An Abstract Approach to Dissipation

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Abstract

We suggest that a very natural mathematical framework for the study of dissipation —in the sense of Willems, Moylan and Hill, and others— is that of indefinite quasimetric spaces. Several basic facts about dissipative systems are seen to be simple consequences of the properties of such spaces. Quasimetric spaces provide also one natural context for optimal control problems, and even for "gap" formulations of robustness.

1. Introduction

We provide an abstract framework in which to study dissipative systems in the sense of [3,2]. Classically, given a continuous-time system Σ and a supply function $w(x,u)$ defined on states and control values, dissipativity means that there exists a nonnegative V on states, called a storage function, so that the inequality $V(x(t)) - V(x(0)) \leq \int_0^t w(x(s),u(s))\,ds$ holds along all possible trajectories. In problems involving L^2 gains (that is, "H^∞" problems), the state-space is Euclidean and the supply function w has the form $\gamma^2 \|u\|^2 - \|x\|^2$; if we also assume that $V(0) = 0$ then dissipativity implies that γ is less or equal to the L^2 gain (operator norm) of the zero-state response. In this manner, dissipation (and the characterization of extremal storage functions as solutions of indefinite Hamilton-Jacobi equations) becomes central to the calculation of operator norms and solving disturbance attenuation problems. Alternatively, if w involves a product of functions of states and inputs, passive systems result.

The starting observation for the present note is that the dissipation inequality is equivalent to asking that $V(z) - V(x) \leq W(x,z)$ for all x, z, where

$$W(x,z) := \inf \left\{ \int_0^t w(x_u(s),u(s))\,ds \right\} \qquad (1)$$

and the minimization is over all t and all controls u of length t that steer x to z, with $x_u(\cdot)$ the corresponding trajectory. (The value of $W(x,z)$ is defined as $+\infty$ if z cannot be reached from x.) Note that this optimal transfer cost W satisfies the triangle inequality and so defines a generalized "distance" between states. Oriented distances, called *quasi*-(pseudo)metrics, were proposed independently by Wilson and Nyemytzki in 1931 and much studied since (we only need to generalize to allow for possibly negative values).

Working with W rather than w, the selection of controls and particular trajectories become irrelevant, and the pure cost structure is abstracted. Moreover, several basic foundational results regarding dissipation are almost trivial to establish when working with W. These ideas are of obvious potential interest in other areas of control as well, e.g. optimal control, graph search, and robustness (directed gap metric). It also allows to naturally define dissipation concepts for hybrid systems.

2. IQMs

An *indefinite quasi (pseudo)metric* (IQM) on a set X is a function $W : X \times X \to \mathbb{R} \cup \{+\infty\}$ so that $W(x,x) = 0$ for all x and $W(x,z) \leq W(x,y) + W(y,z)$. for all x, y, z. An IQM *space* is a pair (X, W) (just "X" if W is clear from context) consisting of set X and an IQM W on X.

If $W(x,z) \geq 0$ for all x, z, one has (cf. [1]) a *quasi-pseudo-metric space*. Metric spaces result when W takes only nonnegative values, is symmetric, and $W(x,z) \neq 0$ for all $x \neq z$. None of these properties is required here. It is convenient (and consistent with the intended control applications) to define the *reachability relation* Reach on the set X as the binary relation given by: $x \text{Reach} y \iff W(x,y) < \infty$ (read "x can be steered to y, or controlled to y, or y is reachable from x"). Fix from now on an IQM space (X, W).

By definition, for each $y \in X$, $V = W_y : X \to \mathbb{R} \cup \{+\infty\}$ given by $V(x) := W(y,x)$ satisfies $V(y) = 0$ and $V(z) \leq V(x) + W(x,z)$ for all $x, z \in X$. The concepts associated to dissipativity in control systems arise from asking when such a function is nonnegative. We'll say that $x_0 \in X$ is an *initial point* for (X, W) if $W(x_0, z) \geq 0$ for all $z \in X$. The (possibly empty) set of initial points is denoted by $\mathcal{I}(X, W)$ (or just $\mathcal{I}$). Let $S \subseteq X$. The IQM space (X, W) is *subintegrable with respect to* S (abbreviated SIWRT S) if $S \subseteq \mathcal{I}$. A function $V : X \to [0, +\infty]$ so that

$$V(z) \;\leq\; V(x) + W(x,z) \quad \text{for all } x, z \in X\,. \qquad (2)$$

is a *subpotential* (or a *storage function*, or a *subintegral*) for W. If also $V|_S \equiv 0$, V is a *subpotential (for W) with respect to* the subset S (abbreviated SPWRT S). The subpotentials for (X, W) form a convex set. Standard terminology in control theory is "dissipativity" (typically with respect to $S = \{0\}$ in $\mathbb{R}^n$).

For each two nonempty subsets $A, B \subseteq X$, write $W(A,B) := \inf\{W(a,b), a \in A, b \in B\} \in \mathbb{R} \cup \{\pm\infty\}$. (If $A = \{x\}$, $W(x,B)$, or $W^B(x)$; if $B = \{x\}$, $W(A,x)$ or $W_A(x)$). Let $\underline{V} := -W^X$; then $\underline{V}$ is a subpotential

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for W. In general, $\underline{V}$ is a SPWRT S whenever (X, W) is SIWRT S, and conversely, (X, W) is SIWRT any subset of the zero set of $\underline{V}$. Note that $\underline{V} \equiv 0$ if $W \geq 0$; this happens in particular for standard optimal control problems, and for quasi-pseudo-metric spaces. It is minimal in the sense that $\underline{V} \leq V$ for every subpotential V.

Assume now that a subset $S \subseteq X$ has been fixed, and let $\overline{V} := W_S$, so that $\overline{V}|_S \leq 0$. Then (X, W) is SIWRT S if and only if $\overline{V}$ is a SPWRT S. Moreover, $\underline{V} \leq V \leq \overline{V}$ for every SPWRT S. The main abstract properties of subintegrability are as follows:

Theorem. *For any $S \subseteq X$, the following are equivalent: (1) W is* SIWRT *S. (2) $W(x_0, x) \geq 0$ for all $x_0 \in S$, $x \in X$. (3) $\overline{V} \geq 0$. (4) $\underline{V}|_S \equiv 0$. (5) $\overline{V}$ is a* SPWRT *S. (6) $\underline{V}$ is a* SPWRT *S. (7) There is some* SPWRT *S. (8) $\underline{V} \leq \overline{V}$.*

The following are equivalent: *(1) There is a finite-valued subpotential. (2) $\underline{V}: X \to [0, +\infty)$. (3) W has finite-loss, i.e., for each $x \in X$ the set $\{W(x, z), z \in X\}$ is bounded below.* When there is reachability from S: $W(x_0, x) < \infty$ for all $x_0 \in S$, $x \in X$, all subpotential functions are finite.

A particular case of subpotential is as follows. We say that a function $V: X \to [0, +\infty)$ is a *potential* (or an *integral*) for W if $V(z) = V(x) + W(x, z)$ for all x, z. The IQM W is *independent of path* if $W(x, z) = W(x, y) + W(y, z)$ for all x, y, z, and it is *antisymmetric* if $W(x, z) = -W(z, x)$ for all x, z (which implies that W is finite). The following are equivalent: *(1) There is a potential V for W. (2) W is independent of path and finite-loss. (3) W is antisymmetric and finite-loss.*

One can often normalize potentials. If there is a potential V for W which attains its minimum at a set S, then W is SIWRT S. Conversely, if W is SIWRT S, and if there is a potential for W, then there one with $V|_S \equiv 0$. In general, if W is finite-valued and finite-loss, then $W = \max_{\lambda \in \Lambda} W_\lambda$, where the W_λ's are IQMs which satisfy the integrability conditions above.

If W admits a potential V and is SIWRT S then $\overline{V} = \underline{V}$, and thus there a *unique* possible SPWRT S. On the other hand, a necessary condition for $\overline{V} = \underline{V}$ is that for all $x \in X$ it hold that $W(S, R(x)) = 0$, which one can interpret as an "asymptotic controllability to S" property: it says that from each point x one can get as "close" as desired to the set S.

Continuous time systems $\dot{x} = f(x, u)$ for which W is as in Equation (1) provide many examples. If $w(x, u) = \gamma^2 |u|^2 - |x|^2$, then W being SIWRT $S = \{0\}$ means that $\gamma \leq$ the L^2 gain of the system. On the other hand, one can construct examples of totally nonholonomic systems without drift ($\dot{x} = \sum_i u_i g_i(x)$, Lie algebra rank condition holds) for which w depends only on u and W is finite and antisymmetric, providing a source of examples of unique SPWRT's.

Observe also that if X is also a metric space so that $\mathrm{dist}(x, z)$ small implies that also $W(x, z)$ is small (a "strong local controllability" condition) then every subpotential is continuous with respect to the original metric.

The possible IQMs on a given set form a convex cone. Given an IQM W, we can define the new IQMs $W_+ := \max\{W, 0\}$. and $W_+^1 := \min\{W_+, 1\}$. The latter is a quasi-pseudo-metric in the sense of [1]. Defining $W^\dagger(x, y) := W(y, x)$ makes $(X, W^\dagger)$ into another IQM space. the conjugate or transpose of (X, W). Taking $\widetilde{W} := \max\{W, W^\dagger\}$ provides a symmetrization of (X, W). This is a pseudometric except for the possible infinite values, so $\min\{\widetilde{W}, 1\}$ is a pseudometric in the usual sense. Many examples are obtained starting from any IQM space (X, W) (for instance, a metric space) and any transitive and reflexive relation R on X. Then $W_R(x, y) := W(x, y)$ if xRy and $+\infty$ otherwise, defines a new IQM space $X_R = (X, W_R)$.

One may not want to differentiate between the metrics corresponding to supply functions w giving rise to the same notion of nearness (energy required for a state transfer "small" for W_1 iff small for W_2); this leads to topologies induced by IQMs. For each $x \in X$ and $\varepsilon \in \mathbb{R}$. consider the ball of radius ε: $B_\varepsilon(x) := \{y \mid W(x, y) < \varepsilon\}$. These sets form a basis for the (right) topology induced by (X, W). The topology induced by W is the same as the topology induced by W_+^1, so the topological spaces induced by IQMs are precisely the same as what are called *quasi-pseudo-metrizable spaces* in the literature. Analogously to the Zariski topology in algebraic geometry, such spaces are typically far from being Hausdorff, and in fact few separation axioms hold in general. The quasimetrization problem is that of characterizing those topologies arising in this fashion. An analogue of A. Weil's theorem on uniformization is valid: a quasi-uniformity $\mathcal{U}$ admits a quasi-pseudo-metric if and only if $\mathcal{U}$ admits a countable base. The topology induced by W is of class T_1 (points are closed sets) iff the one for W_+^1 is, which is in turn equivalent to asking that W_+^1 be a quasimetric (as opposed to a quasi-pseudo-metric).

One may define "quasinormed spaces" by starting with a real vector space X and postulating a subadditive map $x \mapsto \|x\|$ (e.g. a gauge or Minkowski functional for a not-necessarily balanced subset of X). Then $\|x - y\|$ defines an IQM on X. Starting from any finite-valued W, the map $y \mapsto \{W_x(y), x \in X\}$ provides an isometry from (X, W) into a generalized l^∞ quasinormed space.

The distance-to-set map $W_A : X \to \mathbb{R} \cup \{\pm\infty\} : x \mapsto W(A, x)$ (resp., W^A) is upper (resp., lower) semicontinuous. From these properties one establishes elementary existence theorems for optimization on IQM spaces.

References

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