An Example of a GAS System Which Can Be Destabilized by an Integrable Perturbation

Eduardo D. Sontag and Mikhail Krichman

***Abstract*—A construction of a globally asymptotically stable time-invariant system which can be destabilized by some integrable perturbation is given. Besides its intrinsic interest, this serves to provide counterexamples to an open question regarding Lyapunov functions.**

***Index Terms*—Asymptotic stability, integrable perturbations, intergral stability.**

I. Introduction

We present a construction of a time-invariant system $\dot{x} = f(x)$ which has the origin as a globally asymptotically stable (GAS) equilibrium, yet there is some integrable function $d(\cdot)$ such the system $\dot{x}(t) = f(x(t)) + d(t)$ admits unbounded solutions.

Based on one-dimensional intuition, one might have expected that such examples cannot exist, so there is an intrinsic interest to this question. However, this work was, in fact, motivated by a problem posed by L. Praly. The question concerns the existence of continuously differentiable Lyapunov functions with globally bounded gradients. If there is such a Lyapunov function V for $\dot{x} = f(x)$, then solutions of $\dot{x} = f(x) + d(t)$ are bounded [since $dV(x(t))/dt \leq c|d(t)|$, where c is a bound on the norm of the gradient, so $V(x(t))$ is bounded], and this implies in turn that solutions converge to the origin. In their work [7], Praly and Arcak analyze output feedback with an observer/controller structure for systems $\dot{x} = f(x, u)$, $y = h(x)$. The observer takes the general form

$$\dot{\hat{x}} = f(\hat{x}, u) + \kappa(\cdots)(y - \hat{y})$$

where $\kappa(\cdots)$ is an appropriate term, and $\hat{y} = h(\hat{x})$, and the construction guarantees that $y - \hat{y}$ is in L^1. The output feedback is obtained with the "certainty equivalent" control $u = \phi(\hat{x})$, where ϕ is a globally asymptotically stabilizing state feedback law. They impose the technical condition that there is a C^1 Lyapunov function V whose gradient is bounded when multiplied by the term κ; a sufficient condition for this, when κ is bounded, is to have boundedness of the gradient. The question was if this technical assumption is needed, in the sense that, conceivably, the fact that $y - \hat{y}$ is in L^1 might suffice to complete the proof. Our counterexample implies that boundedness of the gradient needs, indeed, to be stated as an assumption.

Not unrelated to the observer question is the more abstract question of studying stability of the system $\dot{x} = f(x + e)$, where $e = e(t)$ represents a "measurement error" and the nominal system $\dot{x} = f(x)$ is known to be globally asymptotically stable. One might have hoped that the system remains stable when e does not vary too fast, in an appropriate sense. For instance, one might ask that e be an absolutely continuous function, i.e., that $d := \dot{e}$ be in L^1. Since the change of variables $z := x + e$ transforms $\dot{x} = f(x + e)$ to $\dot{z} = f(z) + d$, we see that, unfortunately, such constraints on variations are not sufficient to guarantee stability.

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In this context, it is worth remarking that Freeman gave an example in [4] of a controlled system $\dot{x} = g(x, K(x))$ with the property that the origin is globally asymptotically stable but so that, for a suitable function of time $e(t)$, with $e(t)$ converging to zero, finite escape times exist for the system $\dot{x} = g(x, K(x+e))$. (Moreover, Freeman showed that this happens for all possible feedback stabilizers K, for the given g.) Note the subtle difference with the problem considered in this note: we are interested in the system $\dot{x} = f(x + e) = g(x + e, K(x + e))$, instead of the system $\dot{x} = g(x, K(x + e))$.

The question studied here is related to, but different from, ideas from input-to-state stability (ISS) (see [8]) and more specifically *integral* ISS (see [1], [2], and [9]). One knows that a system $\dot{x} = f(x, u)$ might well be GAS yet not be integral-ISS, meaning very roughly (see [9] for the precise definition) that "integrable" inputs (integrability is defined with respect to $\mathcal{K}$-function classes) may destabilize the system. An important difference is that, here, we look for systems of the very special form $\dot{x} = f(x) + u$, and we insist upon L^1 norm.

Much closer to this note is the early work of Vrkoč in the 1950s, who studied these same questions. In fact, Vrkoč, in [10], introduced a notion of "integral stability" which essentially amounts to the requirement that systems remain stable under L^1 inputs, and established Lyapunov characterizations of this as well as related properties. (See [3], [5], and [6] for more recent references along these lines). In his paper, Vrkoč gave an example of a *time-varying* system $\dot{x} = f(t, x)$ which has the origin as a GAS point but which is destabilized by some integrable perturbation, and he implied that a counterexample exists for autonomous systems as well. However, we have been unable to find one in the literature, and there seems to be no way to adapt his time-varying example to build a time-invariant one. Thus, we produce an example from scratch.

We will show the existence of a smooth vector field f on $\mathbb{R}^2$, with $f(0) = 0$ and so that the equilibrium 0 is globally asymptotically stable, and with the following property: for any $\varepsilon > 0$, there is some function $\mathbf{d}$ such that $\|\mathbf{d}\| < \varepsilon$ in L^1 and $\dot{x} = f(x) + \mathbf{d}$ admits an unbounded solution.

II. Intuition

Let us first give the intuitive idea behind the counterexample. The basic idea is to start with a system having a trajectory which looks like that shown in Fig. 1 (this will be built from a linear spiral, under an appropriate coordinate change which looks like an "accordion" in the x direction). The system will be GAS, provided that the points P_i go to $\pm\infty$, since all other trajectories are "trapped" inside this one. However, we construct the system in such a manner that the distances between the points $P_1, P_2, \ldots$ labeled by integers $k = 1, 2, \ldots$ satisfy $|P_1 - P_2| = \delta_1$, $|P_3 - P_4| = \delta_2$, $|P_5 - P_6| = \delta_3$, $|P_7 - P_8| = \delta_4$, $|P_9 - P_{10}| = \delta_5$, and so forth, with $\sum \delta_i < \infty$.

Now, if $\mathbf{d}$ is an input which applies an impulse of magnitude δ_1 when crossing the point labeled 1 (landing at 2), then an impulse of magnitude $-\delta_2$ when crossing the point labeled 3 (landing at 4), an impulse of magnitude δ_3 when crossing the point labeled 5 (landing at 6), an impulse of magnitude $-\delta_4$ when crossing the point labeled 7 (landing at 8), and so on, the perturbed system will have a trajectory that diverges. (Of course, impulses have to be approximated by functions).

III. Construction of the System

We will start with the two-dimensional system $\tilde{\Sigma}$ given by:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -2x - 2y \end{aligned} \tag{1}$$

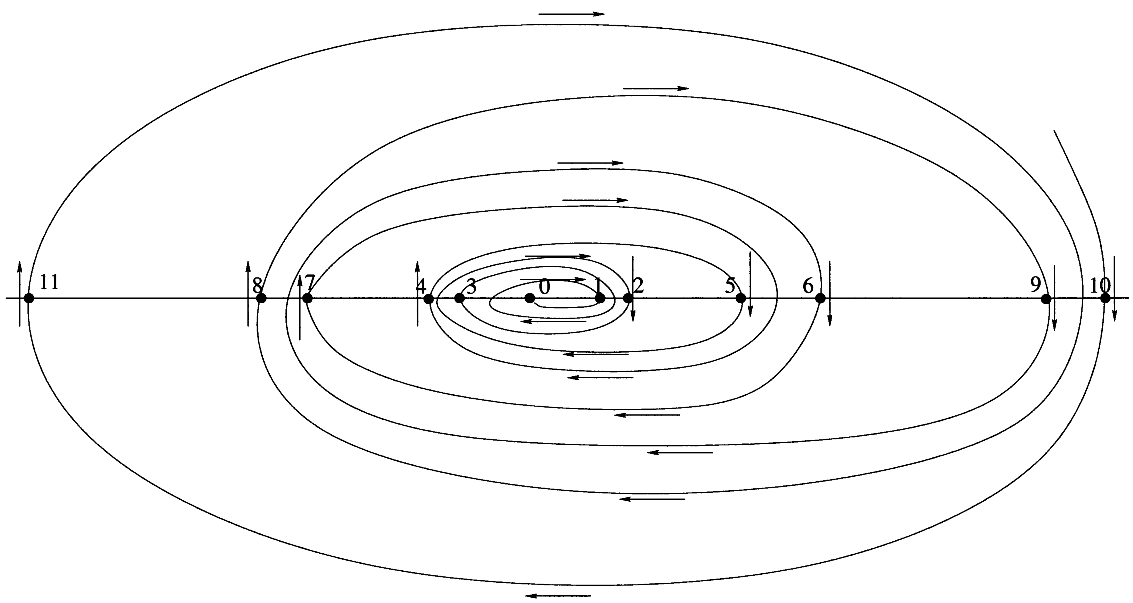

Fig. 1. Intuition.

(so x satisfies the linear second-order equation $\ddot{x}+2\dot{x}+2x=0$, whose general solution is $x=C_1e^{-t}\sin t+C_2e^{-t}\cos t$) and will obtain our counterexample after an appropriate smooth coordinate change in $\mathbb{R}^2$. The original system has convergent spiral trajectories, and the coordinate change will be of the "accordion" type mentioned in the intuitive description. Note that, in polar coordinates, the system is equivalently described by

$$\begin{aligned}\dot{\theta} &= -1-\cos^2\theta-\sin(2\theta)\\ \dot{r} &= -r(\cos\theta\sin\theta+2\sin^2\theta).\end{aligned} \tag{2}$$

For the initial conditions $y(0)=0$, $x(0)=x_0$ the solution is

$$\begin{aligned}x(t) &= x_0e^{-t}(\cos t+\sin t)\\ y(t) &= -2x_0e^{-t}\sin t.\end{aligned}$$

Thus

$$(r(t))^2=(x(t))^2+(y(t))^2=x_0^2e^{-2t}(1+4\sin^2 t+\sin(2t))$$

and

$$\begin{aligned}&\frac{d}{dt}\left((r(t))^2\right)\\ &\quad= -2x_0^2e^{-2t}(1+4\sin^2 t+\sin(2t))\\ &\qquad+x_0e^{-2t}(8\sin t\cos t+2\cos(2t))\\ &\quad= 2x_0^2e^{-2t}(-1-4\sin^2 t-\sin(2t)+2\sin(2t)+\cos(2t))\\ &\quad= 2x_0^2e^{-2t}(\sin(2t)-6\sin^2 t).\end{aligned}$$

When, in particular, the initial states have the form $x(0)=e^{2K\pi}$ and $y(0)=0$, for some fixed integer K, the following properties hold.

1) The solution $x=e^{2K\pi-t}(\cos t+\sin t)$ will hit the positive x axis at all points $e^{2k\pi}$, for integers $k\le K$, and the negative x axis at points $-e^{(2k-1)\pi}$, $k\le K$.
2) For any integer ℓ, when $t=\ell\pi$, $|x(t)|=r(t)=e^{(2K-\ell)\pi}$ and

$$r'(s)\ge 0 \qquad \text{for all } s\in[t,\,t+\tau]$$

where we are denoting $\tau=\arctan(1/3)>\pi/12$, so that, for all $s\in[t,\,t+\pi/12]$ we have

$$\begin{aligned}r(t)\le r(s)&\le r(t)e^{-\pi/12}\sqrt{1+4\sin^2(\pi/12)+\sin(\pi/6)}\\ &<1.03r(t).\end{aligned}$$

We next construct the diffeomorphism. For this, we start by picking, for each $j\ge 1$, a smooth function $\lambda_j\in C^\infty[e^{(6j-3)\pi},\,e^{(6j+3)\pi}]$, with the following properties:

- $\lambda_j(e^{(6j-3)\pi})=e^{(6j-3)\pi}$;
- $\lambda_j(e^{(6j-2)\pi-\pi/2})=e^{6j\pi}-1/2^j$;
- λ_j is linear on $[e^{(6j-2)\pi-\pi/2},\,e^{(6j+2)\pi+\pi/2}]$;
- $\lambda_j(e^{(6j+2)\pi+\pi/2})=e^{6j\pi}+1/2^j$;
- $\lambda_j(e^{(6j+3)\pi})=e^{(6j+3)\pi}$;
- $\lambda_j'(r)>0$ for all $r\in[e^{(6j-3)\pi},\,e^{(6j+3)\pi}]$;
- $\lambda_j'(e^{(6j-3)\pi})=\lambda_j'(e^{(6j+3)\pi})=1$;
- $\lambda_j^{(p)}(e^{(6j-3)\pi})=\lambda_j^{(p)}(e^{(6j+3)\pi})=0$ for all integers $p>1$.

It is easy to see that such functions exist; a partial graph of such a function λ_j is sketched in Fig. 2.

Now, we can define $\lambda(\cdot)$ by gluing the λ_j's together. We let

$$\lambda(r)=r, \qquad r<e^{3\pi}$$

and

$$\lambda(r)=\lambda_j(r) \qquad \text{for } r\in\left[e^{(6j-3)\pi},\,e^{(6j+3)\pi}\right].$$

Next, we define a diffeomorphism S_r of a plane into itself as $(r,\,\theta)\to(\lambda(r),\,\theta)$ in polar coordinates or, equivalently, as

$$S_r(x,\,y)=\left(\frac{x\lambda\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}},\,\frac{y\lambda\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}\right).$$

The diffeomorphism S_r will stretch the right-hand side of the plane.

In the same manner we construct a diffeomorphism for the left-hand side of the plane: We define the function μ as the identity on $[0,\,e^{4\pi}]$, and on each interval $[e^{(6j-2)\pi},\,e^{(6j+4)\pi}]$ we take $\mu(\cdot)=\mu_j(\cdot)$, where

- $\mu_j(e^{6j\pi})=e^{6j\pi}$;
- $\mu_j(e^{6j\pi+\pi/2})=e^{(6j+3)\pi}-1/2^j$;
- μ_j is linear on $[e^{6j\pi+\pi/2},\,e^{(6(j+1)\pi-\pi/2}]$;

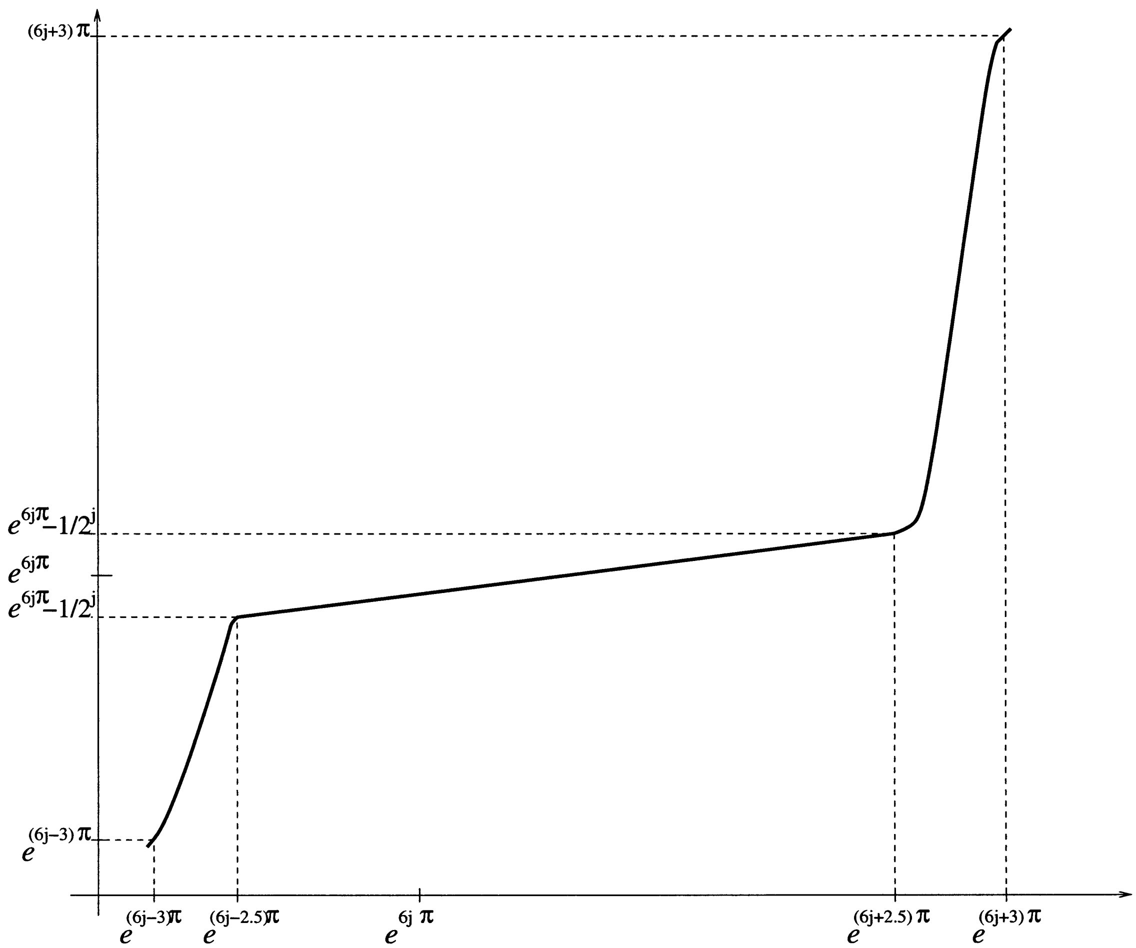

Fig. 2. Graph of λ_j.

- $\mu_j(e^{(6(j+1)\pi - \pi/2}) = e^{(6j+3)\pi} + 1/2^j$;
- $\mu_j(e^{6(j+1)\pi}) = e^{6(j+1)\pi}$;
- $\mu_j'(r) > 0$ for all $r \in [e^{6j\pi}, e^{6(j+1)\pi}]$;
- $\mu_j'(e^{6j\pi}) = \mu_j'(e^{6(j+1)\pi}) = 1$;
- $\mu_j^{(p)}(e^{6j\pi}) = \mu_j^{(p)}(e^{6(j+1)\pi}) = 0$, for all integers $p > 1$.

The diffeomorphism

$$S_l(x, y) = \left(\frac{x\mu\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}, \frac{y\mu\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}} \right)$$

will stretch the left-hand side of the plane.

Let ϕ: $\mathbb{R}^2 \to [0, 1]$ be a C^∞ function such that

$$\phi(x, y) = \begin{cases} 0 & \text{when } x < -1, \\ 1 & \text{when } x > 1 \end{cases}$$

and define S: $\mathbb{R}^2 \to \mathbb{R}^2$ by

$$(\xi, \eta) = S(x, y) := S_r(x, y)\phi(x, y) + S_l(x, y)(1 - \phi(x, y)).$$

Under the diffeomorphism S, the system $\tilde{\Sigma}$ will transform into the desired system, which we denote by Σ. In polar coordinates, Σ is expressed as follows:

$$\begin{aligned} \dot{\vartheta} &= \varphi_\vartheta(\rho, \vartheta) \\ \dot{\rho} &= \varphi_\rho(\rho, \vartheta) \end{aligned} \tag{3}$$

where, excluding the strip $|\rho \cos \vartheta| < 1$

$$\begin{aligned} \varphi_\vartheta(\rho, \vartheta) &= -1 - \cos^2 \vartheta - \sin(2\vartheta) \\ \varphi_\rho(\rho, \vartheta) &= -\lambda^{-1}(\rho)/\lambda^{-1'}(\rho)(\cos \vartheta \sin \vartheta + 2 \sin^2 \vartheta) \end{aligned}$$

for the right half-plane $\rho \cos \vartheta > 1$, or

$$\varphi_\rho(\rho, \vartheta) = -\mu^{-1}(\rho)/\mu^{-1'}(\rho)(\cos \vartheta \sin \vartheta + 2 \sin^2 \vartheta)$$

for the left half-plane $\rho \cos \vartheta < -1$.

Obviously, if $(r(t), \theta(t))$ is the solution of (2), then on all of $\{(\rho, \vartheta)\colon\ |\rho \cos \vartheta| \geq 1\}$ the corresponding solution of Σ can be written as $\vartheta(t) = \theta(t)$, and $\rho(t) = \lambda(r(t))$ or $\mu(r(t))$ chosen appropriately.

IV. Construction of the Disturbance

Now, fix an arbitrary $\varepsilon > 0$. We will find an initial state $(\xi_0, 0)$ and design a disturbance $\mathbf{d} \in L^1$ with $\|\mathbf{d}\| < \varepsilon$, such that the ensuing trajectory of the system:

$$\begin{aligned} \dot{\vartheta} &= \varphi_\vartheta(\rho, \vartheta) \\ \dot{\rho} &= \varphi_\rho(\rho, \vartheta) + \mathbf{d} \end{aligned} \tag{4}$$

will tend to ∞.

We write $x(\overline{x}, t)$ for the value at time t of the solution of IVP (1) with $x(0) = \overline{x}$ and $y(0) = 0$; write $r^+(\overline{r}, t)$ $(r^-(\overline{r}, t))$ for the value at time t of the solution of IVP (2) with $r(0) = \overline{r}$, $\theta(0) = 0$ $(\theta(0) = -\pi)$; write $\xi(\overline{\xi}, t)$ and $\rho^+(\overline{\rho}, t)$ $(\rho^-(\overline{\rho}, t))$ for the corresponding solutions of Σ; and write $\rho^+(\overline{\rho}, \mathbf{d}, t)$ $(\rho^-(\overline{\rho}, \mathbf{d}, t))$ for a solution of (4) with $\rho(0) = \overline{\rho}$, $\vartheta(0) = 0$ $(\vartheta(0) = -\pi)$, and disturbance $\mathbf{d}$.

Find an integer K such that $\varepsilon/4 > 2^{-K+1}$ and $2^{-K+1} < \pi/12$. Let $x_0 = e^{(6K-2)\pi}$, so that $\xi_0 = \lambda(e^{(6K-2)\pi})$; and let also $x_1 = e^{(6K+2)\pi}$ and $\xi_1 = \lambda(x_1)$.

Recall that, by property 2), $x_1 \leq r(x_1, t) < 1.03x_1$ and $x_0 \leq r(x_0, t) < 1.03x_0$ for all $t \in [0, \pi/12]$. In particular, for any $t \in [0, \pi/12]$, and with $x_0 = e^{(6K-2)\pi}$, $x_1 = e^{(6K+2)\pi}$, both $r(x_0, t)$ and $r(x_1, t)$ will be in the interval $[e^{(6K-2.5)\pi}, e^{(6K+2.5)\pi}]$. Then, by construction of λ, both $\rho(\xi_0, t) = \lambda(r(x_0, t))$ and $\rho(\xi_1, t) = \lambda(r(x_1, t))$ will belong to the interval $[e^{6K\pi} - 2^{-K}, e^{6K\pi} + 2^{-K}]$, so that

$$\rho^+(\xi_1, t) - \rho^+(\xi_0, t) < 2^{-K+1}, \qquad t \in [0, \pi/12].$$

Therefore, there must exist a positive $\tau_0 < 2^{-K+1}$ such that if $\mathbf{d}_0 := 1_{[0, \tau_0]}$, then

$$\rho^+(\xi_1, \tau_0) = \rho^+(\xi_0, \mathbf{d}_0, \tau_0).$$

So

$$\rho^+(\xi_0, \mathbf{d}_0, \pi) = \rho^+(\xi_1, \pi) = -\mu\left(e^{(6K+1)\pi}\right).$$

Let $\xi_2 := -\mu(e^{(6K+1)\pi})$, $\xi_3 := -\mu(e^{(6K+5)\pi})$.

Next, take a disturbance $\mathbf{d}_1 = 1_{[\pi, \pi+\tau_1]}$, with some $\tau_1 < 2^{-K+1}$ such that

$$\rho^-(\xi_3, \tau_1) = \rho^-(\xi_2, \mathbf{d}_1(\cdot + \pi), \tau_1).$$

Then

$$\begin{aligned}\rho^+(\xi_0, \mathbf{d}_0 + \mathbf{d}_1, 2\pi) &= \rho^-(\xi_2, \mathbf{d}_1(\cdot + \pi), \pi) \\ &= \rho^-(\xi_3, \pi) = \lambda\left(e^{(6K+4)\pi}\right).\end{aligned}$$

Generally, for each $k \geq 0$, we let

$$\begin{aligned}\xi_{4k} &:= \lambda\left(e^{(6(K+k)-2)\pi}\right) \\ \xi_{4k+1} &:= \lambda\left(e^{(6(K+k)+2)\pi}\right) \\ \xi_{4k+2} &:= -\mu\left(e^{(6(K+k)+1)\pi}\right) \\ \xi_{4k+3} &:= -\mu\left(e^{(6(K+k)+5)\pi}\right)\end{aligned}$$

and choose $\tau_{2k} \leq 2^{-K-k+1}$ and $\tau_{2k+1} \leq 2^{-K-k+1}$ so that

$$\rho^+(\xi_{4k+1}, \tau_{2k}) = \rho^+(\xi_{4k}, \mathbf{d}_{2k}(\cdot + 2k\pi), \tau_{2k})$$

and

$$\rho^-(\xi_{4k+3}, \tau_{2k+1}) = \rho^-(\xi_{4k+2}, \mathbf{d}_{2k+1}(\cdot + (2k+1)\pi), \tau_{2k+1})$$

with $\mathbf{d}_l := 1_{[0, \tau_l]}$.

Finally, let $\mathbf{d} := \sum_l \mathbf{d}_l$. Then

$$\int \mathbf{d}(t)\, dt = \sum \tau_l \leq 4/2^{K-1} < \varepsilon$$

and

$$\lim_{t \to +\infty} \rho^+(\xi_0, \mathbf{d}, t) = \infty.$$

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A Note on Global Output Regulation of Nonlinear Systems in the Output Feedback Form

Qi Gong and Wei Lin

***Abstract*—This note shows how the adaptive control method developed recently for nonlinearly parameterized systems can be used to solve the problem of global output regulation, for nonlinear systems in the so-called output-feedback form with unknown parameters and exogenous signals belonging to a compact set whose bound is also unknown.**

***Index Terms*—Adaptive nonlinear control, global output regulation, output feedback.**

I. Introduction and Preliminaries

In this note, we consider the problem of global output regulation for nonlinear systems of the form

$$\begin{aligned}\dot{x} &= F(\mu)x + G(y, \omega, \mu) + g(\mu)u \\ \dot{y} &= H(\mu)x + K(y, \omega, \mu) \\ \dot{\omega} &= S\omega \\ e &= y - q(\omega, \mu)\end{aligned} \tag{1.1}$$

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