On Characterizations of Input-to-State Stability with Respect to Compact Sets

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Abstract. Previous characterizations of ISS-stability are shown to generalize without change to the case of stability with respect to sets. Some results on ISS-stabilizability are mentioned as well.

Key Words. Set Stability; input-to-state stability; Lyapunov functions; control Lyapunov functions; practical stability

1. Introduction

System stability with respect to input perturbations is one of the central issues to be studied in control. During the past few years, the property called "input to state stability" (ISS) has been proposed (originally in (Sontag, 1989*a*)) as a foundation for the study of such problems, and has been subsequently employed by many authors in areas ranging from robust control to highly nonlinear small-gain theorems (see for instance (Jiang *et al.*, 1994; Tsinias, 1989)). In the paper (Sontag and Wang, 1995), the authors established the equivalence among several natural characterizations of the ISS property, stated in terms of dissipation inequalities, robustness margins, and Lyapunov-like functions.

The ISS property was originally stated for stability with respect to a given equilibrium state of interest. On the other hand, in many applications it is of interest to study stability with respect to an *invariant set* $\mathcal{A}$, where $\mathcal{A}$ does not necessarily consist of a single point. Examples of such applications include problems of robust control and the various notions usually encompassed by the term "practical stability." This motivated the study of the "set" version of the ISS property, originally in (Sontag and Lin, 1992), and developed with applications to the study of parameterized families of systems in (Lin *et al.*, 1995). Given the interest in set-ISS, it is an obvious question to ask whether the equivalent characterizations given in (Sontag and Wang, 1995) for the special case $\mathcal{A}$ = equilibrium extend to the more general set case. It is the main purpose of this paper to point out that, at least for the case of *compact* invariant sets $\mathcal{A}$, the results in (Sontag and Wang, 1995) indeed can be generalized with little or no change; in particular, ISS with respect to a given $\mathcal{A}$ is equivalent to the existence of an "ISS-Lyapunov function" relative to $\mathcal{A}$. (Most of the proofs valid in the equilibrium case extend easily to the set case.) The property of there existing *some* invariant compact set $\mathcal{A}$ so that there is ISS with respect to $\mathcal{A}$ will be called the "compact-ISS" property; various remarks about this notion are included, including the equivalence with what is sometimes called "practical stability." Finally, we also sketch some applications of the ISS notion to feedback design and disturbance attenuation.

2. Set Input to State Stability

We deal with systems of the following general form:

$$\dot{x} = f(x, u)\,, \tag{Σ}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz map, and we interpret x and u as functions of time $t \in \mathbb{R}$, with values $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, for some positive integers n and m. (Generalizations to systems evolving on manifolds, and/or restricted control value sets, are also of interest, but in this paper, all spaces are Euclidean.) From now on, one such system is assumed given.

A *control* or *input* is a measurable locally essentially bounded function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. We use the notation $\|u\|$ to indicate the L^m_∞-norm of u, and $|\cdot|$ for Euclidean norm in $\mathbb{R}^n$ and $\mathbb{R}^m$. For each $\xi \in \mathbb{R}^n$ and each $u \in L^m_\infty$, we denote by $x(t, \xi, u)$ the trajectory of the system Σ with initial state $x(0) = \xi$ and the input u. (This solution is a priori only defined on some maximal interval $[0, T_{\xi,u})$, with $T_{\xi,u} \leq +\infty$, but the main definition will include the requirement that $T_{\xi,u} = +\infty$.) We recall in an appendix the no-

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tions of positive definite and class $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ comparison functions.

Let $\mathcal{A}$ be a nonempty subset of $\mathbb{R}^n$. For each $\xi \in \mathbb{R}^n$, $|\xi|_{\mathcal{A}} = d(\xi, \mathcal{A}) = \inf\{d(\eta,\xi),\ \eta \in \mathcal{A}\}$ will denote the usual point-to-set distance from ξ to $\mathcal{A}$. (So for the special case $\mathcal{A} = \{0\}$, $|\xi|_{\{0\}} = |\xi|$ is the usual Euclidean norm.) We say that such an $\mathcal{A}$ is a *0-invariant set* for Σ, or more precisely, for the associated "zero-input" or "undisturbed" system

$$\dot{x} = f_0(x) = f(x, 0)\,, \qquad (\Sigma_0)$$

if it holds that $x(t, \xi, 0) \in \mathcal{A}, \forall t \geq 0, \forall \xi \in \mathcal{A}$. That is, when the control is $u \equiv 0$, the solution when starting from $\mathcal{A}$ is defined for all $t \geq 0$ and stays in $\mathcal{A}$.

Definition 2.1 The system Σ has *$\mathcal{K}$-asymptotic gain with respect to the nonempty subset $\mathcal{A} \subseteq \mathbb{R}^n$* if there is some $\gamma \in \mathcal{K}$ (an "asymptotic gain") so that, for all u, solutions exist for all $t \geq 0$ and

$$\overline{\lim_{t\to+\infty}} |x(t, \xi, u)|_{\mathcal{A}} \leq \gamma(\|u\|) \qquad (1)$$

uniformly on compacts for ξ.

The uniformity requirement (which, as will be shown in a forthcoming paper, can be substantially relaxed) means, precisely: solutions exist for all initial states, controls, and times, and for each real numbers $r, \varepsilon > 0$, there is some $T = T(r,\varepsilon) \geq 0$ so that $|x(t,\xi,u)|_{\mathcal{A}} \leq \varepsilon + \gamma(\|u\|)$ for all u, all $|\xi| \leq r$, and all $t \geq T$. This generalizes the idea of finite (linear) gain, classically used in input/output stability theory. Clearly, if a system has $\mathcal{K}$-asymptotic gain with respect to a set $\mathcal{A}$, and if $\mathcal{A}'$ is any subset of $\mathbb{R}^n$ containing $\mathcal{A}$, then the system also has $\mathcal{K}$-asymptotic gain with respect to $\mathcal{A}'$.

The main concept to be studied is as follows.

Definition 2.2 Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a nonempty compact set. The system Σ is *input-to-state stable* (ISS) with respect to $\mathcal{A}$ if it has $\mathcal{K}$-asymptotic gain with respect to $\mathcal{A}$ and $\mathcal{A}$ is 0-invariant. □

Note that, in particular, if this property holds then the autonomous system Σ_0 is globally asymptotically stable with respect to $\mathcal{A}$ (in the sense in (Lin *et al.*, to appear)). The ISS property can also be defiinied with respect to non-compact 0-invariant sets $\mathcal{A}$ (cf. (Lin *et al.*, 1995)), but then an additional technical condition must be imposed (see Remark 2.5 below).

For any $\mathcal{A} \subseteq \mathbb{R}^n$, we consider the zero-input orbit from $\mathcal{A}$: $O(\mathcal{A}) := \{\eta : \eta = x(t,\xi,0), t \geq 0, \xi \in \mathcal{A}\}$ and let $\mathbf{O}(\mathcal{A})$ be the closure of $O(\mathcal{A})$.

Lemma 2.3 Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a nonempty compact set. If the system Σ has $\mathcal{K}$-asymptotic gain with respect to $\mathcal{A}$, then it is ISS with respect to $\mathbf{O}(\mathcal{A})$.

Proof: Since $\mathbf{O}(\mathcal{A})$ includes $\mathcal{A}$, the system has $\mathcal{K}$-asymptotic gain with respect to $\mathbf{O}(\mathcal{A})$, and the latter set is 0-invariant (since $O(\mathcal{A})$ is). Thus we only need to prove that $\mathbf{O}(\mathcal{A})$ is compact, or equivalently, that $O(\mathcal{A})$ is bounded. From the $\mathcal{K}$-asymptotic gain property, we know that there exists some $T \geq 0$ so that, for each $\xi \in \mathcal{A}$, $|x(t,\xi,0)|_{\mathcal{A}} \leq 1$ for all $t \geq T$. By continuity at states in $\mathcal{A}$ of solutions of Σ_0 with respect to initial conditions, and compactness of $\mathcal{A}$, there is some constant c so that $|x(t,\xi,0)|_{\mathcal{A}} \leq c$ for all $t \in [0,T]$. Thus $O(\mathcal{A})$ is included in a ball of radius $\max\{1,c\}$. ■

Definition 2.4 The system Σ is *compact*-ISS if it is ISS with respect to some compact set $\mathcal{A}$. □

By Lemma 2.3, this is equivalent to simply asking that system Σ has $\mathcal{K}$-asymptotic gain with respect to some compact set $\mathcal{A}$.

Remark 2.5 The definition of the ISS property is in terms of a uniform attraction property. Just as for systems with no controls (cf. (Bhatia and Szegö, 1970), Theorem 1.5.28, or (Hahn, 1967), Theorem 38.1), compactness of $\mathcal{A}$ insures that the following stability-like property is automatically satisfied as well:

For each $\varepsilon > 0$, there exists a $\delta > 0$ so that

$$|\xi|_{\mathcal{A}} \leq \delta, \|u\| \leq \delta \ \Rightarrow\ |x(t,\xi,u)|_{\mathcal{A}} \leq \varepsilon,\ \ \forall t \geq 0\,.$$

Indeed, assume given $\varepsilon > 0$. Let $T = T(1, \varepsilon/2)$. Pick any $\delta_1 > 0$ so that $\gamma(\delta_1) < \varepsilon/2$; then, if $|\xi|_{\mathcal{A}} \leq 1$ and $\|u\| \leq \delta_1$, and $t \geq T$,

$$|x(t,\xi,u)|_{\mathcal{A}} \leq \varepsilon/2 + \gamma(\|u\|) < \varepsilon\,. \qquad (2)$$

By continuity (at $u \equiv 0$ and states in $\mathcal{A}$) of solutions with respect to controls and initial conditions, and compactness and 0-invariance of $\mathcal{A}$, there is also some $\delta_2 > 0$ so that, if $|\eta|_{\mathcal{A}} \leq \delta_2$ and $\|u\| \leq \delta_2$ then $|x(t,\eta,u)|_{\mathcal{A}} \leq \varepsilon$ for all $t \in [0,T]$. Together with (2), this gives the desired property with $\delta := \min\{1, \delta_1, \delta_2\}$. □

In a manner entirely analogous to the equilibrium case in (Sontag and Wang, 1995), Lemma 2.7, one can show the following characterization in terms of decay estimates:

Lemma 2.6 The system Σ is ISS with respect to the compact set $\mathcal{A}$ if and only if there are a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ so that

$$|x(t, \xi, u)|_{\mathcal{A}} \leq \beta(|\xi|_{\mathcal{A}}, t) + \gamma(\|u\|) \qquad (3)$$

holds for each $t \geq 0$, $u \in L^m_\infty$, and $\xi \in \mathbb{R}^n$. □

Remark 2.7 In the paper (Jiang *et al.*, 1994), a system Σ is said to be "input-to-state practically stable" (ISpS) if there exist a $\mathcal{KL}$-function β, a $\mathcal{K}$-function γ and a constant $c \geq 0$ such that

$$|x(t, \xi, u)| \leq \beta(|\xi|, t) + \gamma(\|u\|) + c \qquad (4)$$

holds for each input $u \in L^m_\infty$ and each $\xi \in \mathbb{R}^n$. *This concept is equivalent to the system being compact-*ISS. Indeed, the concept implies the $\mathcal{K}$-asymptotic gain property with respect to the ball of radius c. Conversely, if the system is ISS with respect to $\mathcal{A}$, and $0 \in \mathcal{A}$ (which may be assumed without loss of generality, by first enlarging the set and then considering $\mathbf{O}(\mathcal{A})$), then it is also ISpS, because $|x(t,\xi,u)| \leq |x(t,\xi,u)|_{\mathcal{A}} + c \leq \beta(|\xi|_{\mathcal{A}}, t) + \gamma(\|u\|) + c \leq \beta(|\xi|, t) + \gamma(\|u\|) + c$ with $c := \sup\{|\xi|, \xi \in \mathcal{A}\}$. □

2.1. *ISS-Lyapunov Functions*

An ISS-*Lyapunov function with respect to* the compact subset $\mathcal{A} \subseteq \mathbb{R}^n$ for system Σ is a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ which satisfies the following conditions: (a) V is proper and positive definite with respect to the set $\mathcal{A}$, that is, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $\xi \in \mathbb{R}^n$,

$$\alpha_1(|\xi|_{\mathcal{A}}) \leq V(\xi) \leq \alpha_2(|\xi|_{\mathcal{A}}), \tag{5}$$

and (b) there exist functions $\alpha_3 \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that

$$\nabla V(\xi) f(\xi, v) \leq -\alpha_3(|\xi|_{\mathcal{A}}) + \sigma(|v|) \tag{6}$$

for all $\xi \in \mathbb{R}^n$ and for all $v \in \mathbb{R}^m$.

Note that such a V is automatically also a Lyapunov function for the zero-input system Σ_0 with respect to $\mathcal{A}$. Also, observe that, since here $\mathcal{A}$ is compact, the existence of a function α_2 as stated is in fact a consequence of the continuity of V.

Remark 2.8 There are two other, equivalent, ways of defining ISS-Lyapunov function. The first is as follows. One asks that V satisfy (5) and that, for some $\mathcal{K}_\infty$-functions α_4 and χ there hold the implication

$$|\xi|_{\mathcal{A}} \geq \chi(|v|) \implies \nabla V(\xi) f(\xi, v) \leq -\alpha_4(|\xi|_{\mathcal{A}}) \tag{7}$$

(for each state $\xi \in \mathbb{R}^n$ and control value $v \in \mathbb{R}^m$). A second variant is to drop the requirement that α_4 be of class $\mathcal{K}_\infty$; that is, one asks only that there is a χ so that

$$|\xi|_{\mathcal{A}} \geq \chi(|v|) \Longrightarrow \nabla V(\xi) f(\xi, v) < 0 \tag{8}$$

for all $\xi \neq 0$ and all $v \in \mathbb{R}^m$. These two variants are equivalent: if (8) holds, then there exists a positive definite function α_4 such that (7) holds (define for instance $\alpha_4(r)$ as the supremum of the values of $\nabla V(\xi) f(\xi, v)$ on the compact subset consisting of all $|\xi|_{\mathcal{A}} = r$ and v so that $\chi(|v|) \leq r$); then, for some properly chosen q, $q(V)$ satisfies (7) with a new $\alpha_4 \in \mathcal{K}_\infty$. Extending the proof of Remark 2.4 in (Sontag and Wang, 1995), it can be shown that all variants of the definition are equivalent. □

Remark 2.9 If $\mathcal{A}$ is not compact, then properties (6) and (7) may no longer be equivalent. As an example, consider the following system: $\dot{x}_1 = -x_1$, $\dot{x}_2 = -x_2 + x_1 q(u - |x_2|)$, where $q : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying $q(r) = 0$ for all $r \leq 0$ and $q(r) > 0$ for all $r > 0$. Let $\mathcal{A}$ be the set $\{(\xi_1, \xi_2) : \xi_2 = 0\}$. Consider the function $V(\xi) = \xi_2^2/2$. Clearly V satisfies (5) (as $|\xi|_{\mathcal{A}} = |\xi_2|$), and $|\xi|_{\mathcal{A}} \geq |v| \implies \nabla V(\xi) f(\xi, v) \leq -\xi_2^2$. But it is easy to see that V fails to satisfy property (6). □

The main result is as follows.

Theorem 1 Let the compact set $\mathcal{A}$ be a 0-invariant set for the system Σ. Then the system is ISS *with respect to $\mathcal{A}$ iff it admits an* ISS-*Lyapunov function with respect to $\mathcal{A}$.*

The proof follows essentially the same lines as the proof of the analogous Theorem 1 (for equilibria) in (Sontag and Wang, 1995). We omit the details for reasons of space, but the main point is that care must be taken to establish all estimates in terms of $|\cdot|_{\mathcal{A}}$ rather than Euclidean norm.

One may modify V in an essentially arbitrary way inside the set $\mathcal{A}$, so as to make V positive definite with respect to subsets of $\mathcal{A}$:

Lemma 2.10 Assume that Σ is ISS with respect to a given compact set $\mathcal{A}$, and let $\widetilde{\mathcal{A}}$ be any nonempty compact subset of $\mathcal{A}$. Then there is a smooth function $\widetilde{V} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ which satisfies the following conditions: (a) $\widetilde{V}$ is proper and positive definite with respect to the set $\widetilde{\mathcal{A}}$, that is, there exist $\widetilde{\alpha}_1, \widetilde{\alpha}_2 \in \mathcal{K}_\infty$ such that for all $\xi \in \mathbb{R}^n$,

$$\widetilde{\alpha}_1(|\xi|_{\widetilde{\mathcal{A}}}) \leq \widetilde{V}(\xi) \leq \widetilde{\alpha}_2(|\xi|_{\widetilde{\mathcal{A}}}), \tag{9}$$

and (b) there exist a function $\widetilde{\alpha}_3 \in \mathcal{K}_\infty$ and a nondecreasing continuous function $\widetilde{\sigma}$ such that

$$\nabla \widetilde{V}(\xi) f(\xi, v) \leq -\widetilde{\alpha}_3(|\xi|_{\widetilde{\mathcal{A}}}) + \widetilde{\sigma}(|v|) \tag{10}$$

for all $\xi \in \mathbb{R}^n$ and for all $v \in \mathbb{R}^m$.

Proof: Let V be an ISS-Lyapunov funtion for Σ satisfying (6). Consider the set $\mathcal{A}_1 = \{\xi : |\xi|_{\mathcal{A}} \geq 1\}$. Since this is disjoint from $\mathcal{A}$, there is a smooth function $\varphi : \mathbb{R}^n \to [0,1]$ so that $\varphi(\xi) = 0$ if $\xi \in \mathcal{A}$ and 1 if $\xi \in \mathcal{A}_1$. Similarly, there is some smooth, nonnegative function λ defined on $\mathbb{R}^n$ which vanishes exactly on $\widetilde{\mathcal{A}}$. Now we define $\widetilde{V}(\xi) := \lambda(\xi)(1 - \varphi(\xi)) + V(\xi)\varphi(\xi)$. By construction, $\widetilde{V}$ is smooth and is proper and positive definite with respect to $\widetilde{\mathcal{A}}$, that is, there are comparison functions as in (9). Furthermore, since $V(\xi) = \widetilde{V}(\xi)$ for $|\xi|_{\mathcal{A}} > 1$, also

$$\nabla \widetilde{V}(\xi) f(\xi, v) \leq -\alpha_3(|\xi|_{\mathcal{A}}) + \gamma(|v|),$$

(where α_3 and γ are the comparison functions associated to V) for all $v \in \mathbb{R}^m$ and all $|\xi|_{\mathcal{A}} > 1$. Since both $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ are compact, there exists some $s_0 \geq 0$ such that $|\xi|_{\widetilde{\mathcal{A}}} \leq |\xi|_{\mathcal{A}} + s_0$; thus

$$\nabla \widetilde{V}(\xi) f(\xi, v) \leq -\tilde{\alpha}_3(|\xi|_{\widetilde{\mathcal{A}}}) + \gamma(|v|), \tag{11}$$

whenever $|\xi|_{\widetilde{\mathcal{A}}} \geq 1 + s_0$, where $\tilde{\alpha}_3$ is any $\mathcal{K}_\infty$ function which satisfies $\tilde{\alpha}_3(r) \leq \alpha_3(r - s_0)$ for all $r \geq s_0 + 1$. The proof is completed by taking any nondecreasing continuous function $\widetilde{\sigma}$ which majorizes both $\gamma(r)$ and the maximum of $\nabla \widetilde{V}(\xi) f(\xi, v)$ over all $|\xi|_{\widetilde{\mathcal{A}}} \leq 1 + s_0, |v| \leq r$. ■

This construction can be applied in particular when $\mathcal{A}$ contains the origin $\widetilde{\mathcal{A}} = \{0\}$ to conclude that if the system Σ is compact-ISS, then Σ admits a *semi*-ISS-*Lyapunov function*, that is, a smooth function V satisfying $\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|)$ for suitable $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and so that

$$\nabla V(\xi) f(\xi, v) \leq -\alpha_3(|\xi|) + \sigma(|v|) \tag{12}$$

for some $\alpha \in \mathcal{K}_\infty$ and nondecreasing continuous function σ. (Note that, in general, σ cannot be picked of class $\mathcal{K}$; there is no need for such a function to vanish at 0. However, one may always write $\sigma \leq c + \widetilde{\sigma}$, for some class-$\mathcal{K}$ function $\widetilde{\sigma}$ and some positive constant c.)

On the other hand, if Σ admits a semi-ISS-Lyapunov function satisfying (12), then it follows that $|\xi| \geq \chi(|v|) \implies \nabla V(\xi) f(\xi, v) \leq -\alpha_3(|\xi|)/2$ where $\chi(r) = \alpha_3^{-1} \circ 2\sigma(r)$. Observe that χ is a nondecreasing function. Using exactly the same arguments as used on page 441 of (Sontag, 1989*a*), one can show that there exist some $\mathcal{KL}$-function β and some nondecreasing function γ such that

$$|x(t, \xi, u)| \leq \beta(|\xi|, t) + \gamma(|u|)$$

for each input u and each initial state ξ. It is easy to see from this that the system is compact-ISS. So we proved the following conclusion:

Corollary 2.11 The system Σ is compact-ISS iff it admits a semi-ISS-Lyapunov function. ∎

3. ISS-Control Lyapunov Functions

In this section, we provide a preliminary discussion of some questions related to input to state *stabilizability* under feedback. (For simplicity, we consider only the case of stabilization with respect to the equilibrium $\mathcal{A} = \{0\}$, but the theory could be developed in more generality.) Now inputs in Σ will be assumed to be partitioned into two components, one of which corresponds to true controls $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, and the other to "disturbances" $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ acting on the system. Furthermore, we will assume that the right-hand side of Σ is affine in u and w. That is to say, we consider systems on $\mathbb{R}^n$ of the general form:

$$\dot{x} = f(x) + G(x)u + D(x)w\,, \tag{13}$$

where f and the columns of the matrices G and D are smooth vector fields. We also assume that $f(0) = 0$.

Our main objective is to study conditions for the existence of a smooth feedback law $u = k(x)$ such that the closed-loop system

$$\dot{x} = f(x) + G(x)k(x) + D(x)w \tag{14}$$

is ISS, with w seen as the external input. As when studying (non-ISS) stabilizability under feedback, we search for conditions expressed in terms of "control Lyapunov functions."

Observe that if there exists a feedback law as above, then, by the main result in (Sontag and Wang, 1995), there exists a positive definite, proper function φ such that the system

$$\dot{x} = f(x) + G(x)k(x) + \varphi(x)D(x)d \tag{15}$$

is uniformly globally asymptotically stable (UGAS) with respect to all $d \in \mathcal{M}_\mathcal{O}$ (= the set of all measurable functions from $\mathbb{R}_{\geq 0}$ to the unit ball in $\mathbb{R}^m$). That is, there exists for solutions of this closed-loop system a $\mathcal{KL}$-function β such that

$$|x(t, \xi, d)| \leq \beta(t, \xi), \qquad \forall\, t \geq 0$$

for all $\xi \in \mathbb{R}^n$ and all $d \in \mathcal{M}_\mathcal{O}$. We next apply to (15) the converse Lyapunov Theorem in (Lin *et al.*, to appear) for systems with no controls but with time-varying parametric uncertainties. We conclude (using the value $u = k(x)$) that there exists also a smooth, proper and positive definite function V such that

$$\inf{}_u \{a(\xi) + B_0(\xi)\, u + \varphi(\xi) B_1(\xi)\, \mathbf{d}\} < 0$$

for all $|\mathbf{d}| \leq 1$ and $\xi \neq 0$, where we are using the notations $a(\xi) = \nabla V(\xi) f(\xi)$, $B_0(\xi) = \nabla V(\xi) G(\xi)$, and $B_1(\xi) = \nabla V(\xi) D(\xi)$. Consequently, for any $\xi \neq 0$,

$$\inf{}_u \{a(\xi) + B_0(\xi)\, u + \varphi(\xi)\, |B_1(\xi)|\} < 0\,. \tag{16}$$

For any given real-valued function V on $\mathbb{R}^n$, let $\mathcal{B}_0$ be the set $\{\xi : \; B_0(\xi) = 0, \xi \neq 0\}$. Thus, for the above V,

$$a(\xi) < 0 \quad \text{if } \xi \in \mathcal{B}_0 \tag{17}$$

and

$$\lim_{\substack{|\xi| \to \infty \\ \xi \in \mathcal{B}_0}} \frac{-a(\xi)}{|B_1(\xi)|} = +\infty. \tag{18}$$

(We make the convention that $-a(\xi)/|B_1(\xi)| = +\infty$ at those points where $B_1(\xi) = 0$. The meaning of the limit along $\mathcal{B}_0$ is the obvious one, namely that values are large as long as $|\xi|$ is large and $\xi \in \mathcal{B}_0$. If $\mathcal{B}_0$ happens to be bounded, this condition is vacuous.)

Motivated by these considerations, we say that a smooth, proper and positive definite function V is an ISS-*control-Lyapunov-function* (ISS-clf) for (13) if (17)-(18) hold. An ISS-clf V is said to satisfy the small control property (scp) if for any $\varepsilon > 0$, there is a $\delta > 0$ such that for each $0 < |\xi| < \delta$, there is some $|v| < \varepsilon$ such that $a(\xi) + B_0(\xi)v < 0$. (To simplify, we also say that the corresponding pair $(a(\xi),\ B_0(\xi))$ "satisfies scp.") These definitions constitute one possible generalization of the corresponding notion for systems with no "w", for which only equation (17) is relevant; see (Sontag, 1989*b*; Isidori, 1995).

Theorem 2 If there is an ISS-clf V for system (13), then there exists a feedback law $k : \mathbb{R}^n \to \mathbb{R}^m$ which is smooth on $\mathbb{R}^n \setminus \{0\}$ and is such that the closed-loop system (14) is ISS. If V satisfies the small control property, then k can be chosen to be continuous at 0.

This result is not really new: (Freeman and Kokotovic, to appear) give a very similar theorem which applies to systems not necessarily affine in controls, but we believe that the method of proof given here is very natural and constructive.

We will say that a feedback k is *almost smooth* if it is smooth on $\mathbb{R}^n \setminus \{0\}$ and continuous at 0.

Remark 3.1 The converse of the above Theorem is also true, that is, if system (13) is ISS-stabilizable by an almost smooth feedback $k(\cdot)$, then there exists an ISS-clf with scp. However, in order to prove the stronger assertion, we would need to make some modifications to the proof of Theorem 1 in (Lin *et al.*, to appear), but we don't have the space to do so in this conference paper. We may sketch the main steps, however. Basically, and using the notations in that paper, one needs to show that, when the 0-invariant set $\mathcal{A}$ used there is compact, the functions g and U constructed for obtaining V are locally Lipschitz outside $\mathcal{A}$ without needing to assume that f is locally Lipschitz on $\mathcal{A}$ (which, when k is almost smooth but not smooth, cannot be guaranteed). To show the Lipschitz continuity of g, one only needs to notice that the trajectories starting outside $\mathcal{A}$ never enter $\mathcal{A}$ in negative time; hence, the original proof is still valid. To prove the Lipschitz condition on U, one first notices that the stability of the system implies that trajectories starting from a compact set always stay in a compact set in positive time. Using this, one can show that $x(t, \xi, d_{\xi,\varepsilon})$ always stays in a compact set disjoint from A, for any $t \in [0, t_{\xi,\varepsilon}]$. This then allows one to apply the Lipschitz condition for f outside $\mathcal{A}$, together with the Gronwall inequality, to conclude that U is locally Lipschitz outside $\mathcal{A}$. □

Sketch of Proof of Theorem 2. Assume that (13) admits an ISS-clf V satisfying (17)–(18). If $\mathcal{B}_0$ is unbounded, we let

$$\psi_1(r) = \inf\left\{-\frac{a(\xi)}{|B_1(\xi)|} : \ B_0(\xi) = 0, |\xi| = r\right\}$$

(values may be infinite). If $\mathcal{B}_0$ is bounded, we let $\psi_1(r)$ be defined in this way for those r for which there is some $\xi \in \mathcal{B}_0$ with $|\xi| \geq r$, but let $\psi_1(r) = \infty$ otherwise. In either case, it holds that $\inf\{\psi_1(r), r \in [a, b]\} > 0$ for each two numbers $0 < a < b$, so there is a smooth $\mathcal{K}_\infty$-function $\psi(r)$ such that $\psi(r) \leq \min\{r^2, \ \psi_1(r)\}$ for all $r \geq 0$. Let $\varphi(\xi) := \psi(|x|)$; this is continuously differentiable everywhere in $\mathbb{R}^n$, because $\psi(r) \leq r^2$. It is clear that the following implication holds:

$$\xi \in \mathcal{B}_0 \implies a(\xi) + \varphi(\xi)|B_1(\xi)| < 0\,.$$

So $\inf_v \{a_1(\xi) + B_0(\xi)\, v\} < 0$ for all $\xi \neq 0$, where $a_1(\xi) := a(\xi) + \varphi(\xi)\,|B_1(\xi)|$. Applying the "universal formula" given in (Sontag, 1989*b*) (see also (Isidori, 1995), Section 9.4) to the pair (a_1, B_0), one obtains a feedback $k_0(\xi) = \alpha(a_1(\xi), |B_0(\xi)|)$, so that for all $\xi \neq 0$,

$$a(\xi) + B_0(\xi)k_0(\xi) + \varphi(\xi)\,|B_1(\xi)| < 0 \qquad (19)$$

where $\alpha(r, s)$ is the function defined by

$$\alpha(r,\, s) = \begin{cases} -\frac{r+\sqrt{r^2+s^4}}{s}\,, & \text{if } s \neq 0\,;\\ 0\,, & \text{if } s = 0\,. \end{cases}$$

As it was shown in (Sontag, 1989*b*), the function α is analytic on the set

$$S = \{(s, r) \in \mathbb{R}^2 : \ r \neq 0 \text{ or } s < 0\}\,.$$

This then implies that k_0 is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$. (Note here that k_0 may fail to be smooth on $\mathbb{R}^n \setminus \{0\}$ because the function $\xi \mapsto |B_1(\xi)|$ may fail to be smooth.) To obtain a feedback that is smooth on $\mathbb{R}^n \setminus \{0\}$, it is sufficient to approximate k_0 on $\mathbb{R}^n \setminus \{0\}$ by a smooth k so that Equation (19) still holds. It then follows that

$$a(\xi) + B_0(\xi)k(\xi) + B_1(\xi)v < 0 \qquad (20)$$

for all $\xi \neq 0$ and all $|v| \leq \psi(|\xi|)$. Therefore, V is an ISS-Lyapunov function for the closed-loop system, and consequently, the closed-loop system is ISS.

To show that if in addition V satisfies the scp then the feedback law can be chosen to be continuous everywhere, observe that one may always choose ψ in such a manner that $\lim_{|\xi|\to 0} \frac{|B_1(\xi)\psi(\xi)|}{a(\xi)} = 0$. For instance, one can pick up any smooth $\mathcal{K}_\infty$-function $\tilde{\psi}$ so that $\tilde{\psi}(r) \leq \min\{\psi(r)(-a(\xi)) : \xi \in \mathcal{B}_0, \ |\xi| = r\}$ in a neighborhood of 0 where $-a(\xi) \leq 1$, and let $\tilde{\psi}(r) \leq \psi(r)$ everywhere. Then replace ψ by $\tilde{\psi}$. With this new choice of ψ and the resulting φ, the pair $(a_1(\xi), B_0(\xi))$ still has the scp, and hence the feedback law k_0 is continuous everywhere, and hence, the approximating function k_1 can be choosen almost smooth.

3.1. *Gain assignment*

Suppose that system (13) admits an ISS-Lyapunov function V, and that the two $\mathcal{K}_\infty$-functions α_1 and α_2 are such that

$$\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|), \quad \forall\, \xi. \qquad (21)$$

The above arguments also show that if there exists a smooth, positive definite and proper function φ such that

$$\inf_u \{a(\xi) + B_0(\xi)u + |B_1(\xi)|\,\varphi(\xi)\} < 0$$

for all $\xi \neq 0$, then there exists a feedback k that is smooth on $\mathbb{R}^n \setminus \{0\}$, such that for any $\xi \neq 0$, $a(\xi) + B_0(\xi)k(\xi) + B_1(\xi)w < 0$ provided $|w| \leq \psi(|\xi|)$, where $\psi(r) = \inf_{|\xi|\leq r}\varphi(\xi)$ is a $\mathcal{K}_\infty$-function. By the proof of Theorem 1 in (Sontag, 1989*a*), (strictly speaking, this is not a particular case of the setup in the first part, because the closed-loop system need not satisfy a Lipschitz property at the origin), one knows that there exists a $\mathcal{KL}$-function β so that for every trajectory $x(t, \xi, w)$ of the closed-loop system, it holds that

$$|x(t, \xi, w)| \leq \beta(|\xi|\,, t) + \gamma(\|w\|)\,. \qquad (22)$$

where $\gamma(r) = \alpha_1^{-1} \circ \alpha_2 \circ \psi^{-1}(r)$. We say that a function γ is a *feedback assignable gain* if there is some k so that an estimate of this form holds (for some β), under some almost smooth feedback law.

When studying the problem of ISS stabilizability for a system of the form (13), a special case is that of "matching uncertainties" when $G = D$; then the equations have the form

$$\dot{x} = f(x) + G(x)(u + w). \qquad (23)$$

In (Sontag, 1989*a*), it was shown that if system (23) is stabilizable by a smooth feedback when $w=0$ in (23), then the system is also ISS-stabilizable by a smooth feedback. In (Praly and Wang, submitted), this result was generalized to obtain the following "gain assignability" result, rederived here in a somewhat simpler manner by means of ISS-Lyapunov functions.

Proposition 3.2 If system (23) is stabilizable by an almost smooth feedback k when $w=0$, then every $\gamma \in \mathcal{K}_\infty$ is an assignable gain.

Proof: By standard converse Lyapunov theorems, the hypotheses imply the existence of a control-Lyapunov function V satisfying (21) and

$$\inf{}_u\{a(\xi)+B(\xi)u\}<0, \quad \forall \xi \neq 0, \tag{24}$$

where $a(\xi)=\nabla V(\xi)f(\xi)$, $B(\xi)=\nabla V(\xi)G(\xi)$, and in addition, V satisfies the scp. Let ψ be a $\mathcal{K}_\infty$-function such that $\psi(r)\geq 2\gamma^{-1}\circ\alpha_1^{-1}\circ\alpha_2(r)$. Let φ be a almost smooth function so that $\varphi(\xi)\geq \psi(|\xi)|/2$ for all $\xi\in\mathbb{R}^n$. It follows from (24) that $\inf{}_u\{a_1(\xi)+B(\xi)u\}<0$, $\forall\xi\neq 0$, where $a_1(\xi)=a(\xi)+|B(\xi)|\,\varphi(\xi)$. Arguing as earlier, one knows that there exists a k that is smooth on $\mathbb{R}^n\setminus\{0\}$ such that

$$a_1(\xi)+B(\xi)k(\xi)<0 \tag{25}$$

for all $\xi\neq 0$. Note again that if the pair (a,B) satisfies the scp, the pair (a_1,B) also satisfies the scp, so the resulting k is continuous everywhere. From (25), it follows that for each $\xi\neq 0$,

$$a(\xi)+B(\xi)k(\xi)+B(\xi)w<0$$

provided $|w|\leq\psi(|\xi|)/2$. From here one obtains the desired conclusion. ∎

Finally, we relate our results to those in (Isidori, 1995), Section 9.5, where the problem of disturbance attenuation with stability is discussed in the context of finiteness of L_2 gains (we assume for this comparison that the output function is $y=x$). It was shown there that if system (13) admits a positive definite proper smooth function V such that for each $\xi\neq 0$,

$$B_0(\xi)=0 \implies a(\xi)+\frac{1}{4\gamma^2}\left|B_1(\xi)\right|^2+|\xi|^2<0, \tag{26}$$

where $\gamma>0$ is some constant, then there exists an almost smooth feedback k such that the closed-loop system is ISS with an ISS-Lyapunov function satisfying

$$\nabla V(f(\xi)+g(\xi)k(\xi)+p(\xi)w)\leq\gamma^2\left|w\right|^2-\xi^2\,.$$

Observe that if (26) holds, then V is also an ISS-Lyapunov function as defined above. This follows immediately from the following observation:

Lemma 3.3 Let $a,b,c:\mathbb{R}^n\to\mathbb{R}$ be continuous functions. Assume that c is positive definite and proper. If $a(\xi)+(b(\xi))^2+c(\xi)<0$ for all $\xi\neq 0$, then $\lim_{|\xi|\to\infty}\frac{-a(\xi)}{|b(\xi)|}=\infty$.

Proof: The Lemma follows directly from the inequality $\rho+\frac{\sigma}{\rho}\geq 2\sqrt{\sigma}$ for all $\rho,\sigma>0$. ∎

4. Appendix

We recall the definitions of the standard classes of comparison functions. A function $\gamma:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is *positive definite* if $\gamma(s)>0$ for all $s>0$, and $\gamma(0)=0$; the function γ is a *$\mathcal{K}$-function* if it is continuous, positive definite, and strictly increasing; and γ is a *$\mathcal{K}_\infty$-function* if it is a $\mathcal{K}$-function and $\gamma(s)\to\infty$ as $s\to\infty$. Finally, a function $\beta:\ \mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is a *$\mathcal{KL}$-function* if for each fixed $t\geq 0$ the function $\beta(\cdot,t)$ is a $\mathcal{K}$-function, and for each fixed $s\geq 0$, $\beta(s,t)$ decreases to zero as $t\to\infty$.

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