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Brief paper

Diagonal stability of a class of cyclic systems and its connection with the secant criterion☆

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Abstract

We consider a class of systems with a cyclic interconnection structure that arises, among other examples, in dynamic models for certain biochemical reactions. We first show that a "secant" criterion for local stability, derived earlier in the literature, is in fact a necessary and sufficient condition for diagonal stability of the corresponding class of matrices. We then revisit a recent generalization of this criterion to output strictly passive systems, and recover the same stability condition using our diagonal stability result as a tool for constructing a Lyapunov function. Using this procedure for Lyapunov construction we exhibit classes of cyclic systems with sector nonlinearities and characterize their global stability properties.

Keywords: Diagonal stability; Vector Lyapunov functions; Passivity; Storage functions; Biochemical reactions

1. Introduction

In this paper we study systems characterized by a *cyclic* interconnection structure as depicted in Fig. 1. An important example where this structure arises is a sequence of biochemical reactions where the end product drives the first reaction as described by the model

$$\begin{aligned}\dot{\chi}_1 &= -f_1(\chi_1) + g_n(\chi_n),\\ \dot{\chi}_2 &= -f_2(\chi_2) + g_1(\chi_1),\\ &\vdots\\ \dot{\chi}_n &= -f_n(\chi_n) + g_{n-1}(\chi_{n-1}).\end{aligned} \tag{1}$$

Tyson and Othmer (1978) and Thron (1991) addressed the situation where $f_i(\cdot)$, $i=1,\ldots,n$, and $g_i(\cdot)$, $i=1,\ldots,n-1$ are increasing functions and $g_n(\cdot)$ is a decreasing function, which means that the intermediate products "facilitate" the next reaction while the end product "inhibits" the rate of the first reaction. To evaluate local stability properties of such reactions Tyson and Othmer (1978) and Thron (1991) analyzed the Jacobian linearization at the equilibrium, which is of the form

$$A = \begin{bmatrix} -\alpha_1 & 0 & \cdots & 0 & -\beta_n \\ \beta_1 & -\alpha_2 & \ddots & & 0 \\ 0 & \beta_2 & -\alpha_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \beta_{n-1} & -\alpha_n \end{bmatrix}, \quad \alpha_i > 0, \quad \beta_i > 0 \tag{2}$$

$i=1,\ldots,n$, and showed that it is Hurwitz if

$$\frac{\beta_1\ldots\beta_n}{\alpha_1\ldots\alpha_n} < \sec(\pi/n)^n. \tag{3}$$

Unlike a *small-gain* condition which would restrict the right-hand side of (3) to be 1, criterion (3) also exploits the phase of the loop and allows the right-hand side to be 8 when $n=3$, 4 when $n=4$, 2.8854 when $n=5$, etc. Furthermore, when α_i's are equal, (3) is also necessary for stability.

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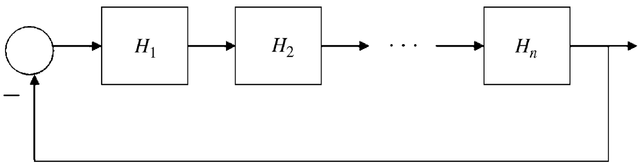

Fig. 1. A cyclic feedback interconnection of systems $H_1, \ldots, H_n$.

The objective of this paper is to extend this stability criterion to classes of nonlinear systems, including (1), by building on a *passivity* interpretation presented recently in Sontag (2006). We first revisit Sontag (2006), which derived an analog of (3) when the blocks in Fig. 1 are *output strictly passive* (Sepulchre, Janković, & Kokotović, 1997; van der Schaft, 2000), and recover the same stability result with a Lyapunov proof that complements the input–output arguments in Sontag (2006). Our Lyapunov function consists in a weighted sum of *storage functions* for each block, with the weights selected judiciously according to a diagonal stability result proved in this paper for the class of matrices (2). This construction resembles the method of *vector Lyapunov functions* in the literature of large-scale systems (Michel & Miller, 1977; Šiljak, 1978), where a Lyapunov function is assembled from a weighted sum of several components.

We next study the case where some of the blocks in Fig. 1 are static *sector* nonlinearities. When such a nonlinearity is time invariant and preceded by a linear, first-order, dynamic block we relax our stability criterion with a special Lyapunov construction that mimics the proof of the Popov criterion (Khalil, 2002). We next apply a similar construction to system (1), and extend the secant condition (3) to become a criterion for global asymptotic stability. Our main assumption in this result is that $f_i(\cdot)$'s and $g_i(\cdot)$'s satisfy a *sector* property, and that the growth ratio of $g_i(\cdot)$ relative to $f_i(\cdot)$ be bounded by a constant that plays the role of β_i/α_i in (3). The next result extends this condition to the case where the state variables are nonnegative quantities as in biochemical reactions.

The results of this paper previewed above all hinge upon our key theorem for diagonal stability of (2), presented in Section 2. Using this theorem, Section 3 studies the cyclic interconnection in Fig. 1, and gives a procedure for selecting the weights in our Lyapunov function construction from storage functions. Section 4 derives a Popov-type relaxed stability criterion for static, time-invariant, sector nonlinearities. Section 5 revisits system (1) and proves global asymptotic stability. Section 6 extends this result to systems with nonnegative state variables. An independent result in Section 7 studies a cascade of output strictly passive systems, and uses our main theorem on diagonal stability to prove an *input feedforward passivity* (IFP) (Sepulchre et al., 1997) property for the cascade, which quantifies the amount of feedforward gain required to re-establish passivity.

2. Main theorem for diagonal stability

The key ingredient for all of the results in this paper is Theorem 1, which states that (3) is a necessary and sufficient condition for *diagonal stability* of (2). This theorem is of independent interest because existing results for diagonal stability of various classes of matrices, such as those surveyed in Redheffer (1985) and Kaszkurewicz and Bhaya (2000) do not address the cyclic structure exhibited by (2). In particular, the sign reversal for β_n in (2) rules out the "M-matrix" condition, which is applicable when all off-diagonal terms are nonnegative.

Theorem 1. *The matrix* (2) *is diagonally stable*; *that is*, *it satisfies*

$$DA + A^{\mathrm{T}}D < 0 \tag{4}$$

for some diagonal matrix $D > 0$, *if and only if* (3) *holds.*

The remaining results of this paper are presented in the form of corollaries to this theorem. Tyson and Othmer (1978) and Thron (1991) studied the characteristic polynomial of (2) and showed that (3) is a sufficient condition for A to be Hurwitz. They further showed that this condition is also necessary when α_i's are equal. Theorem 1 proves that (3) is necessary and sufficient for *diagonal* stability even when α_i's are not equal. This means that if A is Hurwitz but (3) fails, then the Lyapunov inequality $A^{\mathrm{T}}P + PA < 0$ does not admit a diagonal solution.

Proof of Theorem 1. We prove the theorem for the matrix

$$A_0 = \begin{bmatrix} -1 & 0 & \cdots & 0 & -\gamma_1 \\ \gamma_2 & -1 & \ddots & & 0 \\ 0 & \gamma_3 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \gamma_n & -1 \end{bmatrix} \tag{5}$$

because other matrices of the form (2) can be obtained by scaling the rows of this A_0 by positive constants, which does not change diagonal stability. Our task is thus to prove necessity and sufficiency for diagonal stability of the condition

$$\gamma_1 \cdots \gamma_n < \sec(\pi/n)^n, \tag{6}$$

which is (3) for A_0. Necessity follows because the diagonal entries of A_0 are equal, in which case (6) is necessary for A_0 to be Hurwitz (Tyson & Othmer, 1978). To prove that (6) is also sufficient for diagonal stability, we define

$$r := (\gamma_1 \ldots \gamma_n)^{1/n} > 0,$$
$$\Delta := \mathrm{diag}\left\{1, -\frac{\gamma_2}{r}, \frac{\gamma_2\gamma_3}{r^2}, \ldots, (-1)^{n+1}\frac{\gamma_2 \cdots \gamma_n}{r^{n-1}}\right\} \tag{7}$$

and note that

$$-\Delta^{-1}A_0\Delta = \begin{bmatrix} 1 & 0 & \cdots & 0 & (-1)^{n+1}r \\ r & 1 & \ddots & & 0 \\ 0 & r & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & r & 1 \end{bmatrix}. \tag{8}$$

Thus, with the choice

$$D = \Delta^{-2} \tag{9}$$

we get

$$DA_0 + A_0^{\mathrm{T}} D = \Delta^{-1}(\Delta^{-1}A_0\Delta + \Delta A_0^{\mathrm{T}}\Delta^{-1})\Delta^{-1}, \tag{10}$$

which means that $DA + A^{\mathrm{T}}D < 0$ holds if the symmetric part of (8), given by

$$\tfrac{1}{2}(-\Delta^{-1}A_0\Delta - \Delta A_0^{\mathrm{T}}\Delta^{-1}), \tag{11}$$

is positive definite. To show that this is indeed the case, we note that (8) exhibits a *circulant* structure (Davis, 1979) when n is odd, and a *skew-circulant* structure when n is even. In particular, it admits the eigenvalue–eigenvector pairs

$$\lambda_k = 1 + r\mathrm{e}^{\mathrm{i}(2\pi/n)k} v_k = \frac{1}{n}[1\mathrm{e}^{-\mathrm{i}(2\pi/n)k}\mathrm{e}^{-\mathrm{i}2(2\pi/n)k} \ldots \mathrm{e}^{-\mathrm{i}(n-1)(2\pi/n)k}]^{\mathrm{T}}$$

$k = 1, \ldots, n$ when n is odd; and

$$\lambda_k = 1 + r\mathrm{e}^{\mathrm{i}(\pi/n+(2\pi/n)k)},$$
$$v_k = \frac{1}{n}[1\mathrm{e}^{-\mathrm{i}(\pi/n+(2\pi/n)k)}\mathrm{e}^{-\mathrm{i}2(\pi/n+(2\pi/n)k)} \ldots \mathrm{e}^{-\mathrm{i}(n-1)(\pi/n+(2\pi/n)k)}]^{\mathrm{T}}$$

when n is even. Since, in either case, (8) is diagonalizable with the unitary matrix $V = [v_1 \ldots v_n]$, the eigenvalues of the symmetric part (11) coincide with the real parts of λ_k's above. Finally, because

$$\min_{k=1,\ldots,n} Re\{1 + r\mathrm{e}^{\mathrm{i}(2\pi/n)k}\} = \min_{k=1,\ldots,n} Re\{1 + r\mathrm{e}^{\mathrm{i}(\pi/n+(2\pi/n)k)}\} = 1 - r\cos(\pi/n), \tag{12}$$

we conclude that if (6) holds, that is $r < \sec(\pi/n)$, then all eigenvalues of (11) are positive and, hence, (11) is positive definite and (10) is negative definite. □

3. Application to output strictly passive systems

The linear stability criterion (3) has been extended in Sontag (2006) to the feedback interconnection of Fig. 1 where H_i's are characterized by the *output strict passivity* (OSP) property (van der Schaft, 2000; Sepulchre et al., 1997):

$$-\mu_i \leqslant -\|y_i\|^2 + \gamma_i\langle u_i, y_i\rangle, \tag{13}$$

where $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote, respectively, the norm and inner product in the extended L_2 space, and $\mu_i \geqslant 0$ represents a bias due to initial conditions. Using this property, Sontag (2006) proves stability under the secant condition (6).

Unlike the input–output proof given in Sontag (2006), we now assume that a *storage function* V_i is available for each block in Fig. 1, and show that a weighted sum of these V_i's,

$$V = \sum_{i=1}^{n} d_i V_i, \tag{14}$$

where $d_i > 0$ are chosen following the procedure below, is a Lyapunov function for the closed-loop system. Indeed, a storage function verifying the OSP property (13) satisfies

$$\dot{V}_i \leqslant -y_i^2 + \gamma_i u_i y_i, \tag{15}$$

which when substituted in (14) along with the interconnection conditions

$$u_1 = -y_n, \quad u_i = y_{i-1}, \quad i = 2, \ldots n,$$

results in

$$\dot{V} \leqslant y^{\mathrm{T}} DA_0 y = \tfrac{1}{2}\, y^{\mathrm{T}}(A_0^{\mathrm{T}}D + DA_0)y, \tag{16}$$

where A_0 is as in (5) and D is a diagonal matrix comprising the coefficients d_i in (14). It then follows from Theorem 1 that if (6) holds then positive d_i's that render the right-hand side of (16) negative definite indeed exist:

Corollary 1. *Consider the feedback interconnection in Fig. 1 and let u_i, x_i and y_i denote the input, state vector, and output of each block H_i. Suppose, further, there exist C^1 storage functions $V_i(x_i)$, satisfying (15) with $\gamma_i > 0$ along the state trajectories of each block. Under these conditions, if (6) holds then there exist $d_i > 0$, $i = 1, \ldots, n$, such that the Lyapunov function (14) satisfies*

$$\dot{V} = \sum_{i=1}^{n} d_i \dot{V}_i \leqslant -\varepsilon|y|^2 \tag{17}$$

for some $\varepsilon > 0$.

In this corollary we showed how to construct a Lyapunov function for the interconnection in Fig. 1 from storage functions for the individual blocks. We do not discuss the various stability properties that can be established with the resulting Lyapunov function. Numerous results are available in the literature, including the *zero state detectability* notion for the state x_i from the output y_i (see e.g. Sepulchre et al., 1997) which allows one to establish asymptotic stability from (17) when the right-hand side is only semidefinite.

Corollary 1 still holds when some of the blocks are static nonlinearities satisfying the sector condition

$$0 \leqslant -y_i^2 + \gamma_i u_i y_i, \quad \gamma_i > 0, \tag{18}$$

rather than the dynamic property (15). To see this we let $\mathscr{I}$ denote the subset of indices i which correspond to dynamic blocks H_i satisfying (15), and employ the Lyapunov function

$$V = \sum_{i\in\mathscr{I}} d_i V_i. \tag{19}$$

For the static blocks, that is H_i, $i \notin \mathscr{I}$, we note from (18) that the sum

$$\sum_{i\notin\mathscr{I}} d_i(-y_i^2 + \gamma_i u_i y_i), \quad d_i > 0 \tag{20}$$

is nonnegative and, hence,

$$\dot{V} \leqslant \sum_{i\in\mathscr{I}} d_i\dot{V}_i + \sum_{i\notin\mathscr{I}} d_i(-y_i^2 + \gamma_i u_i y_i) \leqslant \sum_{i=1}^{n} d_i(-y_i^2 + \gamma_i u_i y_i) = y^{\mathrm{T}}(DA_0 + A_0^{\mathrm{T}}D)y. \tag{21}$$

Then, as in Corollary 1, condition (6) insures existence of a $D > 0$ such that $\dot{V} \leqslant -\varepsilon|y|^2$ for some $\varepsilon > 0$.

4. A Popov criterion

A special case of interest is the feedback interconnection in Fig. 2, where H_i, $i = 1, \dots, n$, are dynamic blocks as in (15), and the feedback nonlinearity $\psi(\cdot)$ satisfies the *sector* property:

$$0 \leqslant y_n \psi(y_n) \leqslant \kappa y_n^2, \tag{22}$$

rewritten here as

$$0 \leqslant -\psi(y_n)^2 + \kappa\psi(y_n)y_n. \tag{23}$$

If we treat the feedback nonlinearity as a new block $y_{n+1} = \psi(y_n)$, and note from (23) that it satisfies (18) with $\gamma_{n+1} = \kappa$, we obtain from Corollary 1 and the ensuing discussion the stability condition:

$$\kappa\gamma_1 \dots \gamma_n < \sec(\pi/(n+1))^{(n+1)}. \tag{24}$$

This condition, however, may be conservative because it does not exploit the static nature of the feedback nonlinearity. Indeed, using the Popov criterion, Tyson and Othmer (1978) obtained a relaxed condition in which $n+1$ in the right-hand side of (24) is reduced to n when H_i's are first-order linear blocks $H_i(s) = \beta_i/(s+\alpha_i)$ and the feedback nonlinearity is time invariant.

To extend this result to the case where H_i's are OSP as in (15), we recall that the main premise of the Popov criterion is that a time-invariant sector nonlinearity, when cascaded with a first-order, stable, linear block preserves its passivity properties. This means that, by only restricting H_n to be linear, and combining it with the feedback nonlinearity as in Fig. 3, the relaxed sector condition of Tyson and Othmer (1978) holds even if $H_1, \dots, H_{n-1}$ are nonlinear:

Corollary 2. *Consider the feedback interconnection in Fig. 2 where H_i, $i = 1, \dots, n-1$, satisfy* (15) *with C^1 storage functions V_i and $\gamma_i > 0$, H_n is a linear block with transfer function*

$$H_n(s) = \frac{\beta_n}{s+\alpha_n}, \quad \beta_n > 0, \ \ \alpha_n > 0, \quad \gamma_n := \frac{\beta_n}{\alpha_n}, \tag{25}$$

the feedback nonlinearity $\psi(\cdot)$ is time invariant, satisfies the sector property (22), *and*

$$\psi(y) = 0 \ \Rightarrow \ y = 0. \tag{26}$$

Under these assumptions, if

$$\kappa\gamma_1 \dots \gamma_n < \sec(\pi/n)^n, \tag{27}$$

then there exists a Lyapunov function of the form

$$V = \sum_{i=1}^{n-1} d_i V_i + d_n \int_0^{y_n} \psi(\sigma)\,\mathrm{d}\sigma, \quad d_i > 0, \ i = 1, \dots, n, \tag{28}$$

satisfying

$$\dot{V} \leqslant -\varepsilon|(y_1, \dots, y_{n-1}, \psi(\beta_n y_n))|^2$$

for some $\varepsilon > 0$.

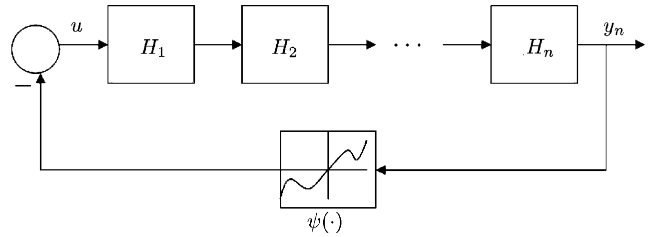

Fig. 2. The feedback interconnection for Corollary 2.

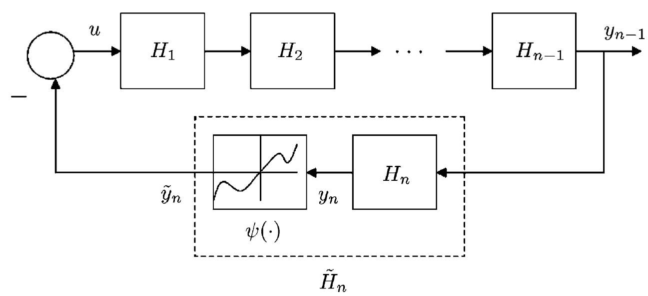

Fig. 3. An equivalent representation of the feedback system in Fig. 2. When H_n is a linear block $H_n(s) = \beta_n/(s+\alpha_n)$, its series interconnection with the $[0, \kappa]$ sector nonlinearity $\psi(\cdot)$ constitutes a dynamic block $\tilde{H}_n$ which satisfies the OSP property (15) with $\tilde{\gamma}_n = \kappa(\beta_n/\alpha_n)$.

Proof. Rather than treat H_n and $\psi(\cdot)$ as separate blocks, we combine them as in Fig. 3:

$$\tilde{H}_n : \begin{cases} \dot{y}_n = -\alpha_n y_n + y_{n-1}, \\ \tilde{y}_n = \psi(\beta_n y_n), \end{cases} \tag{29}$$

and define

$$V_n = \frac{\kappa}{\alpha_n} \int_0^{\beta_n y_n} \psi(\sigma)\,\mathrm{d}\sigma, \tag{30}$$

which is positive definite from (22) and (26), and satisfies

$$\dot{V}_n = -\kappa\beta_n y_n \psi(\beta_n y_n) + \kappa\gamma_n \psi(\beta_n y_n) y_{n-1}. \tag{31}$$

Because $-\kappa\beta_n y_n \psi(\beta_n y_n) \leqslant -\psi(\beta_n y_n)^2$ from (23), we conclude

$$\begin{aligned} \dot{V}_n &\leqslant -\psi(\beta_n y_n)^2 + \kappa\gamma_n \psi(\beta_n y_n) y_{n-1} \\ &= -\tilde{y}_n^2 + \kappa\gamma_n \tilde{y}_n y_{n-1}, \end{aligned} \tag{32}$$

which shows that $\tilde{H}_n$ is OSP as in (15), with $\tilde{\gamma}_n = \gamma_n\kappa$. The result then follows from Corollary 1. □

Corollary 2 can be further generalized to the situation where other nonlinearities exist in between the blocks H_i, $i = 1, \dots, n$, in Fig. 2. If such a nonlinearity is preceded by a first-order linear block then the two can be treated as a single block, thus reducing n in the right-hand side of (6).

5. A class of nonlinear cyclic systems

We now study the system

$$\begin{aligned}\dot{x}_1 &= -a_1(x_1) - b_n(x_n),\\ \dot{x}_2 &= -a_2(x_2) + b_1(x_1),\\ &\vdots\\ \dot{x}_n &= -a_n(x_n) + b_{n-1}(x_{n-1}),\end{aligned} \tag{33}$$

which encompasses the linear system (2) where $a_i(x_i) = \alpha_i x_i$ and $b_i(x_i) = \beta_i x_i$. Using the stability criterion of Corollary 1 and the construction of storage functions as in Corollary 2 from integrals of nonlinear interconnection terms, we obtain the following result:

Corollary 3. *Consider system* (33) *where $a_i(\cdot)$ and $b_i(\cdot)$ are continuous functions satisfying*

$$x_i a_i(x_i) > 0, \quad x_i b_i(x_i) > 0 \;\; \forall x_i \neq 0, \tag{34}$$

and suppose there exist constants $\gamma_i > 0$ such that

$$\frac{b_i(x_i)}{a_i(x_i)} \leqslant \gamma_i \quad \forall x_i \neq 0. \tag{35}$$

If these γ_i's satisfy (6) *then the equilibrium $x = 0$ is asymptotically stable. If, further, the functions $b_i(\cdot)$ are such that*

$$\lim_{|x_i|\to\infty} \int_0^{x_i} b_i(\sigma)\,\mathrm{d}\sigma = \infty, \tag{36}$$

then $x = 0$ is globally asymptotically stable.

In this corollary we only assumed continuity for $a_i(\cdot)$ and $b_i(\cdot)$, which does not guarantee uniqueness of solutions. Uniqueness, however, is not essential for asymptotic stability because we construct a Lyapunov function in the proof, from which we can obtain stability and convergence estimates that apply uniformly to all solutions.

Proof of Corollary 3. We view system (33) as the feedback interconnection of Fig. 1 where the ith block is now given by

$$H_i : \begin{cases} \dot{x}_i = -a_i(x_i) + u_i, \\ y_i = b_i(x_i). \end{cases} \tag{37}$$

To show that this H_i is OSP as in (15) we let

$$V_i(x_i) = \gamma_i \int_0^{x_i} b_i(\sigma)\,\mathrm{d}\sigma, \tag{38}$$

and note that it yields

$$\dot{V}_i = -\gamma_i b_i(x_i) a_i(x_i) + \gamma_i b_i(x_i) u_i. \tag{39}$$

We next multiply both sides of (35) by $b_i(x_i)a_i(x_i)$ which is nonnegative from (34), and obtain the inequality

$$-\gamma_i b_i(x_i) a_i(x_i) \leqslant -b_i(x_i)^2 \tag{40}$$

which, when substituted in (39), results in the OSP estimate (15). Asymptotic stability then follows from Corollary 1 with the Lyapunov function

$$V = \sum_{i=1}^{n} d_i V_i = \sum_{i=1}^{n} d_i \gamma_i \int_0^{x_i} b_i(\sigma)\,\mathrm{d}\sigma. \tag{41}$$

If (36) holds then this Lyapunov function is *proper* and, thus, proves global asymptotic stability. □

6. Extension to systems with nonnegative state variables

The motivation for the earlier studies (Tyson & Othmer, 1978; Thron, 1991) is a sequence of biochemical reactions in which the end product inhibits the first reaction, thus yielding the cyclic structure studied in this paper. In such reaction models the state variables represent concentrations of substances, which are nonnegative quantities. We now extend the result of the previous section to system (1) where the state vector χ evolves in the positive orthant $\mathbb{R}^n_{\geqslant 0}$, and $f_i(\cdot)$ and $g_i(\cdot)$ are continuous functions satisfying the following assumptions:

(A1) For all $i = 1, \ldots, n$, $f_i(0) = 0$ and

$$f_i(\chi_i) \geqslant 0, \quad g_i(\chi_i) \geqslant 0 \;\; \forall \chi_i \geqslant 0. \tag{42}$$

(A2) There exists a unique equilibrium χ^* with $\chi_i^* \geqslant 0$, and $\forall \chi_i \neq \chi_i^*$

$$(\chi_i - \chi_i^*)(f_i(\chi_i) - f_i(\chi_i^*)) > 0, \quad i = 1, \ldots, n, \tag{43}$$

$$(\chi_i - \chi_i^*)(g_i(\chi_i) - g_i(\chi_i^*)) > 0, \quad i = 1, \ldots, n-1, \tag{44}$$

$$(\chi_n - \chi_n^*)(g_n(\chi_n) - g_n(\chi_n^*)) < 0. \tag{45}$$

(A3) There exist constants $\gamma_i > 0$ such that $\forall \chi_i \neq \chi_i^*$

$$\frac{g_i(\chi_i) - g_i(\chi_i^*)}{f_i(\chi_i) - f_i(\chi_i^*)} \leqslant \gamma_i, \quad i \neq n, \tag{46}$$

$$-\frac{g_n(\chi_n) - g_n(\chi_n^*)}{f_n(\chi_n) - f_n(\chi_n^*)} \leqslant \gamma_n. \tag{47}$$

Assumption (A1) insures invariance of the positive orthant $\mathbb{R}^n_{\geqslant 0}$. To extend Corollary 3 to this system we note that the change of variables

$$x_i := \chi_i - \chi_i^* \tag{48}$$

brings (1) into the form (33), where

$$a_i(x_i) := f_i(\chi_i) - f_i(\chi_i^*), \quad i = 1, \ldots, n, \tag{49}$$

$$b_i(x_i) := g_i(\chi_i) - g_i(\chi_i^*), \quad i = 1, \ldots, n-1, \tag{50}$$

$$b_n(x_i) := -g_n(\chi_n) + g_n(\chi_n^*) \tag{51}$$

satisfy (34) and (35) from assumptions (A2) and (A3), respectively. Combining the Lyapunov arguments of Corollary 3 with the invariance of the positive orthant we obtain the following result:

Corollary 4. *Consider system* (1) *and suppose assumptions* (A1)–(A3) *hold. If the* γ_i*'s in* (A3) *satisfy* (6) *then the equilibrium* $\chi = \chi^*$ *is asymptotically stable. If, further, the functions* $g_i(\cdot)$ *are such that*

$$\lim_{\chi_i \to \infty} \int_{\chi_i^*}^{\chi_i} g_i(\sigma)\,\mathrm{d}\sigma = \infty, \tag{52}$$

then $\chi = \chi^*$ *is asymptotically stable with region of attraction* $\mathbb{R}^n_{\geqslant 0}$.

A sufficient condition for (49)–(51) in (A2) to hold is that the functions $f_i(\cdot)$, $i = 1, \ldots, n$, and $g_i(\cdot)$, $i = 1, \ldots, n-1$, be strictly increasing and $g_n(\cdot)$ be strictly decreasing. Under this assumption, the sector properties (49)–(51) hold regardless of the value of χ^* and, thus, knowledge of the equilibrium is not needed to verify (49)–(51). Furthermore, this assumption also guarantees that an equilibrium, when it exists, is unique as stipulated in (A2). To see this, consider (1) with $g_n(\chi_n)$ in the first equation replaced by an arbitrary input u. Then, since the remaining $f_i(\cdot)$'s and $g_i(\cdot)$'s are strictly increasing, there exist a subset of the input space and a map defined on this subset from u to χ that annihilates the right-hand side of (1). Because this map defines an increasing function from u to χ_n, and because the feedback $u = g_n(\chi_n)$ is decreasing, their graphs can intersect at most one point and, hence, the closed-loop equilibrium must be unique.

Similarly, it is not difficult to show that assumption (A3) holds if f_i's and g_i's are continuously differentiable and, satisfy for all $\chi_i \geqslant 0$ the infinitesimal inequalities

$$\frac{\partial g_i(\chi_i)}{\partial \chi_i} \geqslant 0, \quad \frac{\partial f_i(\chi_i)}{\partial \chi_i} \geqslant 0, \quad i \neq n, \tag{53}$$

$$\frac{\partial g_n(\chi_n)}{\partial \chi_n} \leqslant 0, \tag{54}$$

$$\frac{\partial g_i(\chi_i)}{\partial \chi_i} \leqslant \gamma_i \frac{\partial f_i(\chi_i)}{\partial \chi_i}, \quad i \neq n, \tag{55}$$

$$-\frac{\partial g_n(\chi_n)}{\partial \chi_n} \leqslant \gamma_n \frac{\partial f_n(\chi_n)}{\partial \chi_n}. \tag{56}$$

Example. The reaction sequence

$$S_0 \to S_1 \to S_2 \to \cdots \to S_n \to$$

in which the concentration of S_0 is kept constant, and the rate of formation of S_1 from S_0 is inhibited by S_n, gives rise to a dynamic model of the form (1) where χ_i denotes the concentration of S_i, and the functions $f_i(\cdot)$ and $g_i(\cdot)$ satisfy (A1) and (53)–(54). In particular, $g_n(\chi_n)$ implicitly depends on the constant χ_0, and is a decreasing function of χ_n because it represents the formation rate of S_1 from S_0, which is inhibited by S_n.

If there are no losses of the intermediate substances, that is if $f_i(\chi_i) \equiv g_i(\chi_i)$, $i = 1, \ldots, n-1$, then (55) holds with $\gamma_1 = \cdots = \gamma_{n-1} = 1$. This means that, if (56) holds for all $\chi_n \geqslant 0$ with

$$\gamma_n < \sec(\pi/n)^n, \tag{57}$$

and if an equilibrium exists then, from Corollary 4, it is asymptotically stable with region of attraction $\mathbb{R}^n_{\geqslant 0}$.

As an illustration, we apply this global stability criterion to the following model studied in Thron (1991):

$$\dot{\chi}_1 = \frac{p_1 \chi_0}{p_2 + \chi_3} - p_3 \chi_1, \tag{58}$$

$$\dot{\chi}_2 = p_3 \chi_1 - p_4 \chi_2, \tag{59}$$

$$\dot{\chi}_3 = p_4 \chi_2 - \frac{p_5 \chi_3}{p_6 + \chi_3}, \tag{60}$$

where $p_1, \ldots, p_6$ are positive constants. Using

$$g_3(\chi_3) = \frac{p_1 \chi_0}{p_2 + \chi_3} \quad \text{and} \quad f_3(\chi_3) = \frac{p_5 \chi_3}{p_6 + \chi_3} \tag{61}$$

to obtain a γ_3 as in (55), and applying (57) with $n = 3$ we get the stability condition

$$\gamma_3 = \frac{p_1 \chi_0}{p_5 p_6} \max\left\{1, \left(\frac{p_6}{p_2}\right)^2\right\} < 8. \tag{62}$$

This condition is tight because simulations reported in Thron (1991, p. 390) with $p_1 = p_2 = p_5 = p_6 = 1$, $\chi_0 = 9$ (which result in $\gamma_3 = 9$ in (62) above) show unstable oscillations. Unlike the local study in Thron (1991), our condition (62) ensures global asymptotic stability and, furthermore, it is unchanged if the linear reaction rates $p_3\chi_1$ and $p_4\chi_2$ in (58)–(60) are replaced with arbitrary increasing functions $f_1(\chi_1) = g_1(\chi_1)$ and $f_2(\chi_2) = g_2(\chi_2)$, respectively.

7. The shortage of passivity in a cascade of OSP systems

In this section we present a result of independent interest that concerns the cascade interconnection of OSP systems. When the blocks $H_1, \ldots, H_n$ each satisfy the OSP property (15), their cascade interconnection in Fig. 4 inherits the sum of their phases and loses passivity. The following corollary to Theorem 1 quantifies the "shortage" of passivity in such a cascade:

Corollary 5. *Consider the cascade interconnection in Fig.* 4. *If each block* H_i *satisfies* (15) *with a* C^1 *storage function* V_i *and* $\gamma_i > 0$, *then for any*

$$\delta > \gamma_1 \ldots \gamma_n \cos(\pi/(n+1))^{(n+1)}, \tag{63}$$

the cascade admits a storage function of the form (14) *satisfying*

$$\dot{V} \leqslant -\varepsilon |y|^2 + \delta u^2 + u y_n \tag{64}$$

for some $\varepsilon > 0$.

Inequality (64) is an IFP property (Sepulchre et al., 1997) where the number δ represents the gain with which a feedforward path, if added from u to y_n in Fig. 4, would achieve passivity. Corollary 5 thus shows that the cascade of OSP systems

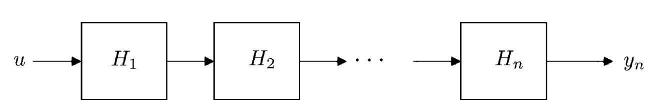

Fig. 4. The cascade interconnection for Corollary 5.

(15) in which $\gamma_i > 0$ represents an "excess" of passivity, satisfies the IFP property (64) with a "shortage" characterized by (63).

Proof of Corollary 5. Using (14), (15), and substituting $u_i = y_{i-1},\ i = 2, \ldots, n$, we rewrite (64) as

$$d_1(-y_1^2+\gamma_1 y_1 u)+\sum_{i=2}^{n} d_i(-y_i^2+\gamma_i y_i y_{i-1})+\delta\left(-u^2-\frac{1}{\delta}u y_n\right) \leqslant -\varepsilon|y|^2. \quad (65)$$

To show that $d_i > 0,\ i = 1, \ldots, n$, satisfying (65) indeed exist, we define

$$\tilde{A} = \begin{bmatrix} -1 & 0 & \cdots & 0 & -\frac{1}{\delta} \\ \gamma_1 & -1 & \ddots & & 0 \\ 0 & \gamma_2 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \gamma_n & -1 \end{bmatrix} \quad (66)$$

and note that the left-hand side of (65) is

$$[u\ \ y^{\mathrm{T}}]\tilde{D}\tilde{A}\begin{bmatrix} u \\ y \end{bmatrix}, \quad (67)$$

where $\tilde{D} := \mathrm{diag}\{\delta,\ d_1, \ldots, d_n\}$. Because $\tilde{A}$ is of the form (2) with dimension $(n+1)$, an application of Theorem 1 shows that a diagonal $\tilde{D}$ rendering (67) negative definite exists if and only if $(\gamma_1 \ldots \gamma_n(1/\delta)) < \sec(\pi/(n+1))^{(n+1)}$. Because this condition is satisfied when δ is as in (63), we conclude that such a $\tilde{D} > 0$ exists and, thus, (64) holds. □

8. Conclusions

The secant condition (3) exploits gain and phase information simultaneously, and proves stability in situations where *small-gain* and *passivity* theorems are not applicable. In this paper we gave several extensions of this condition to classes of nonlinear systems. The key result was a diagonal stability proof, which was used in the rest of the paper as a tool for constructing Lyapunov functions. Further attempts to bridge the gap between passivity and small-gain theorems would be of great interest in nonlinear systems research.

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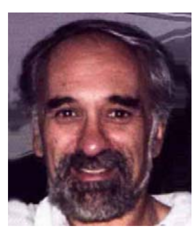

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