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On the Continuity and Incremental-Gain Properties of Certain Saturated Linear Feedback Loops

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Abstract

This paper discusses various continuity and incremental-gain properties for neutrally stable linear systems under linear feedback subject to actuator saturation. The results complement our previous ones, which applied to the same class of problems and provided finite-gain stability.

1 Introduction

In this paper. we continue the study, started in [6], of operator stability properties for saturated-input linear systems. In the previous paper, we studied feedback systems of the form

$$\dot{x} = Ax + B\sigma(Fx + u). \qquad (\Sigma)$$

Here σ denotes a vector of saturation-type functions, each of which satisfies mild technical conditions that are recalled later (at this point, it suffices to say that all reasonable "sigmoidal" maps such as $\sigma(x) = \tanh(x)$ and the standard saturation function $\sigma_0(t) = \mathrm{sign}(t)\min\{|t|, 1\}$ are included). The matrix A is assumed to be neutrally stable and one uses the standard passivity theory choice of feedback F that makes the origin of the unforced closed-loop system $\dot{x} = Ax+B\sigma(Fx)$ globally asymptotically stable. (For instance, if A has all eigenvalues in the imaginary axis and the pair (A, B) is controllable, $F = -B^T P$, where P is a positive definite matrix satisfying $A^T P + PA = 0$.)

It is proved in [6] that this system is finite-L^p-gain stable, that is, the zero-initial state operator $F_{\sigma,p}$ mapping input functions $u(\cdot)$ to solutions $x(\cdot)$ is a well-defined and bounded operator from $L^p([0,\infty), \mathbb{R}^m)$ to $L^p([0,\infty), \mathbb{R}^n)$. The result is valid for each p in the range $[1, \infty]$. Estimates were provided of the operator norms, in particular giving for $p = 2$ an upper bound expressed in terms of the H^∞-norm of the same input-state map for the system in which the saturation σ is not present. We also dealt with partially observed states, generalizing the result to the case where an observer is inserted in the feedback construction. The assumption of neutral stability is critical: we also obtained examples showing that the double integrator cannot be stabilized in this operator sense by any linear feedback, contradicting what may be expected from the fact that such systems are globally asymptotically stabilizable in the state-space sense. (Recently, Lin, Saberi, and Teel in [5] obtained related results, showing in particular that under the restriction that the input signals be bounded one can drop the stability assumption in obtaining finite-gain stability. See also [8, 10] and [9] for state-space stabilization of linear systems subject to saturation, under minimal conditions.)

Finite-gain stability, studied in the above-mentioned papers, means that the "energy" of inputs is amplified by a bounded amount when passing through the system. Another property which is extremely important in the context of feedback systems analysis is that of *incrementally finite gain* ("ifg") stability. In mathematical terms, this latter property is the requirement that the operator $F_{\sigma,p}$ be globally Lipschitz. That is to say, if y_{nom} is the output produced in response to a nominal input u_{nom}, then a new input $u_{\mathrm{nom}} + \Delta u$ produces an output whose energy differs from that of y_{nom} by at most a constant multiple of the energy of the increment Δu. This stronger notion measures sensitivity to input perturbations; for differentiable mappings, one would be asking that the derivative be bounded. In the context of stability, the usual formulations of the small-gain theorem involve ifg stability, because fg stability by itself is not sufficient in order to guarantee the existence and uniqueness of signals ("well-posedness") in a closed-loop system; see [11]. In the recent work [4], it is shown how to generalize the gap metric, so successful in robustness analysis of linear systems, to the context of ifg stability of nonlinear systems. Even stronger properties may sometimes be needed; for instance, the work in [3] requires what the author of that paper calls "differential stability," which means that ifg stability holds and $F_{\sigma,p}$ is Fréchet differentiable as well. Motivated by this, we ask here if stronger properties hold for the feedback configuration studied in [6].

Our results can be summarized in informal terms as follows:

1. The operator $F_{\sigma,p}$ is *continuous* if p is finite, but is not in general continuous for $p = \infty$.

2. $F_{\sigma,p}$ is *locally Lipschitz* under additional assumptions on the saturation (for p finite, a sufficient con-

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dition is that the components of σ be differentiable near the origin; for $p = \infty$ one asks in addition that they be differentiable everywhere, with positive derivative). A much stronger statement than the local Lipschitz property is established –which we call "semiglobal Lipschitz"– as incremental gains are shown to depend only on the norms of the controls; on the other hand, we also show by counterexample that these operators are not generally globally Lipschitz (so ifg stability does not hold).

3. Assume that σ is continuously differentiable. For $p = \infty$, we show that $F_{\sigma,p}$ is Fréchet differentiable (under the assumption that σ' is always positive), but this may fail for finite p. In the latter case, however, we can prove that directional derivatives always exist.

2 Preliminaries

Before stating our results, first we recall some definitions and basic results from [6], including those of "saturation function" and finite gain L^p-stability.

By a *saturation function* ("S-function" for short) we mean any $\sigma : \mathbb{R} \to \mathbb{R}$ which satisfies the following properties:

- σ is locally Lipschitz and bounded;
- $t\sigma(t) > 0$ if $t \neq 0$;
- $\liminf_{t\to 0} \frac{\sigma(t)}{t} > 0$;
- $\limsup_{t\to 0} \frac{\sigma(t)}{t} < \infty$ and $\liminf_{|t|\to\infty} |\sigma(t)| > 0$.

We remark that the locally Lipschitz assumption on σ is not really needed in establishing Theorem (FG) below. This purpose of this condition is only to guarantee that system (1) in Theorem (FG) has uniqueness of solutions for any input u.

All the interesting saturation functions found in usual systems models, including the standard saturation function $\sigma_0(t) = \mathrm{sign}(t)\min\{|t|, 1\}$ as well as the functions $\arctan(t)$ and $\tanh(t)$ are S-functions.

We say that σ is an $\mathbb{R}^n$*-valued S-function* if $\sigma = (\sigma_1, \ldots, \sigma_n)^T$, where each component σ_i is an S-function and

$$\sigma(x) \stackrel{\text{def}}{=} (\sigma_1(x_1), \ldots, \sigma_n(x_n))^T$$

for $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$. (Here we use $(\cdots)^T$ to denote the transpose of the vector $(\cdots)$.)

We now turn to the stability definitions. These can be introduced for any initialized control system

$$\dot{x} = f(x, u), \quad x(0) = 0 . \tag{Σ}$$

The state x and the control u take values in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. With appropriate assumptions on f (for example $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is globally Lipschitz with respect to its argument (x, u)), the solution x of (Σ) corresponding to any input $u \in L^p([0,\infty), \mathbb{R}^m)$ for $1 \leq p \leq \infty$ is well defined for all $t \in [0, \infty)$. When defined for all $t \in [0, \infty)$, we denote this solution x, which is a priori just a locally absolutely continuous (l.a.c for short) function, as $F_{(\Sigma)}(u)$.

For any $1 \leq p \leq \infty$ and any vector function $x \in L^p([0,\infty), \mathbb{R}^n)$, we consider the standard L^p-norm

$$\|x\|_{L^p} := \left(\int_0^\infty \|x(t)\|^p dt\right)^{1/p} \quad (p < \infty),$$
$$\|x\|_{L^\infty} := \text{ess sup}_{0\leq t<\infty}\|x(t)\| .$$

(For vectors in $\mathbb{R}^n$ we use Euclidean norm $\|\xi\| = \left(\sum_{i=1}^n \xi_i^2\right)^{\frac{1}{2}}$. We use the same notation for matrices, that is, $\|S\|$ is the Frobenius norm equal to the square root of the sum of squares of entries, i.e. $\|S\| = \mathrm{Tr}(SS^T)^{1/2}$, where Tr denotes trace.)

We define the L^p*-gain* of a system (Σ) as the norm of the operator $F_{(\Sigma)}$ that maps inputs to solutions, assuming a zero initial state. That is, the L^p-gain of (Σ), to be denoted by G_p, is the infimum (possibly $+\infty$) of the numbers M so that

$$\|F_{(\Sigma)}(u)\|_{L^p} \leq M\|u\|_{L^p}$$

for all $u \in L^p([0,\infty), \mathbb{R}^m)$. (If $F_{(\Sigma)}(u)$ is undefined for any $u \in L^p([0,\infty), \mathbb{R}^m)$, we also write $G_p = \infty$.) When this number is finite, we say that the operator is *finite gain* L^p*-stable* (in more usual mathematical terms, it is a bounded operator).

The main result in [6] concerns the finiteness of the L^p-gain of (Σ) for a specific class of input-saturated linear systems. We quote this result next.

Theorem (FG) *Let A, B be $n \times n, n \times m$ matrices respectively and let σ be an $\mathbb{R}^m$-valued S-function. Assume that A is neutrally stable. Then there exists an $m \times n$ matrix F such that the system*

$$\begin{aligned} \dot{x} &= Ax + B\sigma(Fx + u), \\ x(0) &= 0, \end{aligned} \tag{1}$$

is finite gain L^p-stable for all $1 \leq p \leq \infty$.

By neutral stability, we mean as usual that the origin of the differential equation $\dot{x} = Ax$ is stable in the sense of Lyapunov (not necessarily asymptotically stable, of course; otherwise the result would be trivial from linear systems theory); equivalently, there is a symmetric positive definite matrix Q which provides a solution of the Lyapunov matrix inequality $A^TQ + QA \leq 0$.

The results in this paper will refer to the specific feedback F that is found in the proof of the above-cited result. In order to understand the choice of F (which is the most natural choice of feedback to use in this context), we need to recall the preliminary steps in the proof of Theorem (FG). The first step consisted of the observation that one can assume without loss of generality that the pair (A, B) is controllable, because the trajectories lie in the controllability space $R(A, B)$. Next we applied a change of basis to reduce A to the block-diagonal form

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \tag{2}$$

where A_1 is an $r \times r$ Hurwitz matrix and A_2 is an $(n-r) \times (n-r)$ skew-symmetric matrix. (Recall that A is assumed to be neutrally stable.) Thus one only needs to obtain finite gain L^p-stability of the subsystem corresponding to A_2; then feeding-back a function of these variables doesn't affect the finite gain L^p-stability of the first subsystem. Since A_2 is skew-symmetric and the pair (A_2, B_2) is controllable, the non-saturated closed-loop matrix $\tilde{A} := A_2 - B_2 B_2^T$ is Hurwitz. Therefore, the proof of Theorem (FG) is reduced to showing that the following system:

$$\dot{x} = Ax + B\sigma(-B^T x + u)\,, x(0) = 0 \tag{3}$$

is finite gain L^p-stable for every $1 \leq p \leq \infty$, provided that A is skew-symmetric and $\tilde{A} = A - BB^T$ is Hurwitz. Thus, except for two coordinate changes (first to restrict to the controllability space and then to exhibit the above block structure), the F used in the proof of Theorem (FG) is $F = -B^T$. This is the standard choice of feedback suggested by the passivity approach to control –for a discussion and references see [6].

(For completeness, we point out that, after these trivial preliminary steps, the proof of Theorem (FG) then centers upon the hard part, which consists of finding a suitable "storage function" V_p and establishing for it a "dissipation inequality" of the form

$$\frac{dV_p(x(t))}{dt} \leq -\|x(t)\|^p + \kappa_p \|u(t)\|^p\,, \tag{4}$$

for $x = F_{(\Sigma)}(u)$, where now (Σ) is the system in Equation (1) and $\kappa_p > 0$ is some constant. Surprisingly, a nonsmooth V_p is needed.)

In conclusion, we will denote by

$$\boxed{F_{\sigma,p}\;:\; L^p([0,\infty),\mathbb{R}^m) \longrightarrow L^p([0,\infty),\mathbb{R}^n)}$$

the (nonlinear) input/state operator $F_{(\Sigma)}$ for system (1) when the feedback F is chosen as in the above discussion, for any fixed σ and any fixed p.

3 Regularity Properties of $F_{\sigma,p}$

Now we are able to give the precise statement of the regularity properties of $F_{\sigma,p}$ such as continuity, incremental gains, differentiability, which we will study in this paper.

3.1 Statement of the Incremental Gain Results

Recall that a $\mathcal{K}$-function $g : \mathbb{R}^+ \to \mathbb{R}^+$ is one that is continuous, strictly increasing, and satisfies $g(0) = 0$.

Definition 1 The operator $F_{\sigma,p}$ satisfies the *generalized incremental gain* property (with respect to L^p) if

($\mathbf{GIG}_p$) there exists a $\mathcal{K}$-function g such that for all u, v in $L^p([0,\infty),\mathbb{R}^m)$,

$$\|F_{\sigma,p}(v) - F_{\sigma,p}(u)\|_{L^p} \;\leq\; g\Big(\|v-u\|_{L^p}\Big).$$

It is obvious that $F_{\sigma,p}$ satisfies the $\mathbf{GIG}_p$ property if and only if it is *uniformly* continuous, i.e. iff for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|F_{\sigma,p}(u) - F_{\sigma,p}(v)\|_{L^p} \leq \varepsilon$ whenever $\|u - v\|_{L^p} \leq \delta$. Note that if g is linear, this is the standard "finite incremental gain" property, or in mathematical terms, a global Lipschitz property.

It turns out that $\mathbf{GIG}_p$ is a very strong property. For most S-functions, even smooth ones, the operator $F_{\sigma,p}$ does not satisfy the $\mathbf{GIG}_p$ property. For general S-functions σ, $F_{\sigma,\infty}$ even fails to be continuous. However, for restricted classes of S-functions, more precisely the classes $\mathcal{C}_{(0)}$ and $\mathcal{C}^{1,+}$ defined below, $F_{\sigma,p}$ satisfies the following $\mathbf{SLP}_p$ property (*semiglobal Lipschitz property*):

($\mathbf{SLP}_p$) there exist a $\mathcal{K}$-function g and a constant $c > 0$ so that, for all u, v in $L^p([0,\infty),\mathbb{R}^m)$,

$$\|F_{\sigma,p}(v) - F_{\sigma,p}(u)\|_{L^p} \;\leq\; (c + g(\|u\|_{L^p}))\,\|v-u\|_{L^p}\,.$$

This property clearly implies the continuity of $F_{\sigma,p}$.

The class $\mathcal{C}_{(0)}$ is defined as the class of functions $\sigma : \mathbb{R} \to \mathbb{R}$ which are globally Lipschitz, differentiable at 0 and satisfy

$$\lim_{\substack{t,s\to 0\\ t\neq s}} \frac{\sigma(t) - \sigma(s)}{t-s} = \sigma'(0)\,. \tag{5}$$

An $\mathbb{R}^m$-valued S-function σ belongs to $\mathcal{C}_{(0)}$ if each of its components belongs to $\mathcal{C}_{(0)}$.

The class $\mathcal{C}^{1,+}$ is defined as the class of functions $\sigma : \mathbb{R} \to \mathbb{R}$ which are continuously differentiable and satisfy that σ' is everywhere positive. An $\mathbb{R}^m$-valued S-function σ belongs to $\mathcal{C}^{1,+}$ if each of its components belongs to $\mathcal{C}^{1,+}$.

The main results of this paper are summarized in the next theorem:

Theorem 1 *Let σ be an $\mathbb{R}^m$-valued S-function and let $1 \leq p \leq \infty$. We have*

(A) *For each $1 \leq p < \infty$, the following conclusions hold:*

(i) $F_{\sigma,p}$ is continuous, but in general does not satisfy the $\mathbf{SLP}_p$ property.

(ii) Assume that σ belongs to $\mathcal{C}_{(0)}$. Then $F_{\sigma,p}$ satisfies $\mathbf{SLP}_p$.

(iii) Even for smooth non-decreasing saturation functions σ, $F_{\sigma,p}$ does not in general satisfy the $\mathbf{GIG}_p$ property.

(B) *For $p = \infty$, the following conclusions hold:*

(i') In general, $F_{\sigma,\infty}$ is not continuous.

(ii') Assume that each component of σ is non-decreasing. Then for $n = 1$, $F_{\sigma,\infty}$ is globally Lipschitz. If $n > 1$, even for $m = 1$ and σ non-decreasing, $F_{\sigma,\infty}$ need not be continuous.

(iii') Assume that σ belongs to $\mathcal{C}^{1,+}$. Then $F_{\sigma,\infty}$ satisfies $\mathbf{SLP}_\infty$.

(iv') Even for a smooth $\sigma \in \mathcal{C}^{1,+}$, $F_{\sigma,\infty}$ need not satisfy the $\mathbf{GIG}_\infty$ property.

3.2 Statement of the Differentiability Results

We can also discuss the differentiability properties of $F_{\sigma,p}$. First if σ is an $\mathbb{R}^m$-valued S-function, we say that σ is of class $\mathcal{C}^1$ if each component of σ is of class $\mathcal{C}^1$, i.e. continuously differentiable. We have

Theorem 2 *1. For $p = \infty$ and $\sigma \in \mathcal{C}^{1,+}$, $F_{\sigma,p}$ is Fréchet-differentiable.*

2. For $1 \le p < \infty$ and σ of class $\mathcal{C}^1$ and globally Lipschitz, $F_{\sigma,p}$ is Gâteaux-differentiable.

We give an example in [1] to show that $F_{\sigma,1}$ need not be Fréchet-differentiable even for smooth σ.

If $\sigma \in \mathcal{C}^{1,+}$ and $u, v \in L^\infty([0,\infty), \mathbb{R}^m)$, we will use $DF_{\sigma,\infty}(u).v$ to denote the differential of $F_{\sigma,\infty}$ at u applied to v. For each $1 \le p < \infty$ and σ of class $\mathcal{C}^1$, we use $D_v F_{\sigma,p}(u)$ to denote the Gâteaux-differential of $F_{\sigma,p}$ at $u \in L^p([0,\infty), \mathbb{R}^m)$ in the direction v. It is well known that both $DF_{\sigma,\infty}(u).v$ and $D_v F_{\sigma,p}(u)$ are given by the linearization of (Σ) along the trajectory x of (Σ) corresponding to u (cf. [7]). In other words, $DF_{\sigma,\infty}(u).v$ and $D_v F_{\sigma,p}(u)$ are the respective solutions of the following time-varying initialized systems

$$\left(\Sigma_*(p,\sigma,u)\right) \quad \dot{\xi} = A\xi + B\sigma'(Fx+u)(F\xi+v), \ \ \xi(0) = 0, \tag{6}$$

where F is the $m \times n$ matrix given in the proof of Theorem (FG) and if $\sigma = (\sigma_1, \ldots, \sigma_m)$, $z \in \mathbb{R}^m$, then $\sigma'(z) = \mathrm{diag}(\sigma_1'(z_1), \ldots, \sigma_m'(z_m))$.

4 Proof of Theorem 1 (A) (ii) and (iii)

In this section, we will give the complete proof of (A) (ii) in Theorem 1 in order to illustrate the methods used to establish the above mentioned results. Furthermore, in example (E), we will provide a smooth non-decreasing saturation function for which the corresponding $F_{\sigma,p}$ fails to satisfy the $\mathbf{GIG}_p$ property. All the other positive statements in Theorems 1 and 2 together with the counterexamples illustrturating the negative results in these theorems are given in details in [1].

Let us start with the proof of (A) (ii). From the sketch of the proof of Theorem (FG) we can assume with no loss of generality that A is skew-symmetric and (A, B) is controllable.

Fix now u, v in $L^p([0,\infty), \mathbb{R}^m)$ with $1 \le p < \infty$. We may assume that

$$\|v - u\|_{L^p} \le \|u\|_{L^p}, \tag{7}$$

because if $\|v - u\|_{L^p} > \|u\|_{L^p}$, by the finite gain L^p-stability of $F_{\sigma,p}$ we would have

$$\begin{aligned} \|F_{\sigma,p}(v) - F_{\sigma,p}(u)\|_{L^p} &\le \|F_{\sigma,p}(v)\|_{L^p} + \|F_{\sigma,p}(u)\|_{L^p} \\ &\le G_p(\|v\|_{L^p} + \|u\|_{L^p}) \\ &\le G_p(\|v-u\|_{L^p} + 2\|u\|_{L^p}) \\ &< 3G_p\|v-u\|_{L^p}, \end{aligned}$$

and we would be done.

If $\sigma = (\sigma_1, \ldots, \sigma_m)^T$, define $\tilde{D}$ as the diagonal matrix $\mathrm{diag}(\sigma_1'(0), \ldots, \sigma_m'(0))$ and $\tilde{A}$ as the Hurwitz matrix $A - B\tilde{D}B^T$. Let $P > 0$ satisfy

$$P\tilde{A} + \tilde{A}^T P = -I. \tag{8}$$

Let λ_{max} and λ_{min} be respectively the largest and smallest eigenvalue of P and let

$$\beta = \frac{1}{4\sqrt{m}\|B\|\,\|PB\|}.$$

Since σ belongs to $\mathcal{C}_{(0)}$, there exists an $\alpha > 0$ such that, for $|s| \le \alpha$, $|t| \le \alpha$ and $t \ne s$:

$$\left|\frac{\sigma_i(t) - \sigma_i(s)}{t - s} - \sigma_i'(0)\right| \le \beta \ \text{ for } i = 1, \ldots, m.$$

Fix u and v in $L^p([0,\infty), \mathbb{R}^m)$ for which (7) holds. As in the proof of (iii'), letting $x = F(u)$ and $y = F(v)$, then x, y satisfy

$$\begin{aligned} \dot{x} &= Ax + B\sigma(-B^T x + u), \\ \dot{y} &= Ay + B\sigma(-B^T y + v), \\ x(0) &= y(0) = 0. \end{aligned}$$

Write $z = y - x$, $h = v - u$ and let $\tilde{x}_i, \tilde{y}_i, \tilde{z}_i$ denote respectively the i-th component of $B^T x$, $B^T y$, $B^T z$. We have

$$\begin{aligned} \dot{z} &= Az + BD(t)(-B^T z + h), \\ z(0) &= 0, \end{aligned} \tag{9}$$

where

$$\begin{aligned} D(t) &\stackrel{\text{def}}{=} \mathrm{diag}(d_1(t), \ldots, d_m(t)), \\ d_i(t) &\stackrel{\text{def}}{=} \frac{\sigma_i(-\tilde{y}_i(t) + v_i(t)) - \sigma(-\tilde{x}_i(t) + u_i(t))}{-\tilde{z}_i(t) + h_i(t)}. \end{aligned}$$

(If $\tilde{z}_i(t) - h_i(t) = 0$ we just let $d_i(t) = \sigma_i'(0)$.) Let $K > 0$ be a Lipschitz constant for σ (more precisely, let K be a Lipschitz constant for each component of σ). Then $||d_i||_{L^\infty} \leq K$. So $||D(t)|| \leq \sqrt{m}K$.

Let

$$E = \cup_{i=1}^{m} \left\{ t \left| |d_i(t) - \sigma_i'(0)| > \beta \right. \right\} .$$

Clearly

$$\begin{aligned} E \subseteq \quad \cup_{i=1}^{m} \quad & \left\{ \left\{ t \left| |\tilde{x}_i(t) - u_i(t)| > \alpha \right. \right\} \right. \\ \cup \quad & \left. \left\{ t \left| |\tilde{y}_i(t) - v_i(t)| > \alpha \right. \right\} \right\}. \end{aligned}$$

Therefore, by Tchebychev's inequality we get

$$|E| \leq \tilde{C}(||u||_{L^p}^p + ||v||_{L^p}^p) ,$$

for some constant $\tilde{C} > 0$ independent of u and v. Noticing (7) we have $|E| \leq C||u||_{L^p}^p$, where $C > 0$ is a constant independent of u, v.

If we let $V(z) = z^T P z$ for $z \in \mathbb{R}^n$, where P is defined in (8), we get along the trajectories of (9):

$$\begin{aligned} \dot{V}(z(t)) &= -||z(t)||^2 \\ &- 2z(t)^T PB \left[(D(t) - \tilde{D})B^T z(t) - D(t)h(t) \right] \\ &\leq -\left[1 - 2||B||||PB|| \, ||D(t) - \tilde{D}|| \right] ||z(t)||^2 \\ &+ 2\sqrt{m}K||PB||||z(t)|| \, ||h(t)||. \end{aligned}$$

Therefore, along the trajectories of (9), V satisfies the differential inequality

$$\begin{aligned} \dot{V}(z(t)) &\leq 2\lambda(t)V(z(t)) + 2C_1 V^{1/2}(z(t))||h(t)|| , \\ V(0) &= 0 , \end{aligned} \tag{10}$$

where

$$\lambda(t) = \begin{cases} C_2 & \text{if } t \in E, \\ -C_3 & \text{if } t \notin E, \end{cases}$$

and the constants C_1, C_2 and C_3 are respectively equal to

$$\frac{\sqrt{m}K||PB||}{\lambda_{min}^{1/2}} , \frac{1 + 4\sqrt{m}K||B|| \, ||PB||}{\lambda_{min}} \text{ and } \frac{1}{2\lambda_{max}} .$$

Let $\Lambda(t) = \int_0^t \lambda(s)ds$. From (10), if $W(t) = e^{-2\Lambda(t)}V\left(z(t)\right)$, we obtain

$$\dot{W}(t) \leq 2C_1 W^{1/2}(t)e^{-\Lambda(t)}||h(t)||,$$

and then

$$W^{1/2}(t) \leq C_1 \int_0^t e^{-\Lambda(s)}||h(s)||ds ,$$

which gives

$$V^{1/2}(z(t)) \leq C_1 \int_0^t e^{(\Lambda(t) - \Lambda(s))}||h(s)||ds .$$

But for $z \in \mathbb{R}^n$, $V^{1/2}(z) \geq \lambda_{min}^{1/2}||z||$, and if $t \geq s$, $\Lambda(t) - \Lambda(s) \leq (C_2 + C_3)|E| - C_3(t - s)$. Therefore, we have

$$||z(t)|| \leq C_4 \int_0^t e^{-C_3(t-s)}||h(s)||ds , \tag{11}$$

where $C_4 = \frac{C_1 e^{(C_2+C_3)|E|}}{\lambda_{min}^{1/2}} \left(\leq \frac{C_1 e^{C(C_2+C_3)||u||_{L^p}^p}}{\lambda_{min}^{1/2}} \right)$. We conclude from the previous inequality that

$$||z||_{L^p} \leq \Gamma(||u||_{L^p})||h||_{L^p} ,$$

for some $\Gamma(||u||_{L^p}) > 0$. Then the proof of (A) (ii) is complete.

Let us turn now to example (E). Let σ be a smooth non-decreasingsaturation function that satisfies the following condition. There exists a $\delta > 0$ such that $\sigma(t) = t$ if $|t| \leq \delta$ and $\sigma(t) = \text{sign}(t)$ if $|t| \geq 1 + \delta$. Consider the 1-dimensional system

$$\begin{aligned} \dot{x} &= -\sigma(x + u) , \\ x(0) &= 0 . \end{aligned} \tag{12}$$

Let $1 \leq p < \infty$ be a real number. Let $a > 1 + \delta$, $0 < \varepsilon < \delta$ be two real numbers. Take two inputs $u, v \in L^p([0, \infty), \mathbb{R})$ as follows:

$$\begin{aligned} u(t) &= v(t) = -t - 1 - \delta , \text{ if } 0 \leq t \leq a, \\ u(t) &= -a, \\ v(t) &= -\varepsilon(t - a) - a - \varepsilon , \text{ if } a < t \leq a + 1 , \\ u(t) &= v(t) = 0 , \text{ if } t > a + 1 . \end{aligned}$$

Let x, y be the solutions of (12) corresponding to u, v respectively. Then we have for $a \leq t \leq a + 1$,

$$x(t) = a, y(t) = a + \varepsilon(t - a).$$

and

$$x(t) = 2a + 1 - t, y(t) = 2a + \varepsilon + 1 - t$$

for $a + 1 \leq t \leq 2a - \delta$. Therefore

$$\begin{aligned} \int_0^\infty |y(s) - x(s)|^p ds &> \int_{a+1}^{2a-\delta} |y(s) - x(s)|^p ds \\ &= \varepsilon^p(a - 1 - \delta) . \end{aligned}$$

So, $||y - x||_{L^p} \geq \varepsilon(a - 1 - \delta)^{1/p}$. On the other hand,

$$||v - u||_{L^p} = \varepsilon \Big(\int_a^{a+1} |t - a + 1|^p dt \Big)^{1/p} = \varepsilon \Big(\frac{2^{p+1} - 1}{p + 1} \Big)^{\frac{1}{p}} .$$

Noticing that a and ε could be almost arbitrary, we have shown that for any α, $\beta > 0$, there exist u, v in $L^p([0, \infty), \mathbb{R})$ such that

$$||v - u||_{L^p} \leq \alpha \text{ and } ||F(v) - F(u)||_{L^p} \geq \beta .$$

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