Nonsmooth Control-Lyapunov Functions

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Abstract

It is shown that the existence of a continuous control-Lyapunov function (CLF) is necessary and sufficient for null asymptotic controllability of nonlinear finite-dimensional control systems. The CLF condition is expressed in terms of a concept of generalized derivative that has been studied in set-valued analysis and the theory of differential inclusions with various names such as "upper contingent derivative." This result generalizes to the non-smooth case the theorem of Artstein relating closed-loop feedback stabilization to smooth CLF's. It relies on viability theory as well as optimal control techniques. A "non-strict" version of the results, analogous to the LaSalle Invariance Principle, is also provided.

1. Introduction

We deal with systems of the general form

$$\dot{x}(t) \;=\; f(x(t), u(t)) \tag{1}$$

where the states $x(t)$ take values in a Euclidean space $\mathbb{X} = \mathbb{R}^n$, the controls $u(t)$ take values in a metric space U, and f is locally Lipschitz. A widely used technique for stabilization of this system to x=0 relies on the use of abstract "energy" or "cost" functions that can be made to decrease in directions corresponding to possible controls. This approach is based on having a "Lyapunov pair" (V, W), consisting of two positive definite functions V, W: $\mathbb{X} \to \mathbb{R}_{\geq 0}$, with V continuously differentiable and proper ("radially unbounded") and W continuous, so that for each state $\xi \in \mathbb{X}$ there is some control-value u=u_ξ with

$$D_{f(\xi,u)}V(\xi) \;\leq\; -W(\xi)\,. \tag{2}$$

Here, $D_vV(\xi) = \nabla V(\xi).v$ is the directional derivative of V in the direction of the vector v. This property guarantees that for each state ξ there is some control $u(\cdot)$ such that, solving the initial-value problem (1) with $x(0) = \xi$, the resulting trajectory satisfies $x(t)\to 0$ as $t\to +\infty$. The argument is standard; see for instance the textbook [9]. A function V which is part of a Lyapunov pair is generically called a *control-Lyapunov function*, henceforth abbreviated "CLF."

The CLF paradigm is extremely powerful. It suggests the search for stabilizing inputs by iteratively solving a static nonlinear programming problem: when at state ξ, find u such that Equation (2) holds. The idea underlies feedback control design (see the references in [10], and e.g. the many examples in Section 3.6 of the textbook [8]), the optimal control approach of Bellman, "artificial intelligence" techniques based on position evaluations in games and "critics" in learning programs, and can be found in "neural-network" control design (see e.g. [6]). An obvious fundamental question arises: is the existence of a continuously differentiable CLF *equivalent* to the possibility of driving every state asymptotically to zero?

If the question is stated in this form, then it is well-known that the answer is negative. For instance, if controls are in $\mathbb{R}^m$, and $f(x, u) = f_0 + \sum_{i=1}^m u_i f_i(x)$ is affine in u, the existence of a CLF would imply that there is some feedback law u=$k(x)$ so that the origin is a globally asymptotically stable state for the closed-loop system $\dot{x} = f(x, k(x))$ and k is continuous on $\mathbb{R}^n \setminus \{0\}$. (This was proved by Artstein in [1]; cf. also [4, 7, 11]). But continuous feedback may fail to exist, even for very simple controllable systems (see e.g. [9], Section 4.8).

The first main result of this paper —Theorem 1— says that the question has a positive answer provided that we relax the differentiability assumption on V to merely continuity and we re-interpret the directional derivative appearing in Equation (2) as a *generalized* directional derivative, well-known in the literature of Set-Valued Analysis and Differential Inclusions, under various names such as the "upper contingent derivative," or "contingent epiderivative". (For technical reasons, one must allow derivatives in directions in the closed convex hull of the velocity set $f(\xi, U)$.) In §2 we present the precise definitions of asymptotic controllability and CLF, and state and prove the equivalence between asymptotic controllability and existence of a continuous CLF in the sense of generalized derivatives. The proof follows immediately by combining the main result in [10], which gave a necessary condition expressed in terms of Dini derivatives of trajectories, with results from [2].

Thus, asymptotic controllability implies the existence of a "Lyapunov function" in the strict sense that derivatives are negative for nonzero states. In analogy with ordinary differential equations, one may ask when the existence of a "weak CLF," for which W is only required to be nonnegative, suffices for the converse. We answer this with a control theory version of the LaSalle Invariance Principle, stated in §3 — Theorems 2 and 3— and proved in §5, using two technical lemmas about the relationships between local and global decrease, given in §4.

*Supported in part by US Air Force Grant F49620-95-1-0101
†Supported in part by NSF Grant DMS92-02554
‡Full version of paper available by electronic mail.

2. Asymptotic Controllability and CLF's

Throughout this paper, we write $\mathbb{R}_{\geq 0}=\{r \in \mathbb{R} : r\geq 0\}$, and use $\mathcal{I}$ to denote the set of all subintervals I of $\mathbb{R}_{\geq 0}$ such that $0\in I$; thus, $I\in\mathcal{I}$ iff either (i) $I=\mathbb{R}_{\geq 0}$, or (ii) $I=[0,a)$ for some $a>0$, or (iii) $I=[0,a]$ for some $a\geq 0$. If μ is a map, we will use $\mathcal{D}(\mu)$ to denote the domain of μ, and $\mu|S$ to denote the restriction of μ to a subset S of $\mathcal{D}(\mu)$. For any subset S of $\mathbb{R}^n$, we use $\overline{\text{co}}(S)$ to denote the closed convex hull of S.

We consider systems as in (1) and assume that a distinguished element called "0" has been chosen in the metric space U. We let U_ρ denote, for each $\rho \geq 0$, the ball $\{u \,|\, d(u,0) \leq \rho\}$, and assume also that each set U_ρ is compact. (Typically, U is a closed subset of a Euclidean space $\mathbb{R}^m$ and 0 is the origin.) The map $f : \mathbb{X}\times U \to \mathbb{R}^n$ is assumed to be locally Lipschitz with respect to (x,u) and to satisfy $f(0,0) = 0$. (The Lipschitz property with respect to u can be weakened, but we will need to quote results from [10], where this was made as a blanket assumption.) A *control* is a bounded measurable map $u : I_u \to U$, where $I_u \in \mathcal{I}$. We use $\|u\|$ to denote the essential supremum norm of u. i.e.

$$\|u\| = \inf\{\rho \,|\, u(t) \in U_\rho \text{ for almost all } t \in I_u\}\,.$$

To avoid confusion with the sup norm of the controls, we will use $|\xi|$ to denote the Euclidean norm of vectors ξ in the state space $\mathbb{X}$.

We let $\mathbf{S}$ denote the class of all systems (1) that satisfy the above conditions. For a system in $\mathbf{S}$, if $\xi \in \mathbb{X}$ and u is a control u, we let $\phi(t,\xi,u)$ denote the value at time t of the maximally defined solution $x(\cdot)$ of (1) with initial condition $x(0) = \xi$. Then $\phi(t,\xi,u)$ is defined for t in some relatively open subinterval J of I_u containing 0, and either $J = I_u$ or $\lim_{t\to\sup J} |\phi(t,\xi,u)| = +\infty$.

The next definition expresses the requirement that for each state ξ there should be some control driving ξ asymptotically to the origin. As for asymptotic stability of unforced systems, we require that if ξ is already close to the origin then convergence is possible without a large excursion. In addition, for technical reasons, we rule out the unnatural case in which controlling small states requires unbounded controls.

Definition 2.1 The system (1) is *(null-)asymptotically controllable* (henceforth abbreviated "AC") if there exist nondecreasing functions $\theta, \tilde{\theta} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\lim_{r\to 0+} \tilde{\theta}(r) = 0$, with the property that:

- For each $\xi \in \mathbb{X}$ there exist a control $u : \mathbb{R}_{\geq 0} \to U$ and corresponding trajectory $x(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{X}$ such that $x(0) = \xi$, $x(t) \to 0$ as $t \to +\infty$, $\|u\| \leq \theta(|\xi|)$, and $\sup\{|x(t)| : 0 \leq t < \infty\} \leq \tilde{\theta}(|\xi|)$.

Remark 2.2 A routine argument involving continuity of trajectories with respect to initial states shows that the requirements of the above definition are equivalent to the following much weaker pair of conditions:

1. For each $\xi \in \mathbb{X}$ there is a control $u : \mathbb{R}_{\geq 0} \to U$ that drives ξ asymptotically to 0 (i.e. $x(t) := \phi(t,\xi,u)$ is defined for all $t \geq 0$ and $x(t) \to 0$ as $t \to +\infty$);
2. there exists $\rho > 0$ such that for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each $\xi \in \mathbb{X}$ with $|\xi| \leq \delta$ there is a control $u : \mathbb{R}_{\geq 0} \to U_\rho$ that drives ξ asymptotically to 0 and is such that $|\phi(t,\xi,u)| < \varepsilon$ for all $t \geq 0$.

We point out, however, that *Definition 2.1, as stated, makes sense even for the more general class* $\mathbf{S}^*$ *of systems (1) in which f is completely arbitrary (i.e. not necessarily locally Lipschitz or even continuous), and the set of control values is state-dependent, i.e. an additional requirement* $u \in \hat{U}(x)$ *is imposed, where* $\hat{U} : \mathbb{X} \to 2^U$ *is a multifunction with values subsets of U. This includes in particular the situation when* $U = \mathbb{X}$ *and* $f(x,u) = u$, *in which case the system (1) is a differential inclusion* $\dot{x} \in F(x)$. On the other hand, *the formulation in terms of Conditions 1 and 2 above does not make sense for general systems in* $\mathbf{S}^*$ (since $\phi(t,\xi,u)$ need not be well defined), and *the equivalence between the two formulations depends on the fact that each fixed control gives rise to a flow*, which is true for systems in $\mathbf{S}$ but not for systems in $\mathbf{S}^*$.

Throughout the paper, *systems of the form (1) are assumed to be in* $\mathbf{S}$ *unlesss otherwise stated*, so we will use indistinctly the two forms of the definition of AC. The only exception is the end of Section 4, where we will want to compare systems in $\mathbf{S}$ with differential inclusions —which belong to $\mathbf{S}^*$ but not necessarily to $\mathbf{S}$— so we will have to use Definition 2.1 as stated rather than Conditions 1 and 2.

We now introduce an object widely studied in Set-Valued Analysis (cf., for instance, [2], Def. 1 and Prop. 1 of Section 6.1, where it is called the "upper contingent derivative.")

Definition 2.3 For a function $V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, a $\xi \in \mathbb{R}^n$ such that $F(\xi) < +\infty$, and a $v \in \mathbb{R}^n$, the *directional subderivative of* V *in the direction of* v at ξ is

$$D_v^- V(\xi) \;:=\; \liminf_{\substack{t \to 0+ \\ w \to v}} \frac{1}{t}\left[V(\xi + tw) - V(\xi)\right]\;.$$

(The notations $D_+V(\xi)(v)$ and $D_\uparrow V(\xi)(v)$ are used in [2, 5] and [3] respectively, with the same meaning as our $D_v^- V(\xi)$.)

For each fixed ξ, the map $v\mapsto D_v^- V(\xi)$ is lower semicontinuous as an extended-real valued function (cf. [2], page 286); thus $\{v | D_v^- V(\xi) \leq \alpha\}$ is a closed set for any α. Observe that if V is Lipschitz continuous then this definition coincides with that of the classical Dini derivative, that is, $\liminf_{t\to 0+}[V(\xi + tv) - V(\xi)]/t$. However, in our results we will not assume that V is Lipschitz, so this simplification is not possible. Notice also that in the Lipschitz case $D_v^- V(\xi)$ is automatically finite, but for a general function V with finite values it can perfectly well happen that $D_v^- V(\xi) = +\infty$ or $D_v^- V(\xi) = -\infty$. Naturally, $D_v^- V(\xi)$ is the usual directional derivative $\nabla V(\xi).v$ if V is differentiable at ξ.

We are now ready to define what it means for a function V to be a CLF. Essentially, we want the directional derivative $D_v^- V(\xi)$ in some —ξ-dependent— control direction v to be negative for each nonzero state ξ. More precisely, we will require $D_v^- V(\xi)$ to be bounded above by a negative function of the state and, in the nonconvex case, we will allow v to belong to the convex closure of the set of control directions.

A function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ is *positive definite* if $V(0) = 0$ and $V(\xi) > 0$ for $\xi \neq 0$, and *proper* if $V(\xi) \to \infty$ as $|\xi| \to \infty$.

Definition 2.4 A *Lyapunov pair* for the system (1) is a pair (V, W) consisting of a continuous, positive definite, proper function $V : \mathbb{X} \to \mathbb{R}$ and a nonnegative continuous function $W : \mathbb{X} \to \mathbb{R}$, for which there exists a nondecreasing $\nu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with the property that for each $\xi \in \mathbb{X}$ there is a $v \in \overline{\text{co}}(f(\xi, U_{\nu(|\xi|)}))$ such that

$$D_v^- V(\xi) \;\leq\; -W(\xi)\;. \tag{3}$$

Remark 2.5 For the special but very common case when the set of velocities $f(\xi, U_\rho)$ is convex for all ρ (for example if U is a closed convex subset of $\mathbb{R}^m$ and the system (1) is affine in the control), the condition of Definition 2.4 reduces to asking that for each $\xi \neq 0$ there be some control value $u \in U_{\nu(|\xi|)}$ such that $D^-_{f(\xi,u)} V(\xi) \leq -W(\xi)$. If V is differentiable at ξ, then this amounts to requiring that $\min_{u \in U_{\nu(|\xi|)}} [\nabla V(\xi) f(\xi, u)] \leq -W(\xi)$.

Definition 2.6 A *control-Lyapunov function (CLF)* for the system (1) is a function $V : \mathbb{X} \to \mathbb{R}$ such that there exists a continuous positive definite $W : \mathbb{X} \to \mathbb{R}$ with the property that (V, W) is a Lyapunov pair for (1).

Our first main result is as follows:

Theorem 1 *A system Σ of the form (1) is AC if and only if it admits a CLF.*

We prove Theorem 1 in the rest of this section. In the next section, we provide a far more general set of results dealing with Lyapunov pairs for which W is not necessarily positive definite.

2.1. A Previous Result with Relaxed Controls

We first recall the standard notion of relaxed control. If $\rho \geq 0$, a *relaxed U_ρ-valued control* is a measurable map $u : I_u \to \mathbb{P}(U_\rho)$, where $I_u \in \mathcal{I}$ and $\mathbb{P}(U_\rho)$ denotes the set of all Borel probability measures on U_ρ. An ordinary control $t \mapsto u(t)$ can be regarded as a relaxed control in the usual way, using the embedding of the space U_ρ into $\mathbb{P}(U_\rho)$ that assigns to each $w \in U_\rho$ the Dirac Delta measure at w. For $u \in \mathbb{P}(U_\rho)$, we write $f(x, u)$ for $\int_{U_\rho} f(x, w)\, du(w)$. As for ordinary controls, we also use the notation $\phi(t, \xi, u)$ for the solution of the initial value problem that obtains from initial state ξ and relaxed control u, and we denote $\|u\| = \inf\{\rho \,|\, u(t) \in \mathbb{P}(U_\rho)$ for almost all $t \in I_u\}$. The first ingredient in the proof is the following restatement of the main result in [10].

Fact 2.7 A system Σ of the form (1) is AC if and only if there exist two continuous, positive definite functions V, W: $\mathbb{X} \to \mathbb{R}$, V proper, and a nondecreasing ν: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ so that the following property holds: for each $\xi \in \mathbb{X}$ there are a $T > 0$ and a relaxed control ω: $[0, T) \to \mathbb{P}(U_{\nu(|\xi|)})$, so that $x(t) := \phi(t, \xi, \omega)$ is defined for all $0 \leq t < T$ and

$$V(x(t)) - V(\xi) \leq -\int_0^t W(x(\tau))\, d\tau \;\text{ for }\; t \in [0, T). \tag{4}$$

Proof. If there are such V, W, and ν, then for each ξ we may pick a ω so that (4) holds; this implies the inequality $\liminf_{t \to 0+} t^{-1}[V(x(t)) - V(\xi)] \leq -W(\xi)$, which is the sufficient condition for AC given in [10]. Conversely, if the system is AC, then that reference shows that there exist V, W, and ν as above and such that

$$V(\xi) = \min \left\{ \int_0^\infty W(\phi(\tau, \xi, \omega))\, d\tau \;+\; \max\{\|\omega\| - k, 0\} \right\}$$

where the minimum is taken over the set of all relaxed controls ω: $[0, \infty) \to \mathbb{P}(U_{\nu(|\xi|)})$, and k is a constant which arises from the function θ in the definition of AC. (Here we take $W(x) = N(|x|)$, where N: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ from [10] is a strictly increasing, continuous function satisfying also $N(0) = 0$ and $\lim_{r \to +\infty} N(r) = +\infty$, i.e. a function of class $\mathcal{K}_\infty$. The main point of the proof was to construct an N so that the value function V is continuous and for which optimal controls exist.)

This implies property (4), in fact even with $T = +\infty$. Indeed, pick ξ and a minimizing ω. Let $x(\cdot) := \phi(\cdot, \xi, \omega)$ and pick any $t \geq 0$. We may consider the new initial state $x(t)$ and the control $\widetilde{\omega}$ obtained by restricting ω to the interval $[t, \infty)$. Then $V(x(t))$ is bounded above by the cost when using $\widetilde{\omega}$, that is, $V(x(t)) \leq \int_t^\infty W(x(\tau)) d\tau + \max\{\|\widetilde{\omega}\| - k, 0\} \leq \int_t^\infty W(x(\tau)) d\tau + \max\{\|\omega\| - k, 0\} = V(\xi) - \int_0^t W(x(\tau)) d\tau$. ∎

2.2. A Previous Result on Differential Inclusions

Next we recall some concepts from set-valued analysis. We consider set-valued maps (or "multifunctions") between two Hausdorff topological spaces X and Y. A map F from X to subsets of Y is *upper semicontinuous* (abbreviated USC) if for each open subset $V \subseteq Y$ the set $\{x \,|\, F(x) \subseteq V\}$ is open. If U is a compact topological space and $f : X \times U \to Y$ is continuous, then the set valued map $F(x) := F(x, U) = \{f(x, u), u \in U\}$ is USC (see for instance [2], Prop. 1 in Section 1.2).

We will henceforth use the abbreviations *DI* and *USCMCC* for "differential inclusion" and "upper semicontinuous multifunction with compact convex values," respectively.

Let X be a subset of $Y = \mathbb{R}^n$. A *solution* of the DI $\dot{x} \in F(x)$ is by definition a locally absolutely continuous curve $x(\cdot) : I \to X$, where I is an interval, such that $\dot{x}(t) \in F(x(t))$ for almost all $t \in I$.

The second ingredient needed to prove Theorem 1 is from the literature on differential inclusions and viability theory. The relevant results are as follows. (We give them in a slightly stronger form than needed, but still not in full generality: in [2], the function "W" is allowed to depend convexly on derivatives $\dot{x}(t)$, and in some implications less than continuity of V or W is required.) Theorem 1 in Section 6.3 of [2] shows that **2** implies **1** (with $T{=}\infty$ if X is closed and $F(X)$ is bounded), and Proposition 2 in Section 6.3 of [2] says that **1**⇒**2**. (Another good reference is [5]; see in particular Theorem 14.1 there.)

Fact 2.8 Let F be an USCMCC from X into subsets of $\mathbb{R}^n$, where X is a locally compact subset of $\mathbb{R}^n$. Assume that V and W are two continuous functions $X{\to}\mathbb{R}_{\geq 0}$. Let $\tilde{V} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be such that $\tilde{V} \equiv V$ on X, $\tilde{V} \equiv +\infty$ on $\mathbb{R}^n \backslash X$. Then the following properties are equivalent:
1. For each $\xi \in X$ there are a $T > 0$ and a solution of $\dot{x}(t) \in F(x(t))$ defined on $[0, T)$ with $x(0) = \xi$ which is monotone with respect to V and W, that is,

$$V(x(t)) - V(x(s)) + \int_s^t W(x(\tau))d\tau \;\leq\; 0 \qquad (5)$$

for all $0 \leq s \leq t < T$.
2. For each $\xi \in X$ there is some $v \in F(\xi)$ such that $D_v^- \tilde{V}(\xi) \leq -W(\xi)$.

Moreover, if X is closed and $F(X) = \bigcup_{x\in X} F(x)$ is bounded, then one can pick $T = +\infty$ in **2**.

2.3. Proof of Theorem 1

Let Σ be a system of the form (1). Assume that Σ is AC. We apply Fact 2.7, and obtain V, W, and ν. Pick $\xi{\in}\mathbb{X}$. Let T, ω, $x(\cdot)$ be as in Fact 2.7. Then $x(t) - \xi = \int_0^t f(x(s), \omega(s))ds = \int_0^t f(\xi, \omega(s))ds + o(t) \in t.\overline{\text{co}}(f(\xi, U_{\nu(|\xi|)})) + o(t)$. So there is a sequence $\{t_j\}$ such that $t_j > 0$ and $t_j \to 0$, with the property that, if $v_j = t_j^{-1}(x(t_j) - \xi)$, then $v_j \to v$ for some $v \in \overline{\text{co}}(f(\xi, U_{\nu(|\xi|)}))$. On the other hand, (4) implies that $\liminf t_j^{-1}(V(\xi + t_jv_j) - V(\xi)) \leq -W(\xi)$. So $D_v^- V(\xi) \leq -W(\xi)$. Therefore (V, W) is a Lyapunov pair.

Conversely, assume that (V, W) is a Lyapunov pair with W continuous and positive definite, and let ν be as in the definition of Lyapunov pair. For $\xi \in \mathbb{X}$, let X_ξ be the sublevel set $\{x \,|\, V(x) \leq V(\xi)\}$, and write $\hat{\nu}(\xi) = \nu(r(\xi))$, where $r(\xi) = \sup\{|x| : x \in X_\xi\}$. Then let $\hat{\nu}(s) = \sup\{\tilde{\nu}(\xi) : |\xi| \leq s\}$ for $s \geq 0$. For $x \in X_\xi$, define $F_\xi(x) := \overline{\text{co}}(f(x, U_{\hat{\nu}(|\xi|)}))$, and let $\tilde{V}_\xi(x) = V(x)$ for $x \in X_\xi$, $\tilde{V}_\xi(x) = +\infty$ for $x \notin X_\xi$. Then it is clear that F_ξ is an USCMCC. If $x \in \mathbb{X}_\xi$, then Def. 2.4 implies that there is a $v \in \overline{\text{co}}(f(x, U_{\nu(|x|)}))$ such that $D_v^- V(x) \leq -W(x)$. Since $|x| \leq r(\xi)$, we have $\nu(|x|) \leq \tilde{\nu}(\xi) \leq \hat{\nu}(|\xi|)$. So v belongs to $F_\xi(x)$. If $v_j \to v$, $t_j > 0$, $t_j \to 0$, and $t_j^{-1}(V(x + t_jv_j) - V(x)) \to w \leq -W(x)$, then $V(x{+}t_jv_j)$ must be finite for all large j. Therefore $V(x + t_jv_j) = \tilde{V}_\xi(x + t_jv_j)$ for large j. So $D_v^- \tilde{V}_\xi(x) \leq -W(x)$. This shows that Condition **2** of Fact 2.8 holds with $X = X_\xi$, $F = F_\xi$, and $V_\xi = V|X$ in the role of V. Fact 2.8 —together with standard measurable selection theorems— then implies that there is a control $\omega : [0, +\infty) \to \mathbb{P}(U_{\hat{\nu}(\xi)})$ such that Equation (4) holds with $T = +\infty$, $x(t) = \phi(t, x, \omega)$. Since this is true for every ξ, we see that the condition of Fact 2.7 holds (with $\hat{\nu}$ in the role of ν), so Σ is AC.

Remark 2.9 The proof actually shows that in the AC case one has trajectories, corresponding to relaxed controls, which are monotone with respect to V and W, and are defined on the entire $[0, +\infty)$. (Observe that the cost function used in [10] is not additive, because of the term "$\max\{\|\omega\|{-}k, 0\}$", so the dynamic programming principle does not apply, and hence we cannot conclude that *optimal* trajectories are monotone. If desired, this situation could be remedied by redefining the optimal control problem as follows: drop the term $\max\{\|\omega\|{-}k, 0\}$ but instead add a state-dependent control constraint forcing $u(t)$ to be bounded by $\theta(x(t))$.) □

3. Non-Strict Lyapunov Functions

We now state two theorems that, together, generalize LaSalle's Invariance Theorem to our control situation. This requires that we first define the concept of a DBCBP ("decreasing with bounded control and bounded peaking") Lyapunov pair.

Definition 3.1 A Lyapunov pair (V, W) for the system (1) is *DBCBP* if there are nondecreasing functions σ: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, τ: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that $\lim_{s\to 0+} \tau(s) = 0$, and

- For each nonzero $\xi{\in}\mathbb{X}$ such that $W(\xi){=}0$, there exist an $a{>}0$ and a control u: $[0, a]{\to}U$ such that $\|u\|{\leq}\sigma(|\xi|)$, $V(\phi(a, \xi, u)){<}V(\xi)$, and $V(\phi(t, \xi, u)){\leq}\tau(V(\xi))$ for $0{\leq}t{\leq}a$.

A *weak control-Lyapunov function (WCLF)* for the system (1) is a function V such that there exists W for which (V, W) is a DBCBP Lyapunov pair. □

(Notice that (V, W) is automatically DBCBP if W is positive definite. Therefore, every CLF is a WCLF.)

Theorem 2 *If (V, W) is a Lyapunov pair for a system Σ of the form (1), then Σ is AC if and only if (V, W) is DBCBP.*

Theorem 3 *The following three conditions are equivalent for a system Σ of the form (1): (i) Σ is AC, (ii) Σ admits a CLF, (iii) Σ admits a WCLF.*

Of course, the equivalence between (i) and (ii) was the content of Theorem 1.

4. Some Technical Lemmas and Remarks

We now state —without proof— two key lemmas that will be used to derive Theorems 2 and 3, and make some observations on the precise reasons for our various technical assumptions. The lemmas deal with the question whether "local decrease implies global decrease," i.e. whether it is true, given a family $\mathcal{A}$ of arcs in a space $\mathbb{X}$ and a proper nonnegative "height" function V on $\mathbb{X}$, that if from every $\xi \in \mathbb{X}$ with positive height one can "go down" by some positive amount by following arcs in $\mathcal{A}$, then one can actually approach the bottom, i.e. find for each ξ an $\mathcal{A}$-trajectory from ξ that approaches the set $K = V^{-1}(0)$. We discuss this question in a more general abstract setting than that of our control theory situation, where the results will eventually be applied. Wo do this so as to better understand exactly why the technical hypotheses of this paper are needed for all the parts of the theory (that is, the lemmas of this section and Fact 2.7) to work simultaneously. It turns out that

1. Fact 2.7 ("infinitesimal implies local") requires that we take $\mathbb{X} \subseteq \mathbb{R}^n$, but applies to a general DI defined by an USCMCC.

2. The lemmas of this section, on the other hand, say that "local implies global" in a fairly abstract setting, where $\mathbb{X}$ is a metric space and $\mathcal{A}$ is a set of arcs closed under concatenations and time translations, *provided that $\mathcal{A}$ has a "continuity property"* defined below. This property holds in particular for control systems, but can fail for more general DI's arising from USCMCC's. In Remark 4.3, we give an example of such a DI for which local decrease does not imply global decrease.

3. The existence of a *continuous* strict Lyapunov function V also depends crucially on the fact that we are dealing with control systems, as shown by Remark 4.4, where we exhibit an AC DI arising from an USCMCC for which such a V does not exist.

4. Great care is needed in the precise choice of definitions, because some things that may appear obvious are actually false. For example, M. Ortel has shown that if a smooth, positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ is such that for every $\xi \neq 0$ there is a smooth curve $x : [0, a] \to \mathbb{R}^n$ such that $a > 0$, $x(0) = \xi$, and V is strictly decreasing along x, then it can happen that there are points ξ from which there exists no continuous curve approaching 0 along which V is nonincreasing. (Ortel also proved that the above situation cannot arise if V is real analytic.) This shows that, even when V has the "strict local decrease property" (cf. below), one may only be able to approach 0 via trajectories along which V is not monotonic.

To state our lemmas, we first need some definitions. An *arc* in a metric space $\mathbb{X}$ is a continuous curve γ in $\mathbb{X}$ defined on a compact interval. If $\gamma : [a, b] \to \mathbb{X}$ is an arc, and $x = \gamma(a)$, $y = \gamma(b)$, then we say that γ *goes from x to y*. The *concatenation* $\gamma_1 * \gamma_2$ of two arcs $\gamma_j : [a_j, b_j] \to \mathbb{X}$, $j = 1, 2$, is defined if $b_1 = a_2$ and $\gamma_1(b_1) = \gamma_2(a_2)$, and in that case $\gamma_1 * \gamma_2$ is the arc such that $\mathrm{Graph}(\gamma_1 * \gamma_2) = \mathrm{Graph}(\gamma_1) \cup \mathrm{Graph}(\gamma_2)$. If $t \in \mathbb{R}$, the *time t translate* of an arc $\gamma : [a, b] \to \mathbb{X}$ is the arc $\tau_t(\gamma)$ with domain $[a - t, b - t]$, defined by $\tau_t(\gamma)(s) = \gamma(t + s)$. If $\mathcal{A}$ is a set of arcs, we call $\mathcal{A}$ *closed under concatenations* if $\gamma_1 * \gamma_2 \in \mathcal{A}$ whenever $\gamma_j \in \mathcal{A}$, $j = 1, 2$, are such that $\gamma_1 * \gamma_2$ is defined. We say that $\mathcal{A}$ is *closed under time translations* if $\tau_t(\gamma) \in \mathcal{A}$ whenever $\gamma \in \mathcal{A}$ and $t \in \mathbb{R}$. We say that $\mathcal{A}$ has the *continuity property* if, whenever $\gamma : [a, b] \to \mathbb{X}$ is in $\mathcal{A}$, U is an open set containing $\gamma([a, b])$, and W is a neighborhood of $\gamma(b)$, there exists a neighborhood Z of $\gamma(a)$ such that for every $z \in Z$ there is a $\delta \in \mathcal{A}$ —possibly with a different domain— that goes from z to a point in W and whose image is entirely contained in U.

If $\mathcal{A}$ is a set of arcs in a metric space $\mathbb{X}$, a *forward $\mathcal{A}$-trajectory* is a continuous curve $\gamma : I \to \mathbb{X}$, defined on an interval $I \in \mathcal{I}$, with the property that for every $a \in I$ there exists a $b \in I$ such that $a \leq b$ and $\gamma|[0, b] \in \mathcal{A}$. (Notice that if I itself is compact, this just says that $\gamma \in \mathcal{A}$. If I is of the form $[0, L)$ —with $L \leq +\infty$— then $\gamma \notin \mathcal{A}$, but there has to exist a sequence $\{L_j\}$ such that $L_j < L$, $L_j \to L$, and $\gamma|[0, L_j] \in \mathcal{A}$ for each j.)

An *arc system* is a pair $(\mathbb{X}, \mathcal{A})$ such that $\mathbb{X}$ is a metric space and $\mathcal{A}$ is a set of arcs in $\mathbb{X}$ which is closed under concatenations and time translations.

If K is a compact subset of $\mathbb{X}$, we write $d_K(x) = \min\{\mathrm{dist}(x, y) : y \in K\}$. If $0 < L \leq +\infty$, then the *basin of L-attraction* of K for $(\mathbb{X}, \mathcal{A})$ is the set $\mathcal{B}_L(K)$ of all points $\xi \in \mathbb{X}$ with the property that there exists a forward $\mathcal{A}$-trajectory $\gamma : I \to \mathbb{X}$ such that $\gamma(0) = \xi$, $\lim_{t \to \sup I}(d_K(\gamma(t))) = 0$, and $d_K(\gamma(t)) < L$ for all $t \in I$. We just write $\mathcal{B}(K)$ for $\mathcal{B}_{+\infty}(K)$, and call $\mathcal{B}(K)$ the *basin of attraction* of K for $(\mathbb{X}, \mathcal{A})$.

We call K an *asymptotically stable attractor* (ASA) of the arc system $(\mathbb{X}, \mathcal{A})$ if

[ASA] for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\{x : d_K(x) \leq \delta\} \subseteq \mathcal{B}_\varepsilon(K)$.

An ASA K of $(\mathbb{X}, \mathcal{A})$ is a *globally asymptotically stable attractor* (GASA) of $(\mathbb{X}, \mathcal{A})$ if $\mathcal{B}(K) = \mathbb{X}$.

If $\mathbb{X}$ is a metric space, a *function of Lyapunov type* on $\mathbb{X}$ is a nonnegative, continuous function $V : \mathbb{X} \to \mathbb{R}$ such that $\{x : V(x) \leq L\}$ is compact for all $L \in \mathbb{R}$.

We now consider a triple $(\mathbb{X}, \mathcal{A}, V)$ such that $(\mathbb{X}, \mathcal{A})$ is an arc system and V is a function of Lyapunov type on $\mathbb{X}$. We write $K = \{x : V(x) = 0\}$, and seek conditions in terms of V for K to be an ASA of $(\mathbb{X}, \mathcal{A})$.

Let us say that V has the *weak local decrease property* (WLDP) along $\mathcal{A}$ from a point $x \in \mathbb{X}$ if there exist $a > 0$ and $\gamma \in \mathcal{A}$ such that $\mathcal{D}(\gamma) = [0, a]$, $\gamma(0) = x$ and $V(\gamma(a)) < V(x)$. If in addition γ can be chosen so that the function $t \mapsto V(\gamma(t))$ is nonincreasing (resp. strictly decreasing) on $[0, a]$, then we say that V has the *strong* (resp. *strict*) *local decrease property* from x along $\mathcal{A}$.

With $\mathbb{X}$, $\mathcal{A}$, V as above, write $\mathbb{X}_L^V = \{x \in \mathbb{X} : V(x) \leq L\}$, $\mathcal{A}_L^V = \{\gamma \in \mathcal{A} : V(\gamma(t)) < L \text{ for all } t \in \mathcal{D}(\gamma)\}$.

Lemma 4.1 *Let $(\mathbb{X}, \mathcal{A})$ be an arc system, let V be a function of Lyapunov type on $\mathbb{X}$, and write $K = V^{-1}(0)$. Assume that $\mathcal{A}$ has the continuity property. Suppose that $0 < L < \Lambda \leq \infty$ are such that V has the WLDP along $\mathcal{A}_\Lambda^V$ from every $x \in \mathbb{X}_L^V \backslash K$. Then for every $x \in \mathbb{X}_L^V \backslash K$ and every $\tilde{L} > 0$ there exist $a > 0$ and $\gamma \in \mathcal{A}_\Lambda^V$ such that $\mathcal{D}(\gamma) = [0, a]$, $\gamma(0) = x$ and $V(\gamma(a)) \leq \tilde{L}$.*

We will say that the triple $(\mathbb{X}, \mathcal{A}, V)$ has the *D−stability property* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, whenever $x \in \mathbb{X}$ and $V(x) \le \delta$, it follows that there exist $a > 0$ and an arc $\gamma : [0, a] \to \mathbb{X}$ in $\mathcal{A}$ such that $\gamma(0) = x$, $V(\gamma(a)) < V(x)$, and $V(\gamma(t)) < \varepsilon$ for all $t \in [0, a]$. (The "D" stands for "decreasing": D-stability is essentially neutral stability with the extra proviso that V can actually be made to decrease strictly without ever exceeding the bound $V(\xi) < \varepsilon$.)

Lemma 4.2 *Let $(\mathbb{X}, \mathcal{A})$ be an arc system, let V be a function of Lyapunov type on $\mathbb{X}$, and let $K = \{x \in \mathbb{X} : V(x) = 0\}$. Assume that $\mathcal{A}$ has the continuity property. Then: (i) if $0 < L < \infty$, then K is an ASA of $(\mathbb{X}, \mathcal{A})$ such that $\mathbb{X}_L^V \subseteq \mathcal{B}(K)$ if and only if $(\mathbb{X}, \mathcal{A}, V)$ has the D-stability property and V has the WLDP along $\mathcal{A}$ from every $x \in \mathbb{X}_L^V \backslash K$; (ii) K is an ASA of $(\mathbb{X}, \mathcal{A})$ if and only if $(\mathbb{X}, \mathcal{A}, V)$ has the D-stability property and there exists $L > 0$ such that V has the WLDP along $\mathcal{A}$ from every $x \in \mathbb{X}_L^V \backslash K$; (iii) K is a GASA of $(\mathbb{X}, \mathcal{A})$ if and only if $(\mathbb{X}, \mathcal{A}, V)$ has the D-stability property and V has the WLDP along $\mathcal{A}$ from every $x \in \mathbb{X} \backslash K$; (iv) if V has the strong LDP along $\mathcal{A}$ from x for every $x \in \mathbb{X} \backslash K$, then K is a GASA of $(\mathbb{X}, \mathcal{A})$.*

Remark 4.3 In the above lemmas, the continuity property of $\mathcal{A}$ is essential. In particular, *Conclusion (iv) of the Lemma 4.2 can fail if $\mathcal{A}$ is the set of arcs that are solutions of a DI $\dot{\xi} \in F(\xi)$ defined by an USCMCC F.*

For an example of this, we define a set-valued function F from $\mathbb{R}^2$ to $\mathbb{R}^2$ as follows. Let $E_+ \subseteq \mathbb{R}$ be the union of the intervals $[k + 2^{-2m-1}, k + 2^{-2m}]$ for k, m nonnegative integers, and define $E_- \subseteq \mathbb{R}$ to be the union of the intervals $[-k - 2^{-2m-1}, -k - 2^{-2m-2}]$, also for k, m nonnegative integers. For $(x, y) \in \mathbb{R}^2$, let $f(x, y)$ be the vector $(y, -x)$. Define $F(x, y) = \{f(x, y)\}$ if either (i) $y \ne 0$ or (ii) $y = 0$, $x \ge 0$, $x \notin E_+$, or (iii) $y = 0$, $x \le 0$, $x \notin E_-$. Also, let $F(x, y)$ be the convex hull of $f(x, y)$ and $(-1, 0)$ if $y = 0$, $x \in E_+$, and $F(x, y) = \mathrm{co}(f(x, y), (1, 0))$ if $y = 0$, $x \in E_-$. Then F is an USCMCC. Let $\mathcal{A}$ be the set of all arcs that are solutions of $\dot{\xi} \in F(\xi)$. Let $V(x, y) = x^2 + y^2$. Then V has the strong LDP along $\mathcal{A}$ from every point, but there is no forward $\mathcal{A}$-trajectory approaching the origin from *any* point $\xi \in \mathbb{R}^2$ such that $|\xi| > 1$. (In fact, there are no finite-length trajectories going from any point with $|\xi| > 1$ to any point with $|\xi| < 1$.) □

Remark 4.4 We conclude this section by comparing our results for systems in **S** with the corresponding facts for DI's arising from USCMCC's.

Notice first that, as explained in Remark 2.2, the systems corresponding to such DI's are in $\mathbf{S}^*$, so the concept of asymptotic controllability given by Def. 2.1 makes sense for them. Moreover, there are obvious definitions of CLF and WCLF in this case as well. It is easy to see that the implications $(ii) \Rightarrow (i) \Rightarrow (iii)$ of Theorem 3 still hold. (Indeed, the proof of the "if" part of Theorem 1 applies in this case as well, so $(ii) \Rightarrow (i)$. Definition 2.1 clearly implies that if a system is AC then the function $V(\xi) = |\xi|$ is a WCLF —with $W \equiv 0$— so $(i) \Rightarrow (iii)$.) We now show that *the implications $(iii) \Rightarrow (i)$ and $(i) \Rightarrow (ii)$ can fail.* In the example of Remark 4.3, V is a WCLF but the system is not AC, so $(iii) \Rightarrow (i)$ fails. So all we need is an example of an AC system for which there is no CLF. (It is proved in [5] that an AC DI arising from an USCMCC always has a *lower semicontinuous* "CLF." Our definition requires the CLF to be continuous.)

We let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (-y, x)$. Let $S = \mathbb{R}_{\ge 0} \times \{0\}$. Define an USCMCC F on $\mathbb{R}^2$ by letting $F(x, y) = \{f(x, y)\}$ if $(x, y) \notin S$, and $F(x, y) = \mathrm{co}(\{f(x, y), (-1, 0)\})$ if $(x, y) \in S$. Then for every $p \in \mathbb{R}^2$ we can construct a trajectory $\gamma_p : [0, T_p] \to \mathbb{R}^2$ of the DI $\dot{\xi} \in F(\xi)$ such that $\gamma(0) = p$, $\gamma(T_p) = 0$, and $t \mapsto |\gamma_p(t)|$ is nonincreasing. So our DI is AC. However, *there is no continuous function $V : \mathbb{R}^2 \to \mathbb{R}$ such that $\inf_{v \in F(\xi)} D_v^- V(\xi) < 0$ for all $\xi \ne 0$.* (Indeed, let V be such a function. Then $D_{f(x,y)}^- V(x, y) < 0$ if $(x, y) \notin S$. If $r > 0$, then Fact 2.8 —with $W \equiv 0$— easily implies that the function $[0, 2\pi] \ni t \mapsto h_r(t) = V(r \cos t, r \sin t)$ is nonincreasing on $(0, 2\pi)$. Since V is continuous, and $h_r(0) = h_r(2\pi)$, we conclude that h_r is constant. So V is in fact a radial function, i.e. $V(\xi) = \widehat{V}(|\xi|)$ for some continuous $\widehat{V} : \mathbb{R}_{\ge 0} \to \mathbb{R}$. Given $r > 0$, let $\xi = (0, r)$, so that $f(\xi) = (-r, 0)$, and find $w_n \to (-r, 0)$, $h_n \to 0+$, such that $V(\xi + h_n w_n) - V(\xi) \le -c h_n$ for some $c > 0$. Let $r_n = |\xi + h_n w_n|$. Then $|r_n - r| = o(h_n)$ as $n \to \infty$. Pick any $L > 0$, and define $k_n = \frac{h_n}{L}$. Write $r_n = r + k_n s_n$. Then $s_n \to 0$, $k_n \to 0+$, and $\widehat{V}(r + k_n s_n) - \widehat{V}(r) \le -cL k_n$. Therefore $D_0^- \widehat{V}(r) \le -cL$. So $D_0^- \widehat{V}(r) = -\infty$. Since this is true for all $r > 0$, Fact 2.8 —with $W \equiv -1$— yields the existence, for each r, of an $a > 0$ and a solution $\rho : [0, a] \to \mathbb{R}$ of $\dot{\rho} = 0$, such that $\rho(0) = r$ and $\widehat{V}(\rho(a)) < \widehat{V}(r)$. Since $\rho(a) = r$, we have reached a contradiction.) □

5. Proof of Theorems 2 and 3

The implication (ii)⇒(iii) of Theorem 3 is trivial, and the implication (iii)⇒(i) follows from Theorem 2. The implication (i)⇒(ii) of Theorem 3 was proved in §2. So all we need to prove is Theorem 2.

Assume first that Σ is AC and (V, W) is a Lyapunov pair. Choose θ, $\tilde{\theta}$ as in Definition 2.1. Let $\sigma \equiv \theta$. Define

$$\psi(r) = \sup\{|\xi| : V(\xi) \le r\}\,,$$

$$\tau(r) = \sup\{V(\xi) : |\xi| \le \tilde{\theta}(\psi(r))\}\,.$$

It is clear that $0 < \tau(r) < \infty$ for each $r > 0$, and $\lim_{r \to 0+} \tau(r) = 0$.

By construction, given any ξ there is a control u: $[0, \infty) \to U$ for which $\|u\| \le \sigma(\xi)$, such that $V(\phi(t, \xi, u)) \le \tau(V(\xi))$ for all $t \ge 0$, and $\lim_{t \to \infty} \phi(t, \xi, u) = 0$. If $\xi \ne 0$, then we can choose $a > 0$ such that $V(\phi(a, \xi, u)) < V(\xi)$. So all the conditions of Definition 3.1 are satisfied. This completes the proof that asymptotic controllability of Σ implies that every Lyapunov pair is DBCBP.

We now prove the converse. We assume that (V, W) is a DBCBP Lyapunov pair, and begin by defining certain arc systems, in order to apply the lemmas of §4.

Given $\alpha > 0$, let $\mathcal{A}_o^\alpha$ (resp. $\mathcal{A}_r^\alpha$) be the set of all arcs $x : [a, b] \to \mathbb{X}$ that are trajectories for ordinary (resp. relaxed) controls u such that $||u|| \le \alpha$. Then $(\mathbb{X}, \mathcal{A}_o^\alpha)$ and $(\mathbb{X}, \mathcal{A}_r^\alpha)$ are arc systems. Moreover, the theorems on continuous dependence of solutions of O.D.E.'s with respect to the initial condition imply that both $\mathcal{A}_o^\alpha$ and $\mathcal{A}_r^\alpha$ have the continuity property.

We will find an $\bar{\alpha} > 0$ and a nondecreasing function $[\bar{\alpha}, +\infty) \ni \alpha \mapsto \hat{L}(\alpha) \in (0, \infty)$ such that

I For $\alpha \ge \bar{\alpha}$, $\{0\}$ is an ASA of the arc system $(\mathbb{X}, \mathcal{A}_o^\alpha)$, with basin of attraction containing the set $\mathbb{X}_{\hat{L}(\alpha)}^V$;

II. $\lim_{\alpha \to +\infty} \hat{L}(\alpha) = +\infty$.

This will clearly imply our conclusion. Indeed, define $\theta(s) = 1 + \inf\{\alpha \in [\bar{\alpha}, \infty) : V(\xi) \le \hat{L}(\alpha) \text{ for all } \xi \ni |\xi| \le s\}$. Then every $\xi \in \mathbb{X}$ lies in the basin of attraction of $\{0\}$ for $(\mathbb{X}, \mathcal{A}_o^{\theta(|\xi|)})$, and this implies that Condition 1 of Remark 2.2 holds. Moreover, since $\theta(s) \ge \bar{\alpha}$ for all s, and $\{0\}$ is an ASA of $(\mathbb{X}, \mathcal{A}_o^{\bar{\alpha}})$, Condition 2 holds as well.

Now, let ν, σ, τ be the functions given by Definitions 2.4 and 3.1. Define $\hat{\theta}(s) = \max\{\nu(s), \sigma(s)\}$. Also, define $\hat{\mu}(s) = \max\{2V(\xi) + \tau(V(\xi)) : |\xi| \le s\}$. For $L > 0$, let $\hat{\beta}(L) = 2\max\{|\xi| : V(\xi) \le L\}$, $\check{\theta}(L) = \hat{\theta}(\hat{\beta}(L))$, $\mu(L) = \hat{\mu}(\hat{\beta}(L))$. Notice that $\lim_{s \to 0+} \hat{\mu}(s) = 0$ and $\lim_{s \to 0+} \mu(s) = 0$.

For $\alpha > 0$, let $F^\alpha(\xi) = \overline{\text{co}}(f(\xi, U_\alpha))$. Then F^α is an USCMCC, and $\mathcal{A}_r^\alpha$ is exactly the set of arcs that are solutions of $\dot{\xi} \in F^\alpha(\xi)$. For $\beta > 0$, write $\Omega^\beta = \{\xi : |\xi| < \beta\}$.

Fix a $\beta > 0$, and take $\alpha = \hat{\theta}(\beta)$. Then $\nu(\beta) \le \alpha$ and $\sigma(\beta) \le \alpha$.

Let $\xi \in \Omega^\beta$ be such that $W(\xi) \ne 0$. Then Fact 2.8 implies that there exist $a > 0$ and an arc $x : [0, a] \to \Omega^\beta$ which belongs to $\mathcal{A}_r^\alpha$ and satisfies $x(0) = \xi$, $V(x(a)) < V(\xi)$, and $V(x(t)) \le V(\xi)$ for $0 < t \le a$. Using the well known theorems on approximation of relaxed trajectories by ordinary ones, we can then find an ordinary control $u : [0, a] \to U_\alpha$ such that $V(\phi(t, \xi, u)) \le 2V(\xi)$ for $0 \le t \le a$ and $V(\phi(a, \xi, u)) < V(\xi)$.

Next, suppose that $\xi \in \Omega^\beta$, $\xi \ne 0$, but $W(\xi) = 0$. Definition 3.1 gives us a control $u : [0, a] \to U$ for which $||u|| \le \sigma(\beta)$, such that $V(\phi(t, \xi, u)) \le \tau(V(\xi))$ for $0 \le t \le a$ and $V(\phi(a, \xi, u)) < V(\xi)$.

So in both cases we have shown that if $\xi \in \Omega^\beta$ and $\xi \ne 0$, then there exist $a > 0$ and a control $u : [0, a] \to U$ such that $||u|| \le \alpha$, $V(\phi(t, \xi, u)) \le \hat{\mu}(\beta)$ for $0 \le t \le a$ and $V(\phi(a, \xi, u)) < V(\xi)$.

Now let $L > 0$, and take $\beta = \hat{\beta}(L)$. Then $\mathbb{X}_L^V \subseteq \Omega^\beta$, so the preceding conclusion holds in particular for every $\xi \in \mathbb{X}_L^V$. In other words, we have proved that

[#] if $L > 0$, and we let $\beta = \hat{\beta}(L)$, then V has the WLDP along $(\mathcal{A}_o^{\check{\theta}(L)})_{\mu(L)}^V$ from every $\xi \in \mathbb{X}_L^V$.

Given $\varepsilon > 0$, choose $\beta > 0$ such that $\mu(\beta) < \varepsilon$. Then choose $\alpha > \check{\theta}(\beta)$, and find δ such that $X_\delta^V \subseteq \Omega^\alpha$. Then V has the WLDP along $(\mathcal{A}_o^\alpha)_\varepsilon^V$ from every $\xi \in \mathbb{X}_\delta^V$, Since such a δ exists for every ε, we conclude that $(\mathbb{X}, \mathcal{A}_o^\alpha, V)$ has the D-stability property. In particular, if we let $\bar{\alpha}$ be the α that corresponds to $\varepsilon = 1$, we see that $(\mathbb{X}, \mathcal{A}_o^\alpha, V)$ has the D-stability property for every $\alpha \ge \bar{\alpha}$.

Now, given $L > 0$, choose $\hat{\alpha}(L) = \check{\theta}(L) + \bar{\alpha}$. Then V has the WLDP along $\mathcal{A}_o^{\hat{\alpha}(L)}$ from every $\xi \in \mathbb{X}_L^V$, and $(V, \mathcal{A}_o^{\hat{\alpha}(L)})$ has the D-stability property. Lemma 4.2 then implies that $\{0\}$ is an ASA of $(V, \mathcal{A}_o^{\hat{\alpha}(L)})$ with basin of attraction containing $\mathbb{X}_L^V$.

Given $\alpha \ge \bar{\alpha}$, we define $\hat{L}(\alpha) = \frac{1}{2}\sup\{L : \hat{\alpha}(L) \le \alpha\}$ Then $\{0\}$ is an ASA of $(V, \mathcal{A}_o^\alpha)$ with basin of attraction containing $\mathbb{X}_{\hat{L}(\alpha)}^V$, so [I] holds. It is clear that $\hat{L}$ is nondecreasing. Finally, given any $L > 0$, let $\alpha = \hat{\alpha}(2L + 1)$. Then $\hat{L}(\alpha) > L$. So [II] holds as well. □

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