A GENERAL APPROACH TO PATH PLANNING FOR SYSTEMS WITHOUT DRIFT

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Abstract. This paper proposes a generally applicable technique for the control of analytic systems with no drift. The method is based on the generation of "nonsingular loops" that allow linearized controllability. One can then implement Newton and/or gradient searches in the search for a control. A general convergence theorem is proved.

Key words. steering of nonholonomic systems, nonsingular controls, mechanical systems, nonlinear control, nonlinear feedback

1. Introduction. This paper concerns itself with the (approximate) path-planning problem for systems *without drift*:

$$\dot{x} = G(x)u , \tag{1.1}$$

where G is analytic. That is, one wishes to find, for any given initial and target states ξ_0 and ξ_F in $\mathbb{R}^n$, a time $T > 0$ and a control u defined on the interval $[0,T]$, so that u steers ξ_0 to ξ_F (or close to it), for the above system. This is a question that has attracted much interest, being motivated in part by the study of nonholonomic mechanical systems, and powerful techniques have been developed for this purpose (see e.g. [1], [10], [11], and [8]). Our approach, based on a "transversality" theorem which ensures that certain rich classes of controls exist (and are in a sense generic) requires no special structure on the controllability Lie algebra of the system, and can be implemented in principle with little effort.

Some of the intermediate results will be valid for more general analytic systems

$$\dot{x} = f(x,u) , \tag{1.2}$$

(but the main results are for the special case shown above). We assume that (1.2) describes a system with Euclidean state space, that is, states $x(t)$ evolve in $\mathbb{R}^n$. Controls $u(\cdot)$ are $\mathbb{R}^m$-valued measurable and essentially bounded, f is continuously differentiable (later we assume analyticity). In general, given a state $\xi_0 \in \mathbb{R}^n$ and a control $u : [0,T] \to \mathbb{R}^m$ so that the solution $x : [0,T] \to \mathbb{R}^n$ of the equation (1.2) with this control and the initial condition $x(0) = \xi_0$ is defined on the entire interval $[0,T]$,, we say that u is *admissible for x* and denote the state $x(t)$ at time $t \in [0,T]$ as $\phi(t, \xi_0, u)$.

This paper presents the "generic loop" approach and establishes a general convergence theorem. The latter relies on a new result proving the

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existence of such loops with good controllability properties. This approach was largely motivated by related work on time varying feedback laws by [4] and [12]. As with any general procedure, ours may be expected to perform poorly in comparison with techniques that make use of structural information about the system being controlled. One possible application might be in the design of large control actions, bringing the system into regions of the state space where the asumptions required for the more refined techniques hold.

1.1. A review of gradient and Newton methods. Recall that we wish, for any given initial and target states ξ_0 and ξ_F in $\mathbb{R}^n$, to find a time $T > 0$ and a control u defined on the interval $[0, T]$, so that u steers ξ_0 to ξ_F, that is, so that $\phi(T, \xi_0, u) = \xi_F$, at least in an approximate sense. After a change of coordinates, we will assume without loss of generality that $\xi_F = 0$. Numerical techniques for this problem (see e.g. the classic reference [3]) start with a guess of a control, let us say $\overline{u} : [0, T] \to \mathbb{R}^m$, and iteratively improve upon this initial candidate. That is, with the notation

$$\overline{x} = \phi(\cdot, \xi_0, \overline{u}),$$

if the obtained final state $\overline{x}(T)$ is already zero, or is sufficiently near zero, the problem has been solved; otherwise, we search for a perturbation $\Delta\overline{u}$ so that the new control $\overline{u} + \Delta\overline{u}$ brings us closer to the origin.

One obtains different algorithms depending on the choice of the perturbation. The two most classical ones are as Newton and gradient descent. Newton's method proceeds as follows. For any fixed initial state ξ_0, we let

$$\alpha(u) := \phi(T, \xi_0, u).$$

This is understood as a partially defined map from $\mathcal{L}_\infty^m(0, T)$ into $\mathbb{R}^n$, which is continuously differentiable (see e.g. [17], Theorem 1). Expanding to first order, we have:

$$\alpha(\overline{u} + v) = \alpha(\overline{u}) + \alpha_*[\overline{u}](v) + o(v)$$

for any control v for which $\alpha(\overline{u} + v)$ is defined. (We use "$*$" as a subscript to denote differentials.) Assume that we are able to pick v so that

$$\alpha_*[\overline{u}](v) = -\alpha(\overline{u}). \tag{1.3}$$

It then follows that, for small enough $h > 0$, (with the perturbation $\Delta\overline{u} := hv$, $0 < h \ll 1$),

$$\alpha(\overline{u} + hv) = (1 - h)\alpha(\overline{u}) + o(h) \tag{1.4}$$

will be smaller than the state $\alpha(\overline{u})$ reached with the initial guessed $\overline{u}$.

One must then solve equation (1.3) for v. The operator

$$L : v \mapsto \alpha_*[\overline{u}](v) \tag{1.5}$$

provides the solution of the variational equation

$$\dot{z} = A(t)z + B(t)v \quad z(0) = 0, \tag{1.6}$$

where

$$A(t) := \frac{\partial f}{\partial x}(\overline{x}(t), \overline{u}(t))$$

and

$$B(t) := \frac{\partial f}{\partial u}(\overline{x}(t), \overline{u}(t))$$

for each t, that is,

$$Lv = \int_0^T \Phi(T, s)B(s)v(s)\, ds\ ,$$

where Φ denotes the fundamental solution associated to $\dot{X} = A(t)X$. The operator $L : \mathcal{L}_\infty^m(0,T) \to \mathbb{R}^n$ is onto when (1.6) is a controllable linear system on the interval $[0,T]$. In that cse, we say that $\overline{u}$ is a control *non-singular for* ξ_0 relative to the system (1.2). That is, ontoness of $L = \alpha_*[\overline{u}]$ is equivalent to first-order controllability of the original nonlinear system along the trajectory corresponding to the initial state ξ_0 and the control $\overline{u}$. One of the main points of this paper lies in showing that it is not difficult to generate useful nonsingular controls for systems with no drift.

There is in general more than one solution to (1.3). Because of its use in (1.4) where a small v is desirable, and in any case because it is the most natural choice, it is reasonable to pick the least squares solution, that is the unique solution of minimum norm,

$$v := -L^{\#}\alpha(\overline{u}) \tag{1.7}$$

where $L^{\#}$ denotes the pseudoinverse operator (see e.g. [17], Section 3.5, for details; we are using the canonical inner product on $\mathbb{R}^n$, and L_2 norm in $\mathcal{L}_\infty^m(0,T)$, and induced norms for elements and operators). This technique is well-known in numerical control; for instance, the derivation in pages 222-223 of [3], when applied to solving the optimal control problem having the trivial cost criterion $J(u) = 0$ and subject to the final state constraints $x = \psi(x) = 0$, results in formula (1.7), and is derived in the same manner as here.

Instead of solving (1.3) for v via (1.7), one might instead use the steepest descent choice

$$v := -L^{*}\alpha(\overline{u}) \tag{1.8}$$

where L^* is the adjoint of L. Formula (1.8) also results from the above derivation in [3], now when applied using the quadratic cost $J(u) = ||\alpha(u)||^2$

but relaxing the terminal constraints ($\psi \equiv 0$). In place of (1.4), now one has

$$\alpha(\overline{u} + hv) = (I - hLL^*)\alpha(\overline{u}) + o(h), \tag{1.9}$$

where I is the identity operator. If again L is onto, that is, if the control $\overline{u}$ is nonsingular for ξ_0, then the symmetric operator LL^* is positive definite, so $0 < h \ll 1$ will give a contraction as earlier. An advantage in using L^* instead of $L^\#$ is that no matrix inversion is required in this case.

One may also combine these techniques with line searches over the scalar parameter h or, even more efficiently in practice, with conjugate gradient approches (see for instance [9]). Line search corresponds to leaving v fixed and optimizing on the step size h, only recomputing a variation v when no further improvement on h can be found. (The control applied at this stage is then the one for the "best" stepsize, not the intermediate ones calculated during the search.)

There are many reasons for which the above classical techniques may in principle fail: the initial guess $\overline{u}$ may be singular for ξ_0, the iteration may fail to converge, and so forth. The main point of this paper is to show that, for a suitable class of systems, a procedure along the above lines can be guaranteed to work. The systems with which we will deal here are often called "systems without drift" and are those expressed as in Equation (1.1). A result given below shows that for such systems (assuming analytic G) rather arbitrary controls provide the desired nonsingularity, and can hence be used as the basis of the approach sketched above.

The next section describes the basic iterative procedure and proves a convergence result assuming that nonsingular controls exist. After that, we state the existence theorem for nonsingular controls in the analytic case (a proof is given in an Appendix), and explain the application to systems without drift. Several remarks are also provided in the last section, and relationships to time-varying feedback design are briefly discussed.

2. Justification of the iterative method. Here we prove the convergence of the algorithm consisting of repeatedly applying a control to obtain a nonsingular trajectory, at each step perturbing this control by means of a linear technique. As a preliminary step, we establish a few results in somewhat more generality.

LEMMA 2.1. *Let $\mathcal{B}$ be a compact subset of $\mathbb{R}^n$, and let $H > 0$. Assume given*

$$F : \mathcal{B} \times [0, H] \to \mathbb{R}^n$$

and a continuous matrix function

$$D : \mathcal{B} \to \mathbb{R}^{n \times n}$$

so that $D(x)$ is symmetric and positive definite for each x. Assume further that the function

$$g(x,h) := F(x,h) + hD(x)x - x$$

is $o(h)$ uniformly on x, that is, for each $\varepsilon > 0$ there is a $\delta > 0$ so that

$$h < \delta \Rightarrow \|g(x,h)\| < \varepsilon h \quad \text{for all } x \in \mathcal{B} . \tag{2.1}$$

Then the following conclusion holds, for some constant $\lambda > 0$: For each $\varepsilon > 0$ there is some $\delta > 0$ so that, for each $h \in (0,\delta)$ and each $x \in \mathcal{B}$,

$$\|F(x,h)\| < \max\{(1-\lambda h)\|x\|, \varepsilon\} . \tag{2.2}$$

Proof. As $D(x)$ is continuous on x, its singular values also depend continuously on x (see e.g. [17], Corollary A.4.4). Let $2\lambda > 0$ be a lower bound and let $\overline{\lambda}$ be an upper bound for the eigenvalues of $D(x)$. Pick a $k > 2$ so that $k\lambda > 2\overline{\lambda}$. Next fix any $\varepsilon > 0$. Thus there is a $0 < \delta < 1/\overline{\lambda}$ such that, for each $0 < h < \delta$,

$$\|g(x,h)\| < \frac{\overline{\lambda}\varepsilon h}{k} < \frac{\varepsilon}{k} \tag{2.3}$$

for all $x \in \mathcal{B}$ and all the eigenvalues of $hD(x)$ are in the interval $(0,1)$.

Pick any $h \in (0,\delta)$ and any $x \in \mathcal{B}$. Since the eigenvalues of the symmetric matrix $I - hD(x)$ are all again in $(0,1)$, this matrix must be positive definite, so its norm equals its largest eigenvalue. Hence:

$$\|I - hD(x)\| \leq 1 - 2\lambda h .$$

It follows that, for $\|x\| > \varepsilon/2$:

$$\begin{aligned} \|F(x,h)\| &\leq \|(I - hD(x))x\| + \|g(x,h)\| \\ &\leq (1-2\lambda h)\|x\| + \overline{\lambda}\varepsilon h/k \\ &= \left(1 - 2\lambda h + \frac{\overline{\lambda}\varepsilon h}{k\|x\|}\right)\|x\| \\ &< (1-\lambda h)\|x\| , \end{aligned}$$

which implies the desired conclusion. If instead $\|x\| < \varepsilon/2$, then

$$\|F(x,h)\| \leq \|I - hD(x)\|\|x\| + \|g(x,h)\| < \varepsilon/2 + \varepsilon/k < \varepsilon ,$$

so the conclusion holds in that case as well. □

LEMMA 2.2. *Let $\mathcal{B}$ be a closed ball in $\mathbb{R}^n$, centered at the origin, and let $H > 0$. Assume given a map*

$$F : \mathcal{B} \times [0,H] \to \mathbb{R}^n ,$$

with $F(x,0) = x$ for all x, so that F is continuously differentiable with respect to $h \in [0,H]$, with $\frac{\partial F}{\partial h}$ continuous on (x,h), and

$$\frac{\partial F}{\partial h}(x,0) = -D(x)x ,$$

where $D : \mathcal{B} \to \mathbb{R}^{n\times n}$ is a continuous matrix function satisfying that $D(x)$ is symmetric positive definite for each x. Denote $F_h := F(\cdot,h)$. Then the following property holds: For each $\varepsilon > 0$, there is some $\delta > 0$ so that, for each $0 < h < \delta$ there is some positive integer $N = N(h)$ so that

$$\|F_h^N(\mathcal{B})\| < \varepsilon ,$$

where F_h^N denotes the Nth iterate of F_h.

Proof. We wish to use Lemma 2.1. For that purpose, we must verify that in the expansion $F(x,h) = x - hD(x)x + g(x,h)$ the last term is $o(h)$ uniformly on x. But (Lagrange formula):

$$g(x,h) = F(x,h) - F(x,0) - \frac{\partial F}{\partial h}(x,0)\,h = \int_0^1 G(x,h,t)h\,dt$$

where

$$G(x,h,t) := \frac{\partial F}{\partial h}(x,th) - \frac{\partial F}{\partial h}(x,0)$$

and $\frac{\partial F}{\partial h}(x,h)$ is continuous by hypothesis. On the compact set $\mathcal{B} \times [0,H]$, this function is uniformly continuous; in particular it is so at the points of the form $(x,0)$. Thus for each $\varepsilon > 0$ there is some $\delta > 0$ so that whenever $h < \delta$ then $\|G(x,h,t)\| < \varepsilon$ for all $x \in \mathcal{B}$ and all $t \in [0,1]$. Therefore also $\|g(x,h)\| < \varepsilon h$ holds, and Lemma 2.1 can indeed be applied.

Since $\mathcal{B}$ is a ball, the iterates remain in $\mathcal{B}$. So, for each l and each $x \in \mathcal{B}$,

$$\|F_h^l(x)\| < \max\{(1-\lambda h)^l\|x\|, \varepsilon\} .$$

This gives the desired result. □

For each $\xi \in \mathbb{R}^n$ and each control $\overline{u} \in \mathcal{L}_\infty^m(0,T)$ admissible for ξ, we let $L_{\xi,\overline{u}}$ be the linear operator $\mathcal{L}_\infty^m(0,T) \to \mathbb{R}^n$ defined as in (1.5), that is, the reachability map for the time-varying linear system (1.6) that results along the ensuing trajectory. Introducing the matrix functions

$$A = A(x,u) = \frac{\partial f}{\partial x}(x,u) \quad \text{and} \quad B = B(x,u) = \frac{\partial f}{\partial u}(x,u) ,$$

we may consider the following new system (the "prolongation" of the original one):

$$\dot{x} = f(x,u) \tag{2.4}$$

$$\dot{z} = A(x,u)z + B(x,u)v \tag{2.5}$$

seen as a system of dimension $2n$ and control (u,v) of dimension $2m$. Observe that $L_{\xi,\overline{u}}(\overline{v})$ is the value of the z-coordinate of the solution that results at time T when applying controls $\overline{u},\overline{v}$ and starting at the initial state $(\xi,0)$. If we add the equation

$$\dot{Q} = AQ + QA + BB^* \tag{2.6}$$

(superscript $*$ indicates transpose) to the prolonged system, the solution with the above controls and initial state $(\xi,0,0)$ has

$$Q(t) \;=\; \int_0^t \Phi(t,s)B(s)B^*(s)\Phi(t,s)^*\,ds$$

so that (see e.g. [17], Section 3.5) ontoness of $L_{\xi,\overline{u}}$ is equivalent to the Grammian $W = Q(T)$ being positive definite. Note that, by continuous dependence on initial conditions and controls, W depends continuously on $\xi,\overline{u}$.

Similar arguments show that other objects associated to the linearization also depend continuously on $\xi,\overline{u}$, and any state q: application to q of the adjoint, $L^*_{\xi,\overline{u}}q$, which is the same as the function $B(t)^*\Phi(T,t)^*q$, and of the pseudoinverse,

$$L^{\#}_{\xi,\overline{u}}q = L^*W^{-1}q\,.$$

Choose now a control $\overline{u}$ and a closed ball $\mathcal{B} \subseteq \mathbb{R}^n$ so that $\overline{u}$ is admissible for all $\xi \in \mathcal{B}$, and denote $L_{\xi,\overline{u}}$ just as L_ξ. (This is the zero-initial-state reachability map of the linearized system when applying $\overline{u}$ and starting at the state ξ; thus for each ξ, L_ξ is a map from controls into states of the linearized system.) In the next result, the map N_ξ plays the role of a one-sided "approximate inverse" of L_ξ (for each state ξ, N_ξ is a map from states into controls).

COROLLARY 2.1. *Assume that the control $\overline{u}$ is so that*

$$\phi(T,\xi,\overline{u}) \;=\; \xi \quad \text{for all } \xi \in \mathcal{B}\,.$$

Assume given, for each $\xi \in \mathcal{B}$, a map $N_\xi : \mathbb{R}^n \to \mathcal{L}^m_\infty(0,T)$ so that $N_\xi(\xi)$ depends continuously on ξ and so that the operator

$$D(\xi) \;:= L_\xi N_\xi$$

is linear, and in the standard basis is symmetric positive definite and depends continuously on ξ. Pick an $H > 0$ so that $\overline{u} - hN_\xi(\xi)$ is admissible for each $\xi \in \mathcal{B}$ and $h \in [0,H]$, and let

$$F(\xi,h) \;:=\; \phi(T,\xi,\overline{u} - hN_\xi(\xi))\,.$$

Then, for each $\varepsilon > 0$, there is some $\delta > 0$ so that, for each $0 < h < \delta$ there is some positive integer $N = N(h)$ so that

$$\|F_h^N(\mathcal{B})\| \;<\; \varepsilon\,,$$

where $F_h := F(\cdot, h)$.

Proof. Since $\left.\frac{\partial\phi(T,\xi,u)}{\partial u}\right|_{u=\overline{u}}$ is the same as L_ξ, we have that, in general,

$$\left.\frac{\partial\phi\,(T,\xi,\overline{u}-hN_\xi(q))}{\partial h}\right|_{h=0} \;=\; -D(\xi)q\,,$$

so in particular $\frac{\partial F}{\partial h}(\xi,0) = -D(\xi)\xi$, as needed in order to apply Lemma 2.2. Note that $\frac{\partial F}{\partial h}(\xi,h)$ is continuous, as it equals

$$-L_{\xi,\overline{u}-hN_\xi(\xi)}N_\xi(\xi)$$

and each of L and N are continuous on all arguments. □

A $H > 0$ as needed in the statement always exists, by continuity of solutions on initial conditions and controls.

3. Application to case of systems with no drift. We now specialize to the vase of systems without drift (1.1). Rescaling if necessary, we may assume that the system is complete (see next section). To apply the numerical techniques just developed, one needs to find a control $\overline{u}$ which leads to *nonsingular loops*:

- $\overline{u}$ is nonsingular for every state x in a given ball $\mathcal{B}$, and
- $\phi(T,x,\overline{u}) = x$ for all such x.

It is shown later that for analytic systems that have the strong accessibility property, controls which are generic –in a sense to be made precise– are nonsingular for all states. (For analytic systems without drift, Chow's Theorem states that the strong accessibility property is equivalent to complete controllability.) Starting from such a control ω, defined on an interval $[0,T/2]$, one may now consider the control $\overline{u}$ on $[0,T]$ which equals ω on $[0,T/2]$ and is then followed by the antisymmetric extension:

$$\overline{u}(t) = -\omega(T-t)\,,\quad t\in(T/2,T]\,. \tag{3.1}$$

This $\overline{u}$ is as needed: nonsingularity is due to the fact that if the restriction of a control to an initial subinterval is nonsingular for the initial state, the whole control is, and the loop property is an easy consequence of the special form (1.1) in which the control appears linearly.

In practice, one might try using a randomization technique in order to obtain ω, and from there $\overline{u}$. More directly, one might use instead a finite Fourier series with random coefficients:

$$\overline{u}(t) \;=\; \sum_{k=1}^{l} a_k \sin kt\,, \tag{3.2}$$

which automatically satisfies the antisymmetry property (3.1) on the time interval $[0,2\pi]$. There is no theoretical guarantee that such a series will provide nonsingularity, but in any case, experimentally, one may always proceed assuming that indeed all properties hold.

The first application is with $N_x = L_x^\#$, the pseudoinverse discussed earlier. Here $D(x) = I$ is certainly positive definite and continuous on x. The second application is with $N_x = L_x^*$, the adjoint operator, in which case $D(x) = W = Q(T)$, as obtained for the composite system (2.4)-(2.6), and as remarked earlier this is also continuous on x (and positive definite for each x, by nonsingularity).

We may summarize the procedure as follows. The objective is to transfer ξ_0 to a neighborhood of ξ_F.

Step 1. Find an $\overline{u}$ that generates nonsingular loops, in the above sense. Let $\xi := \xi_0$.

Step 2. Calculate the effect of applying $\overline{u}$, starting at ξ, and compute the linearization along the corresponding trajectory, using this in turn in order to obtain the variation that allows modifying $\overline{u}$ by $hN_\xi(\xi)$, as described earlier.

Step 3. The original control $\overline{u}$ is *not* applied to the system (from state ξ), but the perturbed one is. Apply this new control to the system and compute the final state ξ' that results.

Step 4. If ξ' is not close enough to ξ_F, let $\xi := \xi'$, and go to Step 2.

Thus one has a guaranteed convergence in finite time to any arbitrary neighborhood of the origin, for small enough stepsize. One might also combine this approach with line searches, or even conjugate gradient algorithms, as discussed earlier. Such techniques are classical in nonlinear control; see for instance [3], [9]. What appears to be new is the observation that, for analytic systems without drift, generic loops provide nonsingularity. The techniques are also related to the material in [15], which relied on control based on pole-shifting along nonsingular trajectories.

3.1. Rescaling: Obstacles and completeness. By means of rescaling, we are able to deal with workspace obstacles and also to restrict attention to complete systems (no explosion times). The basic method is as follows. Let $\beta : \mathbb{R}^n \to \mathbb{R}$ be any smooth mapping, and consider the new system without drift

$$\dot{x} = \beta(x)G(x)u \,. \tag{3.3}$$

Suppose that one has found a control u, defined on an interval $[0, T]$, so that the state ξ_0 is transferred into the state ξ_F using this control, for the system (3.3). Let $\overline{x}(\cdot)$ be the corresponding trajectory. Then, the new control $v(t) := \beta(\overline{x}(t))u(t)$, when applied to the original system (1.1), also produces the desired transfer. In other words, solving a controllability problem for (3.3) provides immediately a solution to the corresponding problem for the original system. (If one is interested in feedback design, as opposed to open-loop control as in this paper, the same situation holds: a feedback law $u = k(x)$ for (3.3) can be re-interpreted as a feedback law $u = \beta(x)k(x)$ for (1.1).)

Assuming that β never vanishes, the controllability properties of the original and the transformed systems are the same. This follows from the above argument. Alternatively, one may see this from the fact that, for any two vector fields g_1, g_2 and any two smooth scalar functions β_1, β_2,

$$[\beta_1 g_1, \beta_2 g_2] \;=\; \beta_1\beta_2[g_1, g_2] + \beta_1 g_1(\beta_2) g_2 - \beta_2 g_2(\beta_1) g_1 \,.$$

This implies inductively that the Lie algebra generated by the columns of $\beta(x)G(x)$ is included in the C^∞-module generated by the Lie algebra corresponding to the columns of $G(x)$, so the accessibility rank condition for the former implies the same for the latter (and viceversa, by reversing the roles of $\beta(x)G(x)$ and $G(x)$).

This construction is of interest in two ways. First of all, one is often interested in control of systems in such a manner that trajectories avoid a certain subset Q of the state-space (which may correspond to "obstacles" in the workspace of a robot, for instance). If β vanishes exactly on Q, then control design on the complement of Q can be done for the new system (3.3), and controls can then be reinterpreted in terms of the original system, a discussed above. Since β vanishes on Q, no trajectories starting outside Q ever pass through Q (uniqueness of solutions). Of course, in planning motions in the presence of obstacles, the control variations should be chosen so as to move in state space directions which do not lead to collisions. One possible approach is to first design a polyhedral path to be tracked, and then to apply the numerical technique explained in order to closely follow this path.

Reparameterization also helps in dealing with possible explosion times in the original system, a fact that had been previously observed in [8], page 2542. In this case, one might use an $\beta(x)$ so that $\beta(x)G(x)$ has all entries bounded; for instance, $\beta(x)$ could be the chosen as $(1+\sum_{i,j} g_{ij}^2(x))^{-1}$. This means that the new system has no finite escape times, for any bounded control.

3.2. Implementation. For complete systems systems without drift, and using steepest descent variations, the explicit computations are as follows. Start with any $\overline{u}(t), t \in [0,T]$ that satisfies the antisymmetry condition

$$\overline{u}(T-t) \;=\; -\overline{u}(t)\,. \tag{3.4}$$

If $x(\cdot)$ satisfies $\dot{x} = G(x)\overline{u}$ then $z(t) := x(T-t)$ satisfies the same equation; thus from the equality $z(T/2) = x(T/2)$ and uniqueness of solutions it follows that $z = x$. In other words,

$$x(T-t) \;=\; x(t) \tag{3.5}$$

for $t \in [0,T]$. To distinguish the objects which depend explicitely on time from those that depend on the current values of states and controls, use the notation

$$\mathcal{A}(x,u) := \sum_{i=1}^{m} \frac{\partial g_i}{\partial x}(x)u_i$$

where g_i is the ith column of G, u_i is the ith entry of the vector $u \in \mathbb{R}^m$, and the partial with respect to x indicates Jacobian. Note that $\mathcal{A}$ can be calcuted once and for all as a function of the variables x, u, before any numerical computations take place. For each $\overline{u}$, and the trajectory $x(\cdot)$ corresponding to this control and initial state ξ_0, denote

$$A(t) := \mathcal{A}(x(t), \overline{u}(t)), \quad B(t) := G(x(t)).$$

Note that if (3.4), and hence also (3.5), hold then

$$A(T-t) = -A(t), \quad B(T-t) = B(t) \tag{3.6}$$

hold as well. Consider next $\Psi(t) := \Phi(T,t)$, where Φ is the fundamental solution as before, corresponding to a given $\overline{u}$ and $x(\cdot)$ as above.

So Ψ satisfies the matrix differential equation

$$\dot{\Psi}(t) = -\Psi(t)A(t), \quad \Psi(T) = I.$$

Consider the function $\widetilde{\Psi}(t) := \Psi(T-t)$. If $\overline{u}$ satisfies the antisymmetry condition, then $\widetilde{\Psi}$ satisfies the same differential equation as Ψ, from which the equality $\widetilde{\Psi}(T/2) = \Psi(T/2)$ implies $\widetilde{\Psi} = \Psi$. Hence also

$$\Psi(T-t) = \Psi(t) \tag{3.7}$$

and so $\Psi(0) = \Psi(T) = I$. The perturbed control to be applied is $\overline{u} + hv = \overline{u} - hL^*\alpha(\overline{u})$ where $\alpha(\overline{u}) = x(T) = x(0) = \xi_0$ if $\overline{u}$ satisfies the antisymmetry condition. The adjoint operator is $(L^*\xi_0)(t) = B(t)^*\Psi(t)^*\xi_0$.

Summarizing, the control to be applied, which for small h should result in a state closer to the origin than ξ_0, is

$$\boxed{\overline{u}(t) - hG(x(t))^*\Psi(t)^*\xi_0} \quad t \in [0,T]$$

where

$$\begin{aligned} \dot{x}(t) &= G(x(t))\,\overline{u}(t), & x(0) &= \xi_0 \\ \dot{\Psi}(t) &= -\mathcal{A}(x(t),\overline{u}(t))\,\Psi(t), & \Psi(0) &= I. \end{aligned}$$

The equations for the system evolution are as follows (the state variable is now denoted by z in order to avoid confusion with the reference trajectory x):

$$\dot{z}(t) = G(z(t))\,[\overline{u}(t) - hG(x(t))^*\Psi(t)^*\xi_0]$$

for $t \in [0,T]$, with initial condition $z(0) = \xi_0$. In a line-search implementation, one would first compute $z(T)$ for various choices of h; the control is only applied once that an optimal h has been found. Then the procedure can be repeated, using $z(T)$ as the new initial state ξ_0.

Remark. Regarding the number of steps that are needed in order to converge to an ε-neighborhood of the desired target state, an estimate is as follows. For a fixed ball around the origin, and sufficient smoothness, one can see that $h = O(\varepsilon)$ provides the inequality in (2.1), as required for (2.3). Thus, the number of iterations N needed, using such a stepsize, is obtained from (2.2):

$$(1 - c\varepsilon)^N < \varepsilon$$

where c is a constant. Taking logarithms and using $\log(1-x) = x + o(x)$ there results the rough estimate

$$N = O\left(\frac{1}{\varepsilon}\log\left(\frac{1}{\varepsilon}\right)\right).$$

4. Review of universal inputs. In this Section, the systems considered will be of the type (1.2) where $x(t) \in \mathcal{X}$, $u(t) \in \mathcal{U}$, and:

- $\mathcal{X} \subseteq \mathbb{R}^n$ is open and connected, for some $n \geq 1$;
- $\mathcal{U} \subseteq \mathbb{R}^m$ is open and connected, for some $m \geq 1$;
- $f : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$ is real-analytic.

A *control* is a measurable essentially bounded map $\omega : [0,T] \to \mathcal{U}$; it is said to be *smooth* (respectively, *analytic*) if it is infinitely differentiable (respectively, real-analytic) as a function of $t \in [0,T]$. As before, we denote by $\phi(t,x,\omega)$ the solution of (1.2) at time t with initial condition x and control ω. This is defined for all small $t = t(x,\omega) > 0$; when we write $\phi(\cdot,x,\omega)$, we mean the solution as defined on the largest interval $[0,\tau)$ of existence.

Recall that the system (1.2) is said to be *strongly accessible* if for each $x \in \mathcal{X}$ there is some $T > 0$ so that

$$\operatorname{int} \mathcal{R}^T(x) \neq \emptyset,$$

where as usual $\mathcal{R}^T(x)$ denotes the reachable set from x in time exactly T. Equivalently, the system must satisfy the *strong accessibility rank condition*: $\dim \mathcal{L}_0(x) = n$ for all x, where $\mathcal{L}_0$ is the ideal generated by all the vector fields of the type $\{f(\cdot,u) - f(\cdot,v), u,v \in \mathcal{U}\}$ in the Lie algebra $\mathcal{L}$ generated by all the vector fields of the type $\{f(\cdot,u), u \in \mathcal{U}\}$; see [20]. For systems affine in controls:

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) \tag{4.1}$$

the algebra $\mathcal{L}_0$ is the Lie algebra generated by all vector fields $\mathrm{ad}_f^k(g_i)$, $k \geq 0$, $i = 1,\ldots,m$.

Given a state x, a control ω defined on $[0,T]$, and a positive $T_0 \leq T$ so that $\xi(t) = \phi(t,x,\omega)$ is defined for all $t \in [0,T_0]$, we may consider the *linearization along the trajectory* (ξ,ω):

$$\dot{z}(t) = A(t)z(t) + B(t)u(t) \tag{4.2}$$

where $A(t) := \frac{\partial f}{\partial x}(\xi(t),\omega(t))$ and $B(t) := \frac{\partial f}{\partial u}(\xi(t),\omega(t))$ for each t. A control ω will be said to be *nonsingular for* x if the linear time-varying system (4.2) is controllable on the interval $[0,T_0]$, for some $T_0 > 0$. When u is analytic, this property is independent of the particular T_0 chosen, and it is equivalent to a Kalman-like rank condition (see e.g. [17], Corollary 3.5.17). Nonsingularity is equivalent to a Fréchet derivative of $\phi(T_0,x,\cdot)$ having full rank at ω.

If ω is nonsingular for $x \in \mathcal{X}$, and T_0 is as above, then $\mathcal{R}^{T_0}(x)$ has a nonempty interior. This is a trivial consequence of the Implicit Function Theorem (see for instance [17], Theorem 6). Thus, if for each state x there is some control which is nonsingular for x, then (1.2) is strongly accessible. The converse of this fact is also true, that is, if a system is strongly accessible then for each state x there is some control which is nonsingular for x. This converse fact was proved in [16] (the result in that reference is stated under a controllability assumption, which is not needed in the proof of this particular fact; in any case, we review below the proof). The main purpose here is to point out that ω can be chosen *independently* of the particular x, and moreover, a generic ω has this property. We now give a precise statement of these facts.

A control $\omega : [0,T] \to \mathcal{U}$ will be said to be a *universal nonsingular control* for the system (1.2) if it is nonsingular for every $x \in \mathcal{X}$.

THEOREM 4.1. *If (1.2) is strongly accessible, there is an analytic universal nonsingular control.*

Let $\mathcal{C}^\infty([0,T],\mathcal{U})$ denote the set of smooth controls $\omega : [0,T] \to \mathcal{U}$, endowed with the $\mathcal{C}^\infty$ topology (uniform convergence of all derivatives). A *generic* subset of $\mathcal{C}^\infty([0,T],\mathcal{U})$ is one that contains a countable intersection of open dense sets.

THEOREM 4.2. *If (1.2) is strongly accessible, the set of smooth universal nonsingular controls is generic in $\mathcal{C}^\infty([0,T],\mathcal{U})$, for any $T > 0$.*

A proof of this fact was originally given [18]. A proof is also given in an Appendix, in order to make this paper self-contained. The proof is heavily based on the universal input theorem for observability. (The theorem for observability is due to Sussmann, but the result had been successively refined in the papers [7,13,19]; see also [21] for a different proof as well as a generalization involving inputs that are universal even over the class of all possible analytic systems. There is also closely related recent work of Coron ([5]) on generalizations of these theorems.)

5. Remarks. It is worth mentioning certain relations between the results in this paper and recent work on time-varying feedback laws for systems without drift, especially the results in [4] and [12].

In [4], Coron proves, for controllable smooth systems with no drift, that there is a smooth feedback law $u = k(t,x)$, periodic on t and with $k(t,0) \equiv 0$, such that the closed-loop system $\dot{x} = G(x)k(t,x)$ is uniformly globally asymptotically stable. The critical step in his proof is to obtain a smooth family of controls $\{u_x(\cdot), x \in \mathbb{R}^n\}$, where each u_x is defined for all $t \in \mathbb{R}$, so that the following properties are satisfied:

1. $u_x(t+1) = u_x(t)\ \forall x,t$,
2. $u_x(1-t) = -u_x(t)\ \forall x,t$,
3. $u_x(t)$ is C^∞ jointly on (x,t),
4. for each $x \neq 0$, u_x is nonsingular for x,
5. $u_0 \equiv 0$, and
6. $\phi(t,x,u_x)$ is defined for all $t \geq 0$.

Observe that the second and last properties imply that $\phi(1,x,u_x) = x$ for all x. Thus, applying the control u_x with initial state x results in a periodic motion, $\phi(t+1,x,u_x) = \phi(t,x,u_x)$. These properties are used in deriving stabilizing feedbacks in [4].

It is possible to obtain a family of controls as above —at least in the analytic case— using Theorem 4.1. A sketch follows. First note that one may take the system to be complete, as discussed in Section 3, so the last property will be satisfied for any choice of u_x.

Assume that ω is a control which is analytic and universal nonsingular, defined on the interval $[0,1]$. As the system being considered in this case has no drift, it follows that for each nonzero constant c the control $c\omega(ct)$, defined on the interval $[0,1/c]$, is again universal nonsingular. (Indeed, if ξ_0 as any initial state and $x(t) = \phi(t,\xi_0,\omega)$ then $x(ct)$ is the trajectory corresponding to this new control, and the linearization along this trajectory is controllable, because, with the notations in [17], Corollary 3.5.17 and using superscript c to denote the dependence on c, $A^{(c)}(t) = cA(ct)$ and $B_i^{(c)}(t) = c^i B(ct)$ for $i = 0,1,2,\ldots$.) Assume that $c < 1$, so that $c\omega(ct)$ is defined on $[0,1]$. Since the system and the control are both analytic, the restriction of $c\omega(ct)$ to the interval $[0,1/6]$ is again universal and nonsingular. Observe that, by definition of analytic function on a closed interval, this means that $c\omega(ct)$ is in fact defined on some larger interval of the form $(-\varepsilon,1)$, for some $\varepsilon > 0$. Let $\beta : \mathbb{R}^n \to \mathbb{R}$ be a smooth function which is positive for $x \neq 0$, vanishes at the origin, and is bounded by 1.

Consider now, for each $x \neq 0$, the control $u_x(t)$ which is defined on the interval $[0,1/2]$ as follows. On the subinterval $[1/6,1/3]$, this equals

$$\beta(x)\,\omega(\beta(x)(t-1/6))\,.$$

Extend u_x smoothly to $[0,1/6]$ in such a manner that all derivatives vanish at 0. Similarly, extend in the other direction, to $[0,1/2]$, so that all derivatives also vanish at $1/2$. Note that u_x is still a universal nonsingular control, because its restriction to the subinterval $[1/6,1/3]$ is. Also, these extensions can be done in such a manner that u_x depends smoothly on x and is bounded by a constant multiple of $\beta(x)$. Finally, it is trivial to extend by antisymmetry to $[0,1]$ and then periodically to all $t \in \mathbb{R}$, so that all the desired properties hold.

6. An illustration. Though very simple, it is worth understanding our method in the case of the simplest possible example of a system with no drift which is controllable but for which no possible smooth stabilizer exists. This example is due to Brockett ([2]) and appears in most textbooks in some variant or another (see e.g. [17], Example 4.8.14); it is closely related, under a coordinate change, to the "unicycle" or "knife edge" example. The system in question has dimension 3 and two controls; the equations are as follows:

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= xv \end{aligned}$$

(we write x, y, z for the coordinates of the state and u, v for the input coordinates, in order to avoid subscripts). As suggested earlier, periodic controls on intervals $[0, 2\pi]$ symmetric about π are natural. In this case, in particular, the input $\overline{u}$ defined by $u(t) \equiv 0$, $v(t) = \sin(t)$ on this interval is already a universal nonsingular control (as shown next), so we use $\overline{u}$. Nonsingularity is shown as follows. Given any initial state $\xi = (x_0, y_0, z_0)$, the trajectory that results is

$$\begin{aligned} x(t) &= x_0 \\ y(t) &= y_0 - \cos t \\ z(t) &= z_0 + x_0 t\,. \end{aligned}$$

Along this trajectory, the linearized system has matrices

$$A(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sin t & 0 & 0 \end{pmatrix} \quad \text{and} \quad B(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & x_0 \end{pmatrix}.$$

Let $B_0 := B$ and $B_1 := AB_0 - B_0'$ $(= AB$ since B is constant). Since

$$(B_0 B_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x_0 & \sin t & 0 \end{pmatrix}$$

has rank 3 generically, this shows that the linearized system is controllable (see e.g. [17], Corollary 3.5.17).

Newton's method is even simpler in this case (using the above nonsingular control). Indeed, since the first two equations are linear in controls, and the system over all is quadratic in a suitable sense, Newton's method results in exact convergence to zero in just two passes. We prove this fact next. With the above control $\overline{u}$, the pseudoinverse of the reachability map is as follows (letting (x_0, y_0, z_0) be the coordinates of the initial state):

$$L^{\#} = (1/\pi)\begin{pmatrix} 1/2+\cos(t) & -x_0\cos(t) & \cos(t) \\ 0 & 1/2 & 0 \end{pmatrix},$$

so the net control applied is

$$\begin{pmatrix} -\frac{h}{\pi}\left(\frac{x_0}{2} + x_0\cos(t) - x_0\cos(t)y_0 + \cos(t)z_0\right) \\ \sin(t) - \frac{hy_0}{2\pi} \end{pmatrix}.$$

A Newton step is obtained by solving the corresponding differential equations with step size h; this gives the new states: $x_h = x_0 - x_0 h$, $y_h = y_0 - hy_0$, and $z_h = (h/2)(-2x_0y_0 + hx_0y_0 - 2z_0) + z_0$. The stepsize $h = 1$ gives zero values for the first two coordinates after one step, while the last coordinate becomes, under this choice of h, $-x_0y_0/2$. But any state of the form $(0,0,z)$ gets mapped to the origin in one step under the same iteration. In summary, all states are mapped in two iterations to the origin.

APPENDIX

A. Appendix: Proof of nonsingularity result. We first recall the fact, mentioned above, that for each x there is a control nonsingular for x. This can be proved as follows. Pick x, and assume that the system (1.2) is strongly accessible. Let y be in the interior of $\mathcal{R}^T(x)$, for some $T > 0$. It follows from [14], Lemma 2.2 and Proposition 2.3, that there exists some real number $\delta > 0$ and some positive integer k so that y is in the interior of the image of

$$F : \mathcal{U}^k \to \mathcal{X}\,,\ (u_1,\ldots,u_k) \mapsto \exp(\delta f_{u_1})\ldots\exp(\delta f_{u_k})(x)\,,$$

where we are using the notation $\exp(\delta f_u)(z) = \phi(\delta, z, \omega)$ for the control $\omega \equiv u$ on $[0,\delta]$. This map F is smooth, so by Sard's Theorem it must have full-rank Jacobian at some point $(u_1^0,\ldots,u_k^0)$. This implies that the piecewise-constant control ω, defined on $[0,k\delta]$ and equal to the values u_i^0 on consecutive intervals of length δ, is nonsingular for the given state x, as desired.

We next need what is basically a restatement of the main results in [19]:

PROPOSITION A.1. *Consider the (analytic) system (1.2) and assume that $h : \mathcal{X} \to \mathbb{R}$ is a real-analytic function. Let G be the set of states x so* *that, for some control $\omega = \omega(x)$, $h(\phi(\cdot, x, \omega))$ is not identically zero. Then, there exists an analytic control ω^* so that, for every $x \in G$, $h(\phi(\cdot, x, \omega^*))$ is not identically zero; moreover, for each $T > 0$, the set of smooth such controls is generic in $\mathcal{C}^\infty([0,T], \mathcal{U})$.*

Proof. We consider the extended system (with state space $\mathcal{X} \times \mathbb{R}$):

$$\begin{aligned} \dot{x} &= f(x,u) \\ \dot{z} &= 0 \\ y &= zh(x)\,, \end{aligned}$$

which is an analytic system with outputs. Consider two states of the form $(x, 0)$ and $(x, 1)$, with $x \in \mathcal{X}$. A control ω distinguishes these states if and only if $h(\phi(\cdot, x, \omega))$ is not identically zero.

Let ω^* be a control for the extended system which is universal with respect to observability. There are analytic such controls, and the desired genericity holds, by Theorems 2.1 and 2.2 in [19]. Now pick any x in the set G. Then $(x, 0)$ and $(x, 1)$ are distinguishable, and hence ω^* distinguishes among them. This means that $h(\phi(\cdot, x, \omega^*))$ is not identically zero, as desired. □

We now prove Theorems 4.1 and 4.2. Let (1.2) be given, and take the composite system consisting of (2.4) and (2.6) with output $h(x, Q) = \det Q$, This is seen as a system with state space $\mathcal{X} \times \mathbb{R}^{n\times n}$. For an initial state of the form $z = (x, 0)$, and a control ω, the solution $\hat{\phi}$ of the larger system at time t, if defined, is so that

$$h(\hat{\phi}(t, z, \omega)) \;=\; \det\left(\int_0^t \Phi(t,s)B(s)B^*(s)\Phi(t,s)^*\,ds\right)$$

(where Φ denotes the fundamental solution of the linearized equation), so ω is nonsingular for x precisely when $h(\hat{\phi}(t, (x, I, 0), \omega))$ is not identically zero.

By the remarks made earlier, strong accessibility guarantees that every state of the form $(x, I, 0)$ is in the set G defined in Proposition A.1 (for the enlarged system); thus our Theorems follow from the Proposition. ∎

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