a change of coordinates, one may assume without loss of generality that $\xi_F = 0$.

Classical numerical techniques for this problem are based on variations of steepest descent; see for instance [1], or [2] for a recent reference. The basic idea is to start with a guess of a control, say $\overline{u} : [0, T] \to \mathbb{R}^m$, and to improve iteratively on this initial guess. More precisely, let $\overline{x} = \phi(\cdot, \xi_0, \overline{u})$. If the obtained final state $\overline{x}(T)$ is already zero, or is sufficiently near zero, the problem has been solved. Otherwise, we look for a perturbation $\Delta\overline{u}$ so that the new control $\overline{u} + \Delta\overline{u}$ brings us closer to our goal of steering ξ_0 to the origin. The various techniques differ on the choice of the perturbation; in particular, two possibilities are discussed next, later to be analized.

The first is basically Newton's method, and proceeds

E-mail: *sontag@hilbert.rutgers.edu*

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as follows. Denote, for any fixed initial state ξ_0, $\alpha(u) := \phi(T, \xi_0, u)$ thought of as a partially defined map from $\mathcal{L}^m_\infty(0,T)$ into $\mathbb{R}^n$. This is a continuously differentiable map (see e.g. [6], Theorem 1), so expanding to first order there results

$$\alpha(\overline{u}+v) \;=\; \alpha(\overline{u}) + \alpha_*[\overline{u}](v) + o(v)$$

for any other control v so that $\alpha(\overline{u}+v)$ is defined, where we use "$*$" as a subscript to denote differentials. If we can now pick v so that

$$\alpha_*[\overline{u}](v) \;=\; -\alpha(\overline{u}) \tag{3}$$

then for small enough $h > 0$ real,

$$\alpha(\overline{u}+hv) \;=\; (1-h)\alpha(\overline{u}) + o(h) \tag{4}$$

will be smaller than the state $\alpha(\overline{u})$ reached with the initial guess control $\overline{u}$. In other words, the choice of perturbation is $\Delta\overline{u} := hv$, $0 < h \ll 1$.

It remains to solve equation (3) for v. The operator

$$L \,:\, v \mapsto \alpha_*[\overline{u}](v) \tag{5}$$

is the one corresponding to the solution of the variational equation

$$\dot{z} \;=\; A(t)z + B(t)v \quad z(0) = 0, \tag{6}$$

where $A(t) := \frac{\partial f}{\partial x}(\overline{x}(t), \overline{u}(t))$ and $B(t) := \frac{\partial f}{\partial u}(\overline{x}(t), \overline{u}(t))$ for each t, that is, $Lv \;=\; \int_0^T \Phi(T,s)B(s)v(s)\,ds$, where Φ denotes the fundamental solution associated to $\dot{X} = A(t)X$.

The operator L maps $\mathcal{L}^m_\infty(0,T)$ into $\mathbb{R}^n$, and it is onto when (6) is a controllable linear system on the interval $[0,T]$, that is, when $\overline{u}$ is a control *nonsingular for* ξ_0 relative to the system (1). In other words, ontoness of $L = \alpha_*[\overline{u}]$ is equivalent to first-order controllability of the original nonlinear system along the trajectory corresponding to the initial state ξ_0 and the control $\overline{u}$. The main point of this paper will lie in showing that it is not difficult to generate useful nonsingular controls for systems with no drift.

Assuming nonsingularity, there exist then many solutions to (3). Because of its use in (4) where a small v is desirable, and in any case because it is the most natural choice, it is reasonable to pick the least squares solution, that is the unique solution of minimum norm,

$$v \;:=\; -L^{\#}\alpha(\overline{u}) \tag{7}$$

where $L^{\#}$ denotes the pseudoinverse operator (see e.g. [6], Section 3.5, for details; we are using the canonical inner product on $\mathbb{R}^n$, and L_2 norm in $\mathcal{L}^m_\infty(0,T)$, and induced norms for elements and operators).

The technique sketched above is well-known in numerical control. For instance, the derivation in pages 222-223 of [1], when applied to solving the optimal control problem having the trivial cost criterion $J(u) = 0$ and subject to the final state constraints $x = \psi(x) = 0$, results in formula (7), and is derived in the same manner as here.

Alternatively, instead of solving (3) for v via (7), one might use the steepest descent choice

$$v \;:=\; -L^*\alpha(\overline{u}) \tag{8}$$

where L^* is the adjoint of L. Formula (8) also results from the above derivation in [1], now when applied using the quadratic cost $J(u) = \|\alpha(u)\|^2$ but relaxing the terminal constraints ($\psi \equiv 0$). In place of (4), now one has

$$\alpha(\overline{u}+hv) \;=\; (I - hLL^*)\alpha(\overline{u}) + o(h), \tag{9}$$

where I is the identity operator. If again L is onto, that is, if the control $\overline{u}$ is nonsingular for ξ_0, then the symmetric operator LL^* is positive definite, so $0 < h \ll 1$ will give a contraction as earlier. A possible advantage of using L^* instead of $L^{\#}$ is that no matrix inversion is required in this case. On the other hand, one may expect Newton's method to behave better locally and steepest descent to be more effective globally.

It is also possible to combine these techniques with line searches over the scalar parameter h or, even more efficiently in practice, with conjugate gradient approches (see for instance [4]). Line search corresponds to leaving v fixed and optimizing on the step size h, only recomputing a variation v when no further improvement on h can be found. (The control applied at this stage is then the one for the "best" stepsize, not the intermediate ones calculated during the search.)

Of course, in general there are many reasons for which the above classical techniques may fail to be useful in a given application: the initial guess $\overline{u}$ may be singular for ξ_0, the iteration may fail to converge, and so forth. The main point of this paper is to show that, for a suitable class of systems, a procedure along the above lines can be guaranteed to work. The systems with which we will deal here are often called "systems without drift" and are those expressed as in Equation (2). A result given below shows that for such systems (assuming analytic G) rather arbitrary controls provide the desired singularity, and can hence be used as the basis of the approach sketched above.

The next section establishes the basic iterative procedure and proves a convergence result assuming that nonsingular controls exist. After that, we explain the application to systems without drift and state the existence theorem for nonsingular controls in the analytic case (a proof is cited).

2. Justification of the Iterative Method

We now prove the convergence of the algorithm consisting of repeatedly applying a control to obtain a nonsingular trajectory, and at each step perturbing this control by means of a linear technique. As a preliminary step, we establish a few results in somewhat more generality; these are fairly obvious remarks about iterative methods, but we have not found them in the literature in the form needed here.

Lemma 2.1 Let $\mathcal{B}$ be a compact subset of $\mathbb{R}^n$, and let $H > 0$. Assume given $F : \mathcal{B} \times [0, H] \to \mathbb{R}^n$ and a continuous matrix function $D : \mathcal{B} \to \mathbb{R}^{n\times n}$ so that $D(x)$ is symmetric and positive definite for each x. Assume further that the function $g(x,h) := F(x,h) + hD(x)x - x$ is $o(h)$ uniformly on x, that is, for each $\varepsilon > 0$ there is a $\delta > 0$ so that

$$h < \delta \ \Rightarrow\ \|g(x,h)\| < \varepsilon h \quad \text{for all } x \in \mathcal{B}\,. \tag{10}$$

Then the following conclusion holds, for some constant $\lambda > 0$: For each $\varepsilon > 0$ there is some $\delta > 0$ so that, for each $h \in (0,\delta)$ and each $x \in \mathcal{B}$,

$$\|F(x,h)\| \ < \ \max\{(1-\lambda h)\|x\|, \varepsilon\}\,. \tag{11}$$

Proof. Note that since $D(x)$ is continuous on x, its singular values also depend continuously on x (see e.g. [6], Corollary A.4.4). Let $2\lambda > 0$ be a lower bound and let $\overline{\lambda}$ be an upper bound for the eigenvalues of $D(x)$. Pick a $k > 2$ so that $k\lambda > 2\overline{\lambda}$.

Now fix any $\varepsilon > 0$. There is then some $0 < \delta < 1/\overline{\lambda}$ such that, for each $0 < h < \delta$,

$$\|g(x,h)\| \ < \ \frac{\overline{\lambda}\varepsilon h}{k} \ < \ \frac{\varepsilon}{k} \tag{12}$$

for all $x \in \mathcal{B}$ and all the eigenvalues of $hD(x)$ are in the interval $(0,1)$.

Pick any $h \in (0,\delta)$ and any $x \in \mathcal{B}$. As the eigenvalues of the symmetric matrix $I - hD(x)$ are all again in $(0,1)$, this matrix must be positive definite and so its norm equals its largest eigenvalue; thus: $\|I - hD(x)\| \leq 1 - 2\lambda h$. Therefore, for $\|x\| > \varepsilon/2$ it holds that:

$$\begin{aligned}\|F(x,h)\| &\leq \|(I - hD(x))x\| + \|g(x,h)\| \\ &\leq (1-2\lambda h)\|x\| + \overline{\lambda}\varepsilon h/k \\ &= \left(1 - 2\lambda h + \frac{\overline{\lambda}\varepsilon h}{k\|x\|}\right)\|x\| \\ &< (1-\lambda h)\|x\|\,,\end{aligned}$$

which implies the desired conclusion. If instead $\|x\| < \varepsilon/2$, then

$$\|F(x,h)\| \leq \|I - hD(x)\|\|x\| + \|g(x,h)\| < \varepsilon/2 + \varepsilon/k < \varepsilon$$

so the conclusion holds in that case as well. ∎

Observe that continuity of $D(x)$ is only used in guaranteeing that the singular values are bounded above and away from zero.

Lemma 2.2 Let $\mathcal{B}$ be a closed ball in $\mathbb{R}^n$, and let $H > 0$. Assume given a map $F : \mathcal{B} \times [0,H] \to \mathbb{R}^n$, with $F(x,0) = x$ for all x, so that F is continuously differentiable with respect to $h \in [0,H]$, with $\partial F/\partial h$ continuous on (x,h), and $\partial F/\partial h(x,0) = -D(x)x$, where $D : \mathcal{B} \to \mathbb{R}^{n\times n}$ is a continuous matrix function satisfying that $D(x)$ is symmetric positive definite for each x. Denote $F_h := F(\cdot, h)$. Then the following property holds: For each $\varepsilon > 0$, there is some $\delta > 0$ so that, for each $0 < h < \delta$ there is some positive integer $N = N(h)$ so that

$$\|F_h^N(\mathcal{B})\| \ < \ \varepsilon\,,$$

where F_h^N denotes the Nth iterate of F_h.

Proof. In order to apply Lemma 2.1, we only need to check that in the expansion $F(x,h) = x - hD(x)x + g(x,h)$ the last term is $o(h)$ uniformly on x. But, by Lagrange's formula, one has $g(x,h) = F(x,h) - F(x,0) - \frac{\partial F}{\partial h}(x,0)\,h = \int_0^1 G(x,h,t)h\,dt$ where

$$G(x,h,t) \ := \ \frac{\partial F}{\partial h}(x,th) \ - \ \frac{\partial F}{\partial h}(x,0)$$

and $\frac{\partial F}{\partial h}(x,h)$ is continuous by hypothesis. On the compact set $\mathcal{B} \times [0,H]$, this function is uniformly continuous; in particular it is so at the points of the form $(x,0)$. Thus for each $\varepsilon > 0$ there is some $\delta > 0$ so that whenever $h < \delta$ then $\|G(x,h,t)\| < \varepsilon$ for all $x \in \mathcal{B}$. Therefore also $\|g(x,h)\| < \varepsilon h$ holds, and Lemma 2.1 can indeed be applied.

As $\mathcal{B}$ is a ball, the iterates remain in $\mathcal{B}$. So for each N:

$$\|F_h^N\| \ < \ \max\{(1-\lambda h)^N\|x\|, \varepsilon\}\,.$$

This gives the desired result. ∎

For each $\xi \in \mathbb{R}^n$ and each control $\overline{u} \in \mathcal{L}_\infty^m(0,T)$ admissible for ξ, we let $L_{\xi,\overline{u}}$ be the linear operator $\mathcal{L}_\infty^m(0,T) \to \mathbb{R}^n$ defined as in (5), that is, the reachability map for the time-varying linear system (6) that results along the ensuing trajectory. Introducing the matrix functions

$$A = A(x,u) = \frac{\partial f}{\partial x}(x,u) \text{ and } B = B(x,u) = \frac{\partial f}{\partial u}(x,u),$$

we may consider the following new system (the "prolongation" of the original one):

$$\begin{aligned}\dot{x} &= f(x,u) &(13)\\ \dot{z} &= A(x,u)z + B(x,u)v &(14)\end{aligned}$$

seen as a system of dimension $2n$ and control (u,v) of dimension $2m$. Observe that $L_{\xi,\overline{u}}(\overline{v})$ is the value of the z-coordinate of the solution that results at time T when applying controls $\overline{u}, \overline{v}$ and starting at the initial state $(\xi, 0)$. If we add the equation

$$\dot{Q} = AQ + QA + BB^* \tag{15}$$

(superscript $*$ indicates transpose) to the prolonged system, the solution with the above controls and initial state $(\xi, 0, 0)$ has

$$Q(t) \ = \ \int_0^t \Phi(t,s)B(s)B^*(s)\Phi(t,s)^*\,ds$$

so that (see e.g. [6], Section 3.5) ontoness of $L_{\xi,\overline{u}}$ is equivalent to the Grammian $W = Q(T)$ being positive definite. Note that, by continuous dependence on initial conditions and controls, W depends continuously on $\xi, \overline{u}$. Similar arguments show that other objects associated to the linearization also depend continuously on $\xi, \overline{u}$, and any state q: application to q of the adjoint, $L_{\xi,\overline{u}}q$, which is the same as the function $B(t)^*\Phi(T,t)^*q$, and of the pseudoinverse, $L_x^{\#}q = L^*Wq$.

Fix now a control $\overline{u}$ and a closed ball $\mathcal{B} \subseteq \mathbb{R}^n$ so that $\overline{u}$ is admissible for all $\xi \in \mathcal{B}$, and denote $L_{\xi,\overline{u}}$ just as L_ξ.

Corollary 2.3 Assume that the control $\overline{u}$ is so that

$$\phi(T,\xi,\overline{u}) = \xi \quad \text{for all } \xi \in \mathcal{B} .$$

Assume given, for each $\xi \in \mathcal{B}$, a map $N_\xi : \mathbb{R}^n \to \mathcal{L}_\infty^m(0,T)$ so that $N_\xi(\xi)$ depends continuously on ξ and so that the operator $D(\xi) := L_\xi N_\xi$ is linear, and in the standard basis is symmetric and depends continuously on ξ. Pick an $H > 0$ so that $\overline{u} - hN_\xi(\xi)$ is admissible for each $\xi \in \mathcal{B}$ and $h \in [0,H]$, and let $F(\xi,h) := \phi(T,\xi,\overline{u} - hN_\xi(\xi))$. Then, for each $\varepsilon > 0$, there is some $\delta > 0$ so that, for each $0 < h < \delta$ there is some positive integer $N = N(h)$ so that $\|F_h^N(\mathcal{B})\| < \varepsilon$, where $F_h := F(\cdot,h)$.

Proof. Observe that, since $\frac{\partial\phi(T,\xi,u)}{\partial u}$ is the same as L_ξ, we have that, in general,

$$\left.\frac{\partial\phi\left(T,\xi,\overline{u} - hN_\xi(q)\right)}{\partial h}\right|_{h=0} = -D(\xi)q ,$$

so in particular $\frac{\partial F}{\partial h}(\xi,0) = -D(\xi)\xi$, as needed in order to apply Lemma 2.2. Note that $\frac{\partial F}{\partial h}(\xi,h)$ is continuous, as it equals $-L_{\xi,\overline{u}-hN_\xi(x)}N_\xi(x)$ and each of L and N are continuous on all arguments. ∎

3. Application to Systems with no Drift

The application to systems without drift, those that are as in Equation (2), is as follows. As discussed in the next subsection, rescaling if necessary, we may assume that the system is complete. In order to apply the numerical techniques just developed, one needs to find a control $\overline{u}$ which leads to *nonsingular loops*:

- $\overline{u}$ is nonsingular for every state x in a given ball $\mathcal{B}$,
- $\phi(T,x,\overline{u}) = x$ for all such x.

It is shown later that for analytic systems that have the strong accesibility property, controls which are generic –in a sense to be made precise– are nonsingular for all states. (For analytic systems without drift, Chow's Theorem states that the strong accessibility property is equivalent to complete controllability.) Starting from such a control ω, defined on an interval $[0,T/2]$, one may now consider the control $\overline{u}$ on $[0,T]$ which equals ω on $[0,T/2]$ and is then followed by the antisymmetric extension:

$$\overline{u}(t) = -\omega(T-t)\,, \ \ t \in (T/2,T]\,. \qquad (16)$$

This $\overline{u}$ is as needed: nonsingularity is due to the fact that if the restriction of a control to an initial subinterval is nonsingular for the initial state, the whole control is, and the loop property is an easy consequence of the special form (2) in which the control appears linearly.

In practice, one might try using a randomization technique in order to obtain ω, and from there $\overline{u}$. More directly, one might use instead a finite Fourier series with random coefficients:

$$\overline{u}(t) = \sum_{k=1}^{l} a_k \sin 2k\pi t\,, \qquad (17)$$

which automatically satisfies the antisymmetry property (16) on the time interval $[0,1]$. Of course, there is no theoretical guarantee that such a series will provide nonsingularity, for any given finite l; the study of Lie-algebraic conditions that insure it would be of interest. But experimentally, one may always proceed assuming that indeed all properties hold.

The first application is with $N_x = L_x^{\#}$, the pseudoinverse discussed earlier. Here $D(x) = I$ is certainly positive definite and continuous on x.

The second application is with $N_x = L_x^*$, the adjoint operator, in which case $D(x) = W = Q(T)$, as obtained for the composite system (13)-(15), and as remarked earlier this is also continuous on x (and positive definite for each x, by nonsingularity).

To summarize the procedure: First find an $\overline{u}$ that generates nonsingular loops, in the above sense. Now calculate the effect of applying this control, starting at ξ_0, and compute the linearization along the corresponding trajectory, using this in turn in order to obtain the variation that allows modifying $\overline{u}$ by $hN_x(x)$, as described earlier. The original control $\overline{u}$ is not applied to the system, but the perturbed one is. Apply this new control to the system and compute the final state that results. If the state is not close enough to ξ_F, repeat. There is then guaranteed convergence in finite time to any arbitrary neighborhood of the origin, for small enough stepsize. One may also combine this approach with line searches, or even conjugate gradient algorithms, as discussed earlier.

Such techniques are classical in nonlinear control; see for instance [1], [4]. What appears to be new is the observation that, for analytic systems without drift, generic loops provide nonsingularity.

3.1. Rescaling: Obstacles and Completeness

For systems with no drift, a simple rescaling of the equations may be an extremely powerful tool that allows (a) dealing with workspace obstacles and (b) the reduction to systems that are complete (no explosion times). The basic idea is very simple, and is as follows.

Assume that $\beta : \mathbb{R}^n \to \mathbb{R}$ is any smooth mapping, and consider the new system without drift

$$\dot{x} = \beta(x)G(x)u\,. \qquad (18)$$

Suppose that one has found a control u, defined on an interval $[0, T]$, so that the state ξ_0 is transferred into the state ξ_F using this control, for the system (18). Let $\overline{x}(\cdot)$ be the corresponding trajectory. Then, the new control $v(t) := \beta(\overline{x}(t))u(t)$, when applied to the original system (2), also produces the desired transfer. In other words, solving a controllability problem for (18) provides immediately a solution to the corresponding problem for the original system. (If one is interested in feedback design, as opposed to open-loop control as in this paper, the same situation holds: a feedback law $u = k(x)$ for (18) can be re-interpreted as a feedback law $u = \beta(x)k(x)$ for (2).) If β never vanishes, the controllability properties of the original and the transformed systems are the same. This is clear from the above argument.

This construction is of interest in two ways. First of all, one is often interested in control of systems in such a manner that trajectories avoid a certain subset Q of the state-space (which may correspond to "obstacles" in the workspace of a robot, for instance). If β vanishes exactly on Q, then control design on the complement of Q can be done for the new system (18), and controls can then be reinterpreted in terms of the original system, a discussed above. Since β vanishes on Q, no trajectories starting outside Q ever pass through Q (uniqueness of solutions). Of course, in planning motions in the presence of obstacles, the control variations should be chosen so as to move in state space directions which do not lead to collisions. One possible approach is to first design a polyhedral path to be tracked, and then to apply the numerical technique explained in order to closely follow this path.

Reparameterization also helps in dealing with possible explosion times in the original system, a fact that had been previously observed in [3], page 2542. In this case, one might use an $\beta(x)$ so that $\beta(x)G(x)$ has all entries bounded; for instance, $\beta(x)$ could be the chosen as $(1 + \sum_{i,j} g_{ij}^2(x))^{-1}$. This means that the new system has no finite escape times, for any bounded control.

3.2. Some Implementation Questions

Next are derived explicit formulas for the use of the above technique, in the case of systems without drift and when steepest descent variations are used. As just discussed, one may assume that the system is complete.

Assume that $\overline{u}(t), t \in [0, T]$ satisfies the antisymmetry condition

$$\overline{u}(T - t) = -\overline{u}(t). \qquad (19)$$

If $x(\cdot)$ satisfies $\dot{x} = G(x)\overline{u}$ then $z(t) := x(T - t)$ satisfies the same equation; thus from the equality $z(T/2) = x(T/2)$ and uniqueness of solutions it follows that $z = x$. In other words,

$$x(T - t) = x(t) \qquad (20)$$

for $t \in [0, T]$. To distinguish the objects which depend explicitely on time from those that depend on the current values of states and controls, use the notation

$$\mathcal{A}(x, u) := \sum_{i=1}^{m} \frac{\partial g_i}{\partial x}(x)u_i$$

where g_i is the ith column of G, u_i is the ith entry of the vector $u \in \mathbb{R}^m$, and the partial with respect to x indicates Jacobian. Note that $\mathcal{A}$ can be calcuted once and for all as a function of the variables x, u, before any numerical computations take place. For each $\overline{u}$, and the trajectory $x(\cdot)$ corresponding to this control and initial state ξ_0, denote

$$A(t) := \mathcal{A}(x(t), \overline{u}(t)), \quad B(t) := G(x(t)).$$

Note that if (19), and hence also (20), hold then

$$A(T - t) = -A(t), \quad B(T - t) = B(t) \qquad (21)$$

hold as well. Consider next $\Psi(t) := \Phi(T, t)$, where Φ is the fundamental solution as before, corresponding to a given $\overline{u}$ and $x(\cdot)$ as above. Thus, Ψ satisfies the matrix differential equation

$$\dot{\Psi}(t) = -\Psi(t)A(t), \quad \Psi(T) = I.$$

Consider the function $\widetilde{\Psi}(t) := \Psi(T - t)$. If $\overline{u}$ satisfies the antisymmetry condition, then $\widetilde{\Psi}$ satisfies the same differential equation as Ψ, from which the equality $\widetilde{\Psi}(T/2) = \Psi(T/2)$ implies $\widetilde{\Psi} = \Psi$. Hence also

$$\Psi(T - t) = \Psi(t) \qquad (22)$$

and so $\Psi(0) = \Psi(T) = I$. The perturbed control to be applied is $\overline{u} + hv = \overline{u} - hL^*\alpha(\overline{u})$ where $\alpha(\overline{u}) = x(T) = x(0) = \xi_0$ if $\overline{u}$ satisfies the antisymmetry condition. The adjoint operator is $(L^*\xi_0)(t) = B(t)^*\Psi(t)^*\xi_0$. Summarizing, the control to be applied, which for small h should result in a state closer to the origin than ξ_0, is

$$\boxed{\overline{u}(t) - hG(x(t))^*\Psi(t)^*\xi_0} \quad t \in [0, T]$$

where

$$\begin{aligned} \dot{x}(t) &= G(x(t))\,\overline{u}(t), & x(0) &= \xi_0 \\ \dot{\Psi}(t) &= -\mathcal{A}(x(t), \overline{u}(t))\,\Psi(t), & \Psi(0) &= I. \end{aligned}$$

The equations for the system evolution are as follows (the state variable is now denoted by z in order to avoid confusion with the reference trajectory x):

$$\dot{z}(t) = G(z(t))\,[\overline{u}(t) - hG(x(t))^*\Psi(t)^*\xi_0]$$

for $t \in [0, T]$, with initial condition $z(0) = \xi_0$. In a line-search implementation, one would first compute $z(T)$ for various choices of h; the control is only applied once that an optimal h has been found. Then the procedure can be repeated, using $z(T)$ as the new initial state ξ_0.

Remark. Regarding the number of steps that are needed in order to converge to an ε-neighborhood of the desired target state, an estimate is as follows. For a fixed ball around the origin, and sufficient smoothness, one can see that $h = O(\varepsilon)$ provides the inequality in (10), as required for (12). Thus, the number of iterations N needed, using such a stepsize, is obtained from (11): $(1-c\varepsilon)^N < \varepsilon$ where c is a constant. Taking logarithms and using $\log(1-x) = x + o(x)$ there results the rough estimate $N = O\left(\frac{1}{\varepsilon}\log\left(\frac{1}{\varepsilon}\right)\right)$.

4. Universal Inputs

In this Section, the systems considered will be of the type (1) where $x(t) \in \mathcal{X}$, $u(t) \in \mathcal{U}$, and:

- $\mathcal{X} \subseteq \mathbb{R}^n$ is open and connected, for some $n \geq 1$;
- $\mathcal{U} \subseteq \mathbb{R}^m$ is open and connected, for some $m \geq 1$;
- $f : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$ is real-analytic.

A *control* is a measurable essentially bounded map $\omega : [0,T] \to \mathcal{U}$; it is said to be *smooth* (respectively, *analytic*) if it is infinitely differentiable (respectively, real-analytic) as a function of $t \in [0,T]$. As before, we denote by $\phi(t,x,\omega)$ the solution of (1) at time t with initial condition x and control ω. This is defined for all small $t = t(x,\omega) > 0$; when we write $\phi(\cdot,x,\omega)$, we mean the solution as defined on the largest interval $[0,\tau)$ of existence.

Given a state x, a control ω defined on $[0,T]$, and a positive $T_0 \leq T$ so that $\xi(t) = \phi(t,x,\omega)$ is defined for all $t \in [0,T_0]$, we may consider the *linearization along the trajectory* (ξ,ω):

$$\dot{z}(t) \;=\; A(t)z(t) + B(t)u(t) \qquad (23)$$

where $A(t) := \frac{\partial f}{\partial x}(\xi(t),\omega(t))$ and $B(t) := \frac{\partial f}{\partial u}(\xi(t),\omega(t))$ for each t. A control ω will be said to be *nonsingular for* x if the linear time-varying system (23) is controllable on the interval $[0,T_0]$, for some $T_0 > 0$. When u is analytic, this property is independent of the particular T_0 chosen, and it is equivalent to a Kalman-like rank condition (see e.g. [6], Corollary 3.5.17). Nonsingularity is equivalent to a Fréchet derivative of $\phi(T_0,x,\cdot)$ having full rank at ω.

If ω is nonsingular for $x \in \mathcal{X}$, and T_0 is as above, then the rechability set in time T_0 from x has a nonempty interior. This is a trivial consequence of the Implicit Function Theorem (see for instance [6], Theorem 6). Thus, if for each state x there is some control which is nonsingular for x, then (1) is strongly accessible. The converse of this fact is also true, that is, if a system is strongly accessible then for each state x there is some control which is nonsingular for x. This converse fact was already known. The main purpose here is to point out that ω can be chosen *independently* of the particular x, and moreover, a generic ω has this property. We now give a precise statement of these facts.

A control $\omega : [0,T] \to \mathcal{U}$ will be said to be a *universal nonsingular control* for the system (1) if it is nonsingular for every $x \in \mathcal{X}$.

Theorem 1 *If (1) is strongly accessible, there is an analytic universal nonsingular control.*

Let $\mathcal{C}^\infty([0,T],\mathcal{U})$ denote the set of smooth controls $\omega : [0,T] \to \mathcal{U}$, endowed with the $\mathcal{C}^\infty$ topology (uniform convergence of all derivatives). A *generic* subset of $\mathcal{C}^\infty([0,T],\mathcal{U})$ is one that contains a countable intersection of open dense sets.

Theorem 2 *If (1) is strongly accessible, the set of smooth universal nonsingular controls is generic in* $\mathcal{C}^\infty([0,T],\mathcal{U})$*, for any* $T > 0$.

The proof is given [7], and is heavily based on Sussmann's universal input theorem, see [9]. This in turn generalized a weaker result in [5], which would have given only Theorem 1 for compact $\mathcal{X}$; see also this special case in [10], Lemma 4.10).

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