A CHOW PROPERTY FOR SAMPLED BILINEAR SYSTEMS

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An eigenvalue condition is given for the sampled positive-time accessibility of a class of nonlinear systems. This result generalizes a previous result of the author, which applied only to bilinear systems and which only concluded a weaker notion of controllability.

1 Introduction

When a continuous-time system is regulated by a digital computer, control decisions are often restricted to be taken at fixed times $0, \delta, 2\delta, ...$; one calls $\delta > 0$ the *sampling time*. Under what is called zero-th order hold sampled control, the resulting situation can be modeled through the constraint that the inputs applied be constant on intervals of length δ. A prefilter may be applied to the system to smooth out control discontinuities, but mathematically the situation is as with piecewise constant controls.

It is thus of interest to characterize the preservation of basic system properties when the controls are so restricted. For controllability, this problem motivated the results in the classical paper of Kalman, Ho, and Narendra [3]. This studied the case of linear systems and established that controllability when sampling at intervals of length δ is preserved provided that $\delta(\lambda - \mu)$ is not of the form $2k\pi i$, for any pair of distinct eigenvalues of the A matrix. The dual version of this result, for observability, is basically the classical Nyquist-Shannon sampling theorem from digital signal processing, and is usually summarized by the statement that controllability (or observability) is preserved provided that one samples at more than twice the natural frequencies of the system.

In [5], we found a general result which in particular implies, for the class of *bi*linear systems, an analogous property; one now needs that $\delta(\lambda + \lambda' - \mu - \mu')$ not be equal to $2k\pi$, k non zero, for any four eigenvalues of the autonomous dynamics matrix. Thus in the bilinear case, one must sample at more than 4 times (rather than twice) the natural frequencies of the system. The bilinear result was obtained by inducing a linear system on the adjoint representation of a certain Lie algebra associated to the given system. The result was proved for transitivity, often called also the "weak controllability," property. The present work shows how to extend this to the much more interesting *(forward) accessibility* (or "positive Chow") property, with the same eigenvalue condition being sufficient. This extension is possible based on new results on discrete time controllability from [4]. Moreover, we can now generalize the result to deal with a much larger class than that of bilinear systems.

The plan for this paper is as follows. We first give definitions and state the main results. After that we shall recall details of the linear case, giving an abstract proof of the classical result, and finally we shall show how to reduce the nonlinear problem to a suitable linear one on a larger dimensional space.

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2 Frequencies and Accessibility

The systems σ that we shall consider have the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}$ for each t; n is the dimension of the system, m the number of independent controls. We assume also that the vector field f is complete, that is, that solutions of the unforced system are defined for all initial conditions and all $t \geq 0$. Further, we assume that f as well as the g_i are real-analytic.

Fix an equilibrium state $x_0 \in \mathbb{R}^n$ of (1), that is, a zero $f(x_0) = 0$, and consider the linearization $F = (\partial f / \partial x)(x_0)$. We shall say that the *natural frequencies* of the system (1) (about the equilibrium state x_0) are the imaginary parts of the eigenvalues of F, and let $\omega(\sigma, x_0)$, or just ω, be the set of these numbers (counted with multiplicities). Note that since F is real, $-\omega \in \omega$ whenever $\omega \in \omega$. For each nonnegative integer j we denote by

$$\mathcal{B}_j$$

the set of all linear combinations

$$\frac{1}{k} \sum_{i=1}^{n} \rho_i \omega_i \tag{2}$$

with k a nonzero integer, $\omega_1, \ldots, \omega_n$ the natural frequencies, and the ρ_i's nonnegative integers satisfying

$$\sum_{i=1}^{n} \rho_i = 2j + 2.$$

Note that if λ is the largest of the ω_i, (equivalently, the largest absolute value of these,) each element of $\mathcal{B}_j$ is in magnitude bounded by $(2j + 2)\lambda$.

The main results will hold for systems for which f is linear and the g_i are all polynomial vector fields. By $\tilde{\sigma}_d^H$ we denote the class of all systems for which f is linear, $f(x) = Fx$, and the coordinates of all the g_i are homogeneous polynomials of degree exactly d. For instance, $\tilde{\sigma}_0^H$ is the class of all linear systems (the g_i's are constant vectors,) while $\tilde{\sigma}_1^H$ is the class of bilinear systems, treated in [5]. We drop the superscript H to indicate more general polynomials: $\tilde{\sigma}_d$ is the class of all systems for which f is linear and the g_i are polynomials of degree at most d. For such systems, the natural frequencies are just the imaginary parts of the eigenvalues of F.

Under relatively mild conditions (the Poincaré resonance conditions), it is possible to reduce under coordinate changes more general systems, at least locally, to the case where f is linear. However, any structure on the g_i's is then destroyed.

Accessibility

Fix again an equilibrium state x_0. The set of states of σ that can be reached from x_0 in time $T > 0$, using arbitrary (measurable locally integrable) controls $u(\cdot)$ is denoted by $A^T(x_0)$. The system (1) is said to be *(forward) accessible (from x_0)* if it holds that for some $T > 0$ this reachable set has full dimension, more precisely if

$$\text{int } A^T(x) \neq \emptyset \ . \tag{3}$$

Let $\omega > 0$ be any real number. We shall say that σ is *ω-(forward) accessible (from x_0)*, or *(forward) accessible under sampling at frequency ω (from x_0)*, if the set of states

$$A^T_\omega(x_0)$$

reachable from x_0 in time T using controls sampled at that frequency has a nonempty interior. A control $u(\cdot)$ defined on an interval $[0, T]$ is said to be sampled at frequency ω (in radians/sec) iff T is an integer multiple of $\delta := 2\pi/\omega$, say $T = r\delta$, and there are vectors

$$v_1, \ldots, v_r$$

such that $u(t) \equiv v_i$ on the interval $[(i-1)\delta, i\delta)$. With this definition it is clear that ω-accessibility for even a single ω implies accessibility. The following two main theorems provide a converses to this fact; they will be proved later.

Theorem 1 *Let x_0 be an equilibrium state, and assume that $\sigma \in \tilde{\sigma}^H_d$ is accessible from x_0. If $\omega > 0$ is not in $\mathcal{B}_d$ then σ is also ω-accessible.* ∎

Theorem 2 *Let x_0 be an equilibrium state, and assume that $\sigma \in \tilde{\sigma}_d$ is accessible from x_0. If $\omega > 0$ is not in $\mathcal{B}_j$ for any $j \leq d$, then σ is also ω-accessible.* ∎

With λ as above, the following is then an obvious consequence of Theorem 2. For linear systems ($d = 0$), it is a version of the classical Sampling Theorem. For $d = 1$, one recovers in a stronger form the result for bilinear transitivity given in [5].

Corollary 2.0.1 *Accessibility is preserved under sampling for systems in $\tilde{\sigma}_d$ provided that the sampling frequency be larger than $2j+2$ times the largest natural frequency of the system.* ∎

It is possible to generalize these theorems to deal with nonpolynomial g_i's, but no simple corollary as the above would appear to hold.

3 Linear systems

It is worth reviewing the case of linear systems in detail, since it will be useful later. This is basically due to [3]. Let V be a vector space over the reals, and let A be a linear map $V \to V$. For any subspace $\mathcal{B}$ of V, we denote by

$$\{A|\mathcal{B}\}$$

the smallest A-invariant subspace of V which contains $\mathcal{B}$.

Assume from now on that V is finite-dimensional. We may then think of V as a Banach space, when endowed with any fixed norm. (Since all norms are equivalent, the particular norm used will not be important in what follows.) If $f(z)$ is an analytic function defined on an open set which contains the spectrum of A, we let $f(A)$ be the application of f to A as per the spectral mapping theorem. In particular, for each $\delta \in \mathbb{R}$, $e^{\delta A}$ denotes the exponential of the matrix δA, which is well-defined for all A, and $\theta_\delta(A)$ will denote the application to any A of the entire function

$$\theta_\delta(z) := \frac{e^{\delta z} - 1}{z}.$$

If it held that $A = \beta(e^{\delta A})$ for some δ and some analytic function β, then for each vector b it would hold that Ab is a linear combination of (finitely many) terms of the form $e^{k\delta A}b$ (k = nonnegative integers,) and therefore

$$\{A|\mathcal{B}\} = \{e^{\delta A}|\mathcal{B}\} \tag{4}$$

for any subspace $\mathcal{B}$. If it also holds that $\theta_\delta(A)$ is invertible, then also

$$\{A|\mathcal{B}\} = \{e^{\delta A}|\theta_\delta(A)\mathcal{B}\}. \tag{5}$$

This last equality is a consequence of (4) applied to $\mathcal{B} = \theta_\delta(A)\mathcal{B}$ and of the fact that $\mathcal{B} \subseteq \{A|\theta_\delta(A)\mathcal{B}\}$ for any subspace $\mathcal{B}$, which in turn follows from

$$\mathcal{B} = \theta_\delta(A)^{-1}\theta_\delta(A)\mathcal{B}$$

and the fact that $\theta_\delta(A)^{-1}$ is again an analytic function of A (spectral mapping theorem). The reason that (5) is of interest is that when

$$\dot{x} = Ax + Bu \tag{6}$$

is a linear system, the system obtained by sampling with period δ (frequency $\omega = 2\pi/\delta$ rad/sec) is, by the variation of parameters formula, the discrete time system

$$x^+ = e^{\delta A}x + \theta_\delta(A)Bu,$$

and therefore controllability is preserved with this sampling provided that the original system is controllable and (5) holds.

The zeroes of the mapping $\theta_\delta(z)$ are the complex numbers of the form $\frac{2\pi}{\delta}\sqrt{-1}$, so $\theta_\delta(A)$ is certainly invertible if ω is not a natural frequency of (6). Furthermore, the exponential mapping $e^{\delta z}$ is everywhere nonsingular, so there it admits a one-sided inverse $\theta_\delta(z) = (1/\delta)\log z$ defined on the spectrum of $e^{\delta A}$ iff it is one-to-one on the spectrum of A. We thus conclude the fundamental linear sampling theorem:

Proposition 3.0.1 *If $k\omega \neq \omega_1 - \omega_2$ for each two natural frequencies of (6) and each nonzero integer k, then (5) holds.* ∎

Note that the condition in this proposition is precisely the one that insures that the exponential is one-to-one, but that this also implies that ω is not a natural frequency of the system: otherwise, $-\omega$ is also a natural frequency and therefore $2\omega = \omega - (-\omega)$ contradicts the condition.

4 Discrete-time nonlinear accessibility

We shall need to use a result from [4]. For this, we first introduce the following vector fields associated to a system (1) and any fixed real number $\delta > 0$. Fix an $i = 1, \ldots, m$. Then consider

$$X_i(x) := \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} e^{-\delta f} e^{\delta(f+\varepsilon g_i)}(x).$$

These are analytic vector fields; note that completeness of f is used in this definition, since we need to know that $e^{\pm\delta f}(p)$, the flow in time $\pm\delta$ of the vector field f starting at p, is defined for any $p \in \mathbb{R}^n$. (By well-posedness of ode's, this implies that $e^{\delta f} e^{-\delta(f-\varepsilon g_i)}(x)$ is also defined, for small ε.) For small δ, there is an expansion (see for instance [2])

$$X_i(x) = \sum_{i=1}^{\infty} \frac{\delta^i}{i!} ad_f^{i-1}(g_i)(x);$$

here ad_f denotes the Lie bracket linear operator on vector fields,

$$ad_f(Y) := [f, Y].$$

Still for any fixed δ, one considers also the conjugation under the flow of f, also a linear operator on vector fields:

$$Ad_f^\delta(Y)(x) := (e^{-\delta f})_* Y(e^{\delta f}(x)),$$

where "$*$" indicates differential. For small δ, one has also the expansion

$$Ad_f^\delta(Y)(p) := \sum_{i=0}^{\infty} \frac{\delta^i}{i!} ad_f^i(Y).$$

Now let $\mathcal{X}$ be the subspace of vector fields generated by the X_i's, let

$$\mathcal{L}_\delta := \text{ Lie algebra generated by } \{Ad_f^\delta | \mathcal{X}\},$$

and let $\mathcal{L}_\delta(x)$ denote the subspace of the tangent space at $x \in \mathbb{R}^n$ spanned by the vectors $Y(x), Y \in \mathcal{L}_\delta$. The main result which we need from [4] is the following, restated in terms of sampling of a continuous-time system. We state only a sufficient condition; a necessary and sufficient characterization is also given in that reference, but it is somewhat less elegant.

Theorem 3 *The system σ is ω-accessible from the equilibrium state x_0, for $\omega = 2\pi/\delta$, if $\mathcal{L}_\delta(x_0)$ has dimension n.* ∎

Let $\mathcal{B}$ be the span of the vector fields $g_1, \ldots, g_m$. From now on, we make the assumption that

$$V := \{ad_f | \mathcal{B}\} \text{ is finite dimensional.} \tag{7}$$

Note that for systems in the classes $\tilde{\sigma}_d^H$ and $\tilde{\sigma}_d$ this assumption holds true, since homogeneous polynomials of any fixed degree are invariant under ad_f for linear f.

Under assumption (7), one can introduce the operators $e^{\delta A}$ and $\theta_\delta(A)$, for $A = ad_f$ seen as a linear operator on V. These are defined and analytic for all δ, and from their expansions in terms of δ and analytic continuation we conclude that

$$Ad_f^\delta = e^{\delta A}$$

as linear operators on V and that

$$X_i = \theta_\delta(A) g_i$$

for each i. Thus $\mathcal{L}_\delta$ is the Lie algebra generated by the right-hand side of (5), with the present A and $\mathcal{B}$. It follows that if δ is such that (5) holds then $\mathcal{L}_\delta$ is the same as the strong accessibility Lie algebra

$$\mathcal{L}_0 := \{ad_f | \mathcal{B}\}_{LA}$$

and hence it has rank n at x_0 if the original system is accessible from x_0. We thus conclude from proposition 3.0.1:

Proposition 4.0.2 *Let x_0 be an equilibrium state, and assume that σ satisfies (7) and that it is accessible from x_0. If $\omega > 0$ is such that $k\omega \neq \omega_1 - \omega_2$ for each two imaginary parts ω_1, ω_2 of eigenvalues of ad_f on V and each nonzero integer k, then σ is also ω-accessible.* ∎

It is also possible to give a result for nonequilibrium initial states x_0; see [4] for the necessary discrete-time accessibility result.

5 Proofs of the main results

We are only left with translating proposition 4.0.2 to the case of linear $f(x) = Fx$ and polynomial g_i's. For this, we need the following fact (see a proof for instance in [1], lemma 12.2.5).

Proposition 5.0.3 *The eigenvalues of ad_f, for linear $f(x) = Fx$, acting on the space of all vector fields whose entries are homogeneous polynomials of degree d, are given by the expressions*

$$a_1\lambda_1 + \ldots + a_n\lambda_n - \lambda_i$$

for all $i = 1, \ldots, n$ and all nonnegative integers $a_1, \ldots, a_n$ such that

$$a_1 + \ldots + a_n = d, \tag{8}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of F. ∎

Since the space of all polynomial vector fields of degree at most d is a direct sum of the spaces corresponding to homogeneous ones, one has an analogue of the above for polynomials of degree $\leq d$, simply replacing equality by $\leq$ in (8).

Consider first the homogeneous case, theorem 1. Note that the imaginary part of an expression

$$a_1\lambda_1 + \ldots + a_n\lambda_n - \lambda_i$$

is the expression

$$a_1\omega_1 + \ldots + a_n\omega_n - \omega_i,$$

where the ω_i are the natural frequencies. Because of proposition 4.0.2, we need to check the differences

$$a_1\omega_1 + \ldots + a_n\omega_n - \omega_p - (b_1\omega_1 + \ldots + b_n\omega_n - \omega_q) \tag{9}$$

with nonnegative a_i, b_i and $\sum a_i = \sum b_i = d$. An expression as in (9) can be also written as

$$\rho_1\omega_1 + \ldots + \rho_n\omega_n, \tag{10}$$

with all ρ_i nonnegative integers and $\sum \rho_i = 2d+2$. This is because any negative term $-b\omega_i$ can be also written as $b\omega_{i'}$, for some other i' (recall that the set ω of natural frequencies is closed under additive inverses). Conversely, any expression as in (10) with

$$\sum \rho_i = 2\delta + 2$$

can be in turn expressed as in (9), for the same reason. This gives theorem 1. To prove theorem 2, argue in exactly the same way, except that the sums of the a_i and b_i may now be arbitrary up to d each, and therefore the sum of the ρ_i's can be at most equal to $2\delta+2$ and must be at least equal to 2. However, the case of sum = 1 can be reduced to that when the sum is 2 (just compare with $2k$ in (2),) and hence the theorem results.

6 References

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