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An eigenvalue condition for sampled weak controllability of bilinear systems

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Abstract: Weak controllability of bilinear systems is preserved under sampling provided that the sampling period satisfies a condition related to the eigenvalues of the autonomous dynamics matrix. This condition generalizes the classical Kalman–Ho–Narendra criterion which is well known in the linear case.

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1. Introduction and statement of result

This note describes an explicit result for bilinear systems, to be obtained as a consequence of a more general but abstract sampling result given in [6]. It seems appropriate to present this explicit condition separately; the notations and statements are considerably simpler in the bilinear case. The criterion generalizes the classical one for linear systems [5,1,2]. In fact, its proof relies essentially in defining, for a given bilinear system, an associated linear system, corresponding to the adjoint representation of the Lie algebra of the original system, to which the usual criterion is applied.

We first give some general definitions. A (continuous-time, input-linear, analytic) *system* Σ is described by equations

$$\dot{x}(t) = f(x(t)) + \sum_{1}^{m} u_i(t) g_i(x(t)), \tag{1.1}$$

where states $x(t)$ belong to a real-analytic Hausdorff second countable connected n-dimensional manifold $\mathbf{M}$, and controls $u(\cdot) = (u_1(\cdot), \ldots, u_m(\cdot))$ take values in $\mathbb{R}^m$. We assume that $f + \sum u_i g_i$ is an analytic complete vector field for each $u \in \mathbb{R}^m$. A *bilinear* system is one for which $\mathbf{M} = \mathbb{R}^n$ for some n, and the vector fields $f, g_1, \ldots, g_m$ are all linear. Thus there are in the bilinear case matrices F and G_i such that the equations take the form

$$\dot{x} = \left(F + \sum u_i G_i\right) x. \tag{1.2}$$

See [3] for an introduction to systems (1.1) and bilinear systems.

Let $\sigma = (s_1, \ldots, s_r)$ be a sequence of positive real numbers, with $T := \sum s_i$, and let $\mu := (\mu_1, \ldots, \mu_r)$ be a sequence of elements in $\mathbb{R}^m$. Then μ_σ is the control function u of length T defined as follows:

$$u(t) := \mu_i \quad \text{if } t \in [s_0 + \cdots + s_{i-1}, s_0 + \cdots + s_i),$$

$i = 1, \ldots, r$ (denoting $s_0 := 0$). If δ is a positive real and $\mu := (\mu_1, \ldots, \mu_r) \in \mathbb{R}^m$, μ_δ^r is the *δ-sampled* control μ_σ, where $\sigma := (\delta, \ldots, \delta)$ (r times).

If ξ is in $\mathbf{M}$ and $u = \mu_\delta^r$ then $\zeta = \phi[\xi, u]$ is the solution at time T of (1.1) with this $u(\cdot)$ and initial condition $x(0) = \xi$; we say that ζ is *δ-reachable* from ξ (in r steps). The δ-accessibility relation is the equivalence relation generated by δ-reachability, and $A_\delta(\xi)$ is the set of states δ-accessible from ξ. Thus, $\zeta \in A_\delta(\xi)$ iff there are integers $t_1, \ldots, t_k$, and a sequence of states $\xi_0 = \xi, \xi_1, \ldots, \xi_k = \zeta$, such that for each i either ξ_i is δ-reachable from ξ_{i+1} in $-t_i$ steps, or ξ_{i+1} is δ-reachable from ξ_i in t_i steps.

The system Σ is *weakly δ-controllable at* ξ if $A_\delta(\xi)$ is a neighborhood of ξ; it is *weakly sampled controllable at* ξ if it is weakly δ-controllable at ξ for some $\delta > 0$.

Finally, recall that the system (1.1) satisfies the *strong accessibility condition at* ξ iff the Lie algebra L_0 of vector fields on $\mathbf{M}$ generated by

$$\left\{ \mathrm{ad}_f^j(g_i),\ j \geq 0,\ i = 1, \ldots, m \right\} \tag{1.3}$$

has rank n at ξ, i.e.

$$\dim \operatorname{span}\{ X(\xi),\ X \in L_0 \} = n.$$

Here $\mathrm{ad}_f^j(h)$ denotes the Lie bracket $[f, h]$. This

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property is equivalent to a notion of 'zero time weak controllability' at ξ; see [7]. The result to be proved in this note is as follows.

Theorem. *Let Σ be a bilinear system and pick $\xi \in \mathbf{M}$. If Σ is sampled accessible at ξ then it satisfies the strong accessibility condition at ξ. Conversely, if the strong accessibility condition at ξ holds, then Σ is δ-accessible for each $\delta > 0$ such that*

$$\delta[(\lambda_1 + \lambda_2) - (\mu_1 + \mu_2)] \neq 2k\pi \mathrm{i} \text{ for all nonzero } k \in \mathbf{Z} \tag{1.4}$$

for any set of 4 eigenvalues λ_1, λ_2, μ_1, μ_2 of F.

2. Proof of the result

Necessity of the strong accessibility condition follows from Propositions 9.2 and 3.9 in [6]. We now prove the converse.

Formally, we let J^δ be the following linear operator on vector fields on $\mathbf{M}$:

$$J^\delta := \sum_{k=1}^{\infty} (\delta^k/k!)\mathrm{ad}_f^{k-1}.$$

If $\pi : \mathbf{M} \to \mathbf{M}$ is a diffeomorphism, we denote by Ad_π the linear operator on vector fields corresponding to conjugation by π, more precisely:

$$\mathrm{Ad}_\pi X(\xi) := (\pi^{-1})_*(X(\pi(\xi)))$$

for any vector field X and each ξ in $\mathbf{M}$ (where $(\pi^{-1})_*$ denotes the differential of π^{-1} at the point $\pi(\xi)$). Finally, let $b_j = J^\delta g_j$. Even if the series defining b_j does not converge in any reasonable sense, it is still possible to give a well defined meaning to this vector field ([6], Section 8). In any case, we shall only consider the bilinear case, in which convergence is not an issue since we deal with entire functions of matrices. Consider for each δ the Lie algebra of vector fields L_δ generated by the elements

$$\left\{\mathrm{Ad}^k_{\exp[-\delta f]} J^\delta g_j \mid k \geqslant 0,\ j = 1, \ldots, m\right\}.$$

The following result is proved in [6], Corollary 9.8:

Proposition. *If L_δ has full rank at ξ then Σ is weakly δ-controllable at ξ. If Σ satisfies the strong accessibility condition at ξ then there is a $\Delta > 0$ such that for each $0 < \delta < \Delta$, L_δ has full rank at ξ.*

So we need to establish that, in the bilinear case, L_δ has full rank at ξ when the strong accessibility condition and the stated eigenvalue condition hold. Consider the analytic function $\omega(z) := (\mathrm{e}^z - 1)/z$. For any finite dimensional linear operator A, we may consider the following operator:

$$\omega(A) := \sum_{k=1}^{\infty} A^{k-1}/k!.$$

By the spectral mapping theorem, $\omega(A)$ is nonsingular if A has no eigenvalues of the form $2k\pi\mathrm{i}$. For any linear operator A on a space V and elements $v_1, \ldots, v_m$ of V, let

$$\{A \mid v_1, \ldots, v_m\} := \mathrm{span}\{A^k v_i,\ i = 1, \ldots, m,\ k \geqslant 0\}.$$

In particular, let A be the operator ad_F, i.e. $\mathrm{ad}_F(G) := GF - FG$, acting on $M_n(\mathbb{R}) = \{n \times n$ real matrices$\}$. Pick any $\delta > 0$. The vector fields b_i are linear, $b_i(\xi) = B_i\xi$, with

$$B_i = \delta\omega(\delta A)G_i$$

for each i. Also, by the Baker–Campbell–Hausdorff formula we have that $\mathrm{Ad}_{\exp[-\delta f]}b_i$ is a linear vector field corresponding to the matrix

$$\mathrm{e}^{\delta A}B_i.$$

Thus L_δ identifies with the Lie algebra of matrices generated by the following elements:

$$\delta\, \mathrm{e}^{k\delta A}\omega(\delta A)G_i = \delta\omega(\delta A)\, \mathrm{e}^{k\delta A}G_i \tag{2.1}$$

with $i = 1, \ldots, m$ and $k \geqslant 0$. With these notations, in the bilinear case strong accessibility at ξ means then that the set of all vectors of type $M\xi$, with M in

$$\{A \mid G_1, \ldots, G_m\},$$

spans a space of dimension n. We shall prove that, for δ and F satisfying condition (1.4), the linear span of the generators in (2.1) coincides with $\{A \mid G_1, \ldots, G_m\}$. It will follow that L_δ has rank n at ξ.

Assume now that δ and F satisfy condition (1.4). The eigenvalues of A are differences of eigenvalues of F; this follows from matrix equation theory, and is also a standard Lie algebraic fact (see e.g. [4], Chapter I, exercise 16). Thus the eigenvalues of A satisfy the condition that

$$\delta[\lambda - \mu] \neq 2k\pi \mathrm{i} \quad \text{for all nonzero } k \in \mathbf{Z}. \tag{2.2}$$

(This is the condition imposed on the 'A' matrix in linear system theory for the analogous sampled controllability result.) In particular, $\lambda = 2k\pi\mathrm{i}$ cannot be an eigenvalue of the real matrix δA (take $\mu = \bar{\lambda}$), and $\omega(\delta A)$ is invertible. Thus we need to prove simply that

$$\{e^{\delta A} \mid G_1, \ldots, G_m\} = \{A \mid G_1, \ldots, G_m\}.$$

This is precisely what is done when proving the classical linear system result. We give a simple argument here. Consider the exponential function e^z. The condition (2.2) assures that this function is one-to-one in a neighborhood of the spectrum of δA. Since e^z is everywhere nonsingular, it has a well defined inverse (a determination of $\log z$) in a neighborhood of the spectrum of $e^{\delta A}$. It follows that A is a function of $e^{\delta A}$, and the desired equality holds.

3. Remarks

Roughly, the theorem says that sampling at twice the frequency predicted by the linear theory will insure preservation of weak controllability. Actually, a much weaker condition is sufficient. By a simple coordinate change in $\mathbb{R}^m$, we can replace F by any linear combination $F + \Sigma r_i G_i$. The eigenvalue condition for any of these new matrices will then be sufficient. Thus in various senses of genericity, all sampling periods δ preserve accessibility, for fixed F, if the G_i are 'randomly' chosen. Finally, note that one could talk about systems like (1.2) with an added extra linear term on x ('internally biaffine systems'); an analogous eigenvalue condition results in that case as well, or, more generally, for systems with finite dimensional Lie algebra.

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