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Bilinear realizability is equivalent to existence of a singular affine differential i/o equation

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Abstract: For continuous time analytic input/output maps, existence of a singular differential equation relating derivatives of controls and outputs is shown to be equivalent to bilinear realizability. A similar result holds for the problem of immersion into bilinear systems. The proof is very analogous to that of the corresponding, and previously known, result for discrete time.

Keywords: Bilinear systems, Input/output equations, Generating series, Identification.

1. Introduction

Starting with [10] and [11], there have been many results relating the existence of i/o difference equations to finite realizability, for discrete time systems. These results, which provide analogues for nonlinear systems of the fact that a transfer function can be realized by a finite dimensional linear system if and only if it is rational, are useful for instance in the context of identification problems (see for instance [7,3]). Here we show how the simplest case, dealing with state-affine systems, has an analogue in the continuous time case. The proof is basically the same as in the older discrete case, but seems not to have been noticed in the literature (see e.g. [9,2]).

By an (output-) *affine i/o equation* of order k we shall mean an equation of the type

$$\sum_{i=0}^{k} a_i\big(u(t), u'(t), \ldots, u^{(k-1)}(t)\big) y^{(i)}(t) = b\big(u(t), u'(t), \ldots, u^{(k-1)}(t)\big), \qquad (1)$$

where b and each a_i is a polynomial and a_k is not identically zero, satisfied by all pairs of smooth controls u and corresponding outputs y. The equation is (output-) *linear* if b is identically zero. For instance, transfer function descriptions of linear systems correspond in the time domain to affine i/o equations in which all the coefficients a_i are constants independent of u and b is linear. Precise definitions are given below. Note that in the *nonsingular* case when a_k is *always* nonzero, (1) implies finite realizability, since the highest derivative $y^{(k)}$ can then be expressed in terms of lower order derivatives of outputs and controls. However such 'purely recursive' equations do not hold in general for nonlinear i/o maps; one of the main objectives of [11] was precisely the study of singular equations and how they relate to realizability. It is relatively trivial to show that such equations must exist if one assumes finite realizability, but they will in general not be affine in y; this is basically a transcendence degree argument [11]. Bilinear realizability and a linear dimension argument do imply the existence of an affine equation, and our main results will provide a converse of this fact. The more general, non-bilinear case, will probably require techniques from algebraic geometry, as done for discrete time in [11]; recent work [1] shows how such techniques can be applied in a continuous time context.

Roughly, the main results will say that if an equation such as (1) holds for all i/o pairs arising from a (possibly unknown) system, then these are the i/o pairs of a finite dimensional smooth continuous time system, in fact a bilinear one. An equation (1) is by itself *not* sufficient to guarantee such realizability (an example of this fact is discussed later); the knowledge that there is some 'well-posed' system producing the observed behavior is essential. Technically, this hypothesis will be stated in two different versions, one in terms of the existence of a locally convergent Volterra type of expansion and the other in terms

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of the existence of a nonlinear smooth (but not necessarily bilinear) realization. The proof will rely on the notion of observation space, introduced by the author and others, whose finite dimensionality is equivalent to bilinear realizability. Essentially, an equation such as (1) will insure that there is a dense subspace of the observation space which is itself finite dimensional, and then a continuity argument based on the hypothesis that the i/o behavior is in a sense 'well-posed' will provide the desired conclusion.

2. Immersions

We shall provide two versions of the main result, one stated in terms of i/o maps and the other in terms of immersions [5]. Neither result contains the other, since the version for i/o maps will correspond to fixed initial states. We start with immersions. Consider an analytic system

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x), \quad y = h(x), \tag{2}$$

with states $x(t)$ evolving on a manifold M, where for simplicity we assume that outputs $y(t) \in \mathbb{R}$ are scalar and that the system is complete. A $\mathscr{C}^k$ *i/o pair* (u, y) of (2) is a pair of functions defined on some interval $[0, T]$, $T > 0$, such that $u:[0, T] \to \mathbb{R}^m$ is of class $\mathscr{C}^{k-1}$ and $y(t) = h(x(t))$ is the output trajectory corresponding to some initial state, that is, x satisfies the equation (2) for the given u. Note that then y is necessarily of class $\mathscr{C}^k$. We shall say that (2) *satisfies an affine i/o equation* if there exist an integer k and b, $a_0, \ldots, a_k$ as above such that (1) is satisfied for each $\mathscr{C}^k$ i/o pair of the system, for all $0 < t < T$. Here b as well as each of the a_i is a polynomial in km variables, the coordinates of u and of its first $k-1$ derivatives. The system *satisfies a linear i/o equation* if the above holds with $b \equiv 0$. Note that if an equation exists for a given k then, differentiating, there is also a similar equation for each $k' > k$; in fact, the same polynomial a_k can be used for larger k'.

We shall say that (2) *can be immersed in a bilinear system* if its observation space is finite dimensional. Recall that the *(infinitesimal) observation space of* (2) is the linear span $\mathscr{O}_0$ of the functions

$$L_{X_1} \ldots L_{X_l}(h) : M \to \mathbb{R}, \qquad l \geq 0, \quad X_i \in \{f, g_1, \ldots, g_m\}, \tag{3}$$

where $L_X(h)$ denotes the Lie derivative, $\nabla h \cdot X$ in local coordinates, of the function h with respect to the vector field X. It is a basic fact in nonlinear system theory that finite dimensionality of $\mathscr{O}_0$ is equivalent to the existence of a finite dimensional internally bilinear system

$$\dot{x} = \left(A + \sum_{i=1}^{m} u_i F_i\right) x, \quad y = Cx, \tag{4}$$

with $x(t) \in \mathbb{R}^n$ for some n and the A, F_i, C matrices of appropriate sizes, as well as the existence of an analytic map $\theta: M \to \mathbb{R}^n$, such that x_0 and $\theta(x_0)$ give rise to the same input/output behavior, for each $x_0 \in M$. See [8] and especially [5] for details, as well as [11] for the analogous discrete time concept; the terminology observation space was introduced in the latter reference. We now state the first of the results to be proved in this paper.

Theorem 1. *The system* (2) *satisfies an affine i/o equation if and only if it can be immersed in a bilinear system. In that case, it also satisfies a linear i/o equation.*

A somewhat more general result could be given, starting with systems which are not necessarily linear but polynomial or even rational in controls, but the class of systems (2) is probably general enough for most applications. A similar result also holds in the multiinput case ($h: M \to \mathbb{R}^p$, $p > 1$).

The following observation will be useful. If there exist b, $a_0, \ldots, a_k$ so that

$$\sum_{i=0}^{k} a_i\big(u(0), u'(0), \ldots, u^{(k-1)}(0)\big) y^{(i)}(0) = b\big(u(0), u'(0), \ldots, u^{(k-1)}(0)\big) \tag{5}$$

holds for each $\mathscr{C}^k$ i/o pair, then (1) also holds for all t. This is because of time-invariance of (2): given any $t_0 < T$, the pair

$$\tilde{u}(t) := u(t + t_0), \qquad \tilde{y}(t) := y(t + t_0)$$

is also a $\mathscr{C}^k$ i/o pair, defined on $[0, T - t_0]$, and

$$\tilde{u}^{(i)}(0) = u^{(i)}(t_0), \qquad \tilde{y}^{(i)}(0) = y^{(i)}(t_0)$$

for all derivatives. Thus equation (5) for the pair $(\tilde{u}, \tilde{y})$ is equivalent to equation (1) for the original pair (u, y) at time t_0.

For any fixed $T > 0$, we let $\mathcal{U}_T$ denote the set of all essentially bounded functions

$$u : [0, T] \to \mathbb{R}^m$$

endowed *with the L^1 topology* (not the L^∞ topology). It is a standard fact that for each fixed initial state x_0 of (2) and any fixed T, the mapping

$$u \mapsto y \tag{6}$$

is continuous from $\mathcal{U}_T$ into $C[0, T]$ (continuous functions with sup norm).

3. Generating series version

In order to state the i/o version of the result, we need to have a notion of analytic i/o mapping (6). We choose a presentation in terms of Fliess series, but one could also base the approach on Volterra expansions with analytic kernels. Let m be a fixed integer, and consider noncommuting variables $\eta_0, \ldots, \eta_m$. A *power series* in these variables is a formal expression

$$c = \sum \langle c, \eta_\iota \rangle \eta_\iota \tag{7}$$

where the sum is over all possible sequences of indices

$$\iota = (i_1, \ldots, i_l), \quad l \geq 0, \tag{8}$$

with each $i_r \in \{0, \ldots, m\}$, including the empty sequence ε ($l = 0$), and where we denote

$$\eta_\iota := \eta_{i_1} \cdots \eta_{i_l}, \tag{9}$$

and $\eta_\varepsilon := 1$. The coefficients $\langle c, \eta_\iota \rangle$ are real numbers. The set of all formal power series on $\eta_0, \ldots, \eta_m$ forms a real vector space under the coefficientwise operations

$$\langle rc_1 + c_2, \eta_i \rangle = r\langle c_1, \eta_i \rangle + \langle c_2, \eta_i \rangle.$$

We shall say that c is *convergent* if there exist M, $K > 0$ such that, for each sequence ι as in (8),

$$|\langle c, \eta_\iota \rangle| \leq KM^l l!.$$

For each T, each $u \in \mathcal{U}_T$, and each multiindex ι as above, we define inductively the functions $V_\iota = V_\iota[u] \in C[0, T]$ by $V_\varepsilon \equiv 1$ and

$$V_{i_1, \ldots, i_{k+1}}(t) = \int_0^t u_{i_1}(\tau) V_{i_2, \ldots, i_{k+1}}(\tau)\, \mathrm{d}\tau, \tag{10}$$

where $u_i(\tau)$ is the i-th coordinate of $u(\tau)$ for $i = 1, \ldots, m$ and $u_0(\tau) \equiv 1$. It is easy to prove that each operator

$$\mathcal{U}_T \to C[0, T] : u \mapsto V_\iota[u]$$

is continuous. Furthermore, if c is a convergent series and K, M are as above, then for $T < (Mm + M)^{-1}$, the series of functions

$$F_c[u](t) = F[u](t) = \sum \langle c, \eta_\iota \rangle V_\iota(t) \tag{11}$$

is absolutely and uniformly convergent for all $t \in [0, T]$ and all those $u \in \mathcal{U}_T$ such that $\sup |u_i(t)| \leq 1$ for all i; see [6], Chapter III. Thus the operator F_c is also continuous on the subset of $\mathcal{U}_T$ satisfying this magnitude constraint. Further, c is in turn determined by F_c, in the sense that if $F_c = F_d$ for small enough T then $c = d$ (see [6]). If T and u are like this and u is of class $\mathscr{C}^{k-1}$, then $y := F[u]$ is of class $\mathscr{C}^k$; we call such a pair (u, y) a $\mathscr{C}^k$ *i/o pair* associated to c.

We shall say that the convergent series c *satisfies an affine i/o equation* (linear if $b \equiv 0$) if there exist an integer k and $b, a_0, \ldots, a_k$ as above (a_k not identically zero) such that that (1) is satisfied for each $\mathscr{C}^k$ i/o pair of c and each $0 \leq t \leq T$. As before, if there is an equation of order k, then there is also an equation of any order $k' > k$.

For any series c and each monomial $\alpha = \eta_\iota$, the series $\alpha^{-1}c$ is defined by the formula

$$\langle \alpha^{-1}c, \beta \rangle := \langle c, \alpha\beta \rangle.$$

the operation $c \mapsto \alpha^{-1}c$ is linear, and is a noncommutative analogue of a shift. Note that $\alpha_2^{-1}\alpha_1^{-1}c = (\alpha_1\alpha_2)^{-1}c$. If c is convergent, then $\alpha^{-1}c$ is too. In fact, if T is as above, the same T is admissible for $\alpha^{-1}c$. Indeed, assume first that α is one of the variables η_i, and take K, M, T as above. Pick any $\tilde{M} > M$ so that the inequality

$$T < (\tilde{M}m + \tilde{M})^{-1}$$

still holds, and let $\tilde{K}$ be such that $(l+1)M^l < \tilde{K}\tilde{M}^l$ for all nonnegative integers l. It follows that

$$\langle \alpha^{-1}c, \beta \rangle < (K\tilde{K}M)\tilde{M}^l l!$$

for all β of length l. The general case follows by induction on the length of α.

We associate an *observation space* $\mathcal{O}_0 = \mathcal{O}_0(c)$ to each power series c; this is the subspace of the space of formal power series in the $m + 1$ variables η_i spanned by the set of all $\alpha^{-1}c$, for all monomials α. If c is convergent, then each member of $\mathcal{O}_0(c)$ is again convergent (and the same T works).

The series c is called *rational* when $\mathcal{O}_0(c)$ is finite dimensional. For each α, $\alpha^{-1}c$ can be identified with the α-th column of the generalized Hankel matrix of c, and hence rationality is equivalent to the operator (11) being the input/output map of a *bilinear system* (4), for some initial x_0. This equivalence is due to Fliess; see [4,6], as well as [12] for related facts on partial realizations. The second result will be as follows.

Theorem 2. *The convergent series c is rational if and only if it satisfies an affine i/o equation. In that case, it also satisfies a linear i/o equation.*

Actually, if c is any rational formal power series, it is necessarily convergent, and in fact (11) is defined *for all* $T > 0$, not just small T, and all $u \in \mathcal{U}_T$. The observation spaces associated to systems and power series are related via realization theory, but we shall not need this relation explicitly here.

4. Proof of Theorem 1

The proof will be easier to understand once that we introduce a few more subspaces of the set $\mathbb{R}^M$ of all functions $M \to \mathbb{R}$. We shall endow $\mathbb{R}^M$ with the topology of pointwise convergence, the weak topology. The closure of a set S with respect to such a topology will be denoted by clos S. Since $\mathbb{R}^M$ is Hausdorff topological vector space, each finite dimensional subspace is closed.

For each fixed system (2), we let $\mathcal{O}$ denote the space of all (noninfinitesimal) observables. This is the subspace of $\mathbb{R}^M$ generated as follows. To each $T \geq 0$ and each $u \in \mathcal{U}_T$ we associate the observable

$$h^u : M \to \mathbb{R}, \quad h^u(\xi) := h(x(T)),$$

where x is the solution of (2) with $x(0) = \xi$ and control u. For $T = 0$, this is just h. Then $\mathcal{O}$ is defined as the span of all such h^u. (Recall that we are assuming completeness; defining this space is a more delicate matter otherwise, since the solution x may fail to exist.) If we only consider *analytic* controls u, we have the subspace $\mathcal{O}^\omega \subseteq \mathcal{O}$.

If we restrict to *piecewise constant* controls, we have another subspace $\mathcal{O}^{PC}$. Because of the continuity of the maps (6) with respect to the L^1 topology, and the density of piecewise constant as well as of analytic controls in such a topology,

$$\mathcal{O}^\omega \subseteq \mathcal{O} \subseteq \text{clos } \mathcal{O}^\omega \quad \text{and} \quad \mathcal{O}^{PC} \subseteq \mathcal{O} \subseteq \text{clos } \mathcal{O}^{PC}.$$

Thus,

$$\dim \mathcal{O}^{PC} < \infty \iff \dim \mathcal{O}^\omega < \infty. \tag{12}$$

Finally, note that

$$\dim \mathcal{O}^{PC} < \infty \iff \dim \mathcal{O}_0 < \infty, \tag{13}$$

where $\mathcal{O}_0$ is the infinitesimal observation space introduced earlier. This is because the generators (3) of $\mathcal{O}_0$ can be obtained as Taylor coefficients of the possible elements in $\mathcal{O}^{PC}$. The argument is basically a standard one, but it is worth reviewing it carefully. We use the following notation for piecewise constant controls:

$$u := (t_1, \mu_1)(t_2, \mu_2) \cdots (t_k, \mu_k) \tag{14}$$

is the control on $\mathcal{U}_T$, $T = \Sigma t_i$, which has the constant value μ_i on the interval

$$[t_0 + \cdots + t_{i-1}, t_0 + \cdots + t_i]$$

where $t_0 := 0$. We say that u has $k-1$ switches. Each generator of $\mathcal{O}_0$ appears as a mixed derivative with respect to the t_i's and μ_j's, as follows. For any fixed k and ξ, consider $h^u(\xi)$ as a function of $t = (t_1, \ldots, t_k)$ and of $\mu = (\mu_1, \ldots, \mu_k)$, for piecewise constant controls with $k-1$ switches. Because of the assumption that (2) is analytic, h^u is analytic as a function of these. Denoting by μ_{ij} the i-th component of μ_j, $i = 1, \ldots, m$, the following classical formula holds for all mixed partial derivatives which are of at most first-order with respect to each variable separately:

$$\left.\frac{\partial^k}{\partial t_1 \cdots \partial t_k}\right|_{t=0} \left.\frac{\partial^s}{\partial \mu_{i_1 j_1} \cdots \partial \mu_{i_s j_s}}\right|_{\mu=0} h^u(\xi) = L_{X_1} L_{X_2} \cdots L_{X_k}(h)(\xi), \tag{15}$$

for each $0 \leq s \leq k$, each sequence

$$1 \leq j_1 < j_2 < \cdots < j_s \leq k$$

and any $i_1, \ldots, i_s \in \{1, \ldots, m\}$, where

$$X_{j_l} = g_{i_l}, \quad l = 1, \ldots, s,$$

and $X_i = f$ for $i \notin \{j_1, \dots, j_s\}$. This formula is easily established by induction on k. It follows from (15) that $\mathcal{O}_0 \subseteq \text{clos } \mathcal{O}^{\text{PC}}$. More general partial derivatives of $h^u(\xi)$ with respect to the components of t and μ are finite linear combinations of the generators (3) of $\mathcal{O}_0$, so by analyticity we conclude the other inclusion $\mathcal{O}^{\text{PC}} \subseteq \text{clos } \mathcal{O}_0$. Thus (13) holds.

Because of (13) and (12), in order to establish Theorem 1 we need only prove that existence of an i/o equation is equivalent to the finite dimensionality of the space $\mathcal{O}^\omega$. In order to do this, we introduce one last space of functions. For any $\xi \in M$, $u \in \mathcal{U}_T$, and $t \in [0, T]$, we let $h_t^u(\xi)$ be $h(x(t))$, where x solves (2) with control u and $x(0) = \xi$. Note that with this notation, $h_T^u(\xi) = h^u(\xi)$ if $u \in \mathcal{U}_T$. If u is of class $\mathcal{C}^{k-1}$ then $h_t^u(\xi)$ is of class $\mathcal{C}^k$ on t, and application of the chain rule shows that its k-th derivative at 0,

$$\left. \frac{\partial^k h_t^u(\xi)}{\partial t^k} \right|_{t=0}, \tag{16}$$

is a polynomial on $u(0), \dots, u^{(k-1)}(0)$ whose coefficients are analytic functions of ξ. For any sequence $\mu_0, \dots, \mu_{k-1}$ of elements in $\mathbb{R}^m$ we let

$$h^{\mu_0, \dots, \mu_{k-1}}(\xi)$$

be the value of (16), for any $\mathcal{C}^{k-1}$ control u for which

$$u^i(0) = \mu_i, \quad i = 0, \dots, k-1.$$

When $k = 0$, this is just $h(\xi)$. For each $k \geq 0$, $\mathcal{O}^{\omega,k}$ is the span of all the functions $h^{\mu_0, \dots, \mu_{l-1}}$ with $0 \leq l \leq k$, and $\mathcal{O}_0^\omega$ is the union of all these spaces. For analytic controls, $h_t^u(\xi)$ is analytic on t, so these spaces are related by Taylor expansions and hence

$$\mathcal{O}_0^\omega \subseteq \text{clos } \mathcal{O}^\omega \quad \text{and} \quad \mathcal{O}^\omega \subseteq \text{clos } \mathcal{O}_0^\omega.$$

Thus if either is finite dimensional then both spaces are equal. Observe that every $\mathcal{O}^{\omega,k}$ is finite dimensional, because each $h^{\mu_0, \dots, \mu_{k-1}}$ is polynomial in the μ_i's. Assume now that

$$\dim \mathcal{O}_0^\omega = k < \infty.$$

Consider formally the elements

$$h^{\mu_0, \dots, \mu_{l-1}}, \quad l = 0, \dots, k, \tag{17}$$

as rational functions on nm variables (the components of the μ_i's), with coefficients on $\mathcal{O}_0^\omega$. More rigorously, we are looking at these elements as belonging to the tensor product

$$\mathbb{R}(\mu_0, \dots, \mu_{k-1}) \otimes \mathcal{O}_0^\omega,$$

with each of $\mu_0, \dots, \mu_{k-1}$ now thought of as a set of m indeterminates. This is a space of dimension k over the rational function field $\mathbb{R}(\mu_0, \dots, \mu_{k-1})$, so the elements (17) must be linearly dependent over this field. After clearing denominators, there results a nontrivial equation

$$\sum_{l=0}^{k} a_l(\mu_0, \dots, \mu_{k-1}) h^{\mu_0, \dots, \mu_{l-1}} = 0. \tag{18}$$

If $a_k \equiv 0$, one may replace this by an equation with small k. (If the set $\{h^{\mu_0, \dots, \mu_{l-1}}, l < k\}$ is linearly dependent over $\mathbb{R}(\mu_0, \dots, \mu_{k-1})$, it is also dependent over $\mathbb{R}(\mu_0, \dots, \mu_{k-2})$.) Note that equation (18) is the same, with a different notation, as equation (5) with $b \equiv 0$. Thus the (easy) sufficiency part of the theorem is proved.

We now prove the converse. Assume that an i/o equation holds, with some integer k, and hence also for all $k' \geq k$. Because a_k is not identically zero, equation (5) exhibits each $h^{\mu_0, \dots, \mu_{l-1}}$ as a linear combination of the $h^{\mu_0, \dots, \mu_{i-1}}$, $i < l$, and of the constant function 1 (corresponding to the b term), generically on the μ_j's. That is, there exist (open) dense subsets

$$W_l \subseteq \mathbb{R}^{lm},$$

$l = k, k+1, \dots,$ such that

$$h^{\mu_0, \dots, \mu_{l-1}} \in \mathcal{O}^{\omega,k-1} + \text{span}\{1\}$$

for all $(\mu_0, \dots, \mu_{l-1}) \in W_l$.

Since $h^{\mu_0, \dots, \mu_{l-1}}(\xi)$ is continuous on the μ_i's, it follows that also *every* element $h^{\mu_0, \dots, \mu_{l-1}}$, $l \geq k$, is in the finite dimensional space $\mathcal{O}^{\omega,k-1} + \text{span}\{1\}$, and we conclude that $\mathcal{O}_0^\omega = \mathcal{O}^\omega$ is indeed finite dimensional. □

5. Proof of Theorem 2

The 'only if' part is a consequence of Theorem 1. Indeed, rationality implies that there is a bilinear realization (4) of the corresponding operator, and an i/o equation associated to (4) is also an equation for c.

The proof of the converse is structured very similarly to that of Theorem 1, so we provide only an outline. Fix a convergent series c and a $T > 0$ so that the operator $F = F_c$ is defined on the set of controls in $\mathscr{U}_T$ which satisfy the magnitude constraints $\sup|u_i(t)| \leq 1$. Since there is no risk of confusion, we also denote by F the same operator (11) acting on controls of length less than T. Pick any $0 < T_0 < T$. The objects which we define next will be in fact independent of the chosen T_0, but this fact is not needed. For each $\rho < T - T_0$ and each $v \in \mathscr{U}_\rho$ satisfying the magnitude constraints, we associate the operator

$$G^v : \mathscr{U}_{T_0} \to \mathbb{R} : w \to F[wv](T),$$

where wv denotes the concentration of w and v, defined on the interval $[0, T_0 + \rho]$. Then $\mathscr{O}$ is defined as the span of all these operators, $\mathscr{O}^\omega$ as the span of the operators corresponding to analytic v's, and $\mathscr{O}^{\mathrm{PC}}$ as the span using piecewise constant v's. Endowing as earlier the space of all operators $\mathscr{U}_{T_0} \to \mathbb{R}$ with the weak topology, we again conclude that (12) holds.

To see that also (13) holds in this case, we argue as follows. Let $\tilde{\mathscr{O}}_0$ be the subspace spanned by all the operators of the form F_d, $d = \alpha^{-1}c$, for all possible monomials α. Because F_d uniquely determines d, it follows that this space is isomorphic to $\mathscr{O}_0$. The generators of $\tilde{\mathscr{O}}_0$ appear as Taylor coefficients of the elements of $\mathscr{O}^{\mathrm{PC}}$, just as in the state-space case. Indeed, if v is as in (14), then

$$\frac{\partial^k}{\partial t_1 \cdots \partial t_k}\bigg|_{t=0} \frac{\partial^s}{\partial \mu_{i_1 j_1} \cdots \partial \mu_{i_s j_s}}\bigg|_{\mu=0} G^v[w] = F_d[w], \tag{19}$$

with $d = (\zeta_1\zeta_2 \cdots \zeta_k)^{-1}c$, for each $0 \leq s \leq k$, each sequence

$$1 \leq j_1 < j_2 < \cdots < j_s \leq k$$

and any $i_1, \ldots, i_s \in \{1, \ldots, m\}$, where

$$\zeta_{j_l} = \eta_{i_l}, \quad l = 1, \ldots, s,$$

and $\zeta_i = \eta_0$ for $i \notin \{j_1, \ldots, j_s\}$. This formula can be obtained by induction on k. Further, mixed derivatives of higher order are also in $\tilde{\mathscr{O}}_0$, so the proof of (13) is as before. One technical fact that should be checked now, however, is that $G^v[w]$ is indeed analytic on the components of t and μ, at the values where these are all zero. In the state-space case this was trivial, because of the existence of solutions for small negative times; here, one must provide an analytic extension in the form of a value for the concatenation of a control w with a control as in (14) with small *negative* t_i's. But this presents no difficulty when done in terms of the iterated integrals (10).

We are left with showing that existence of an i/o equation implies the finite dimensionality of the space $\mathscr{O}^\omega$ for every T_0 as above. For any fixed such T_0, we introduce operators

$$G^{\mu_0, \ldots, \mu_{k-1}}$$

as before, the corresponding spaces $\mathscr{O}^{\omega,k}$, and their union $\mathscr{O}_0^\omega$. The expressions $G^v[w] \in \mathscr{O}^\omega$ correspond to evaluations at controls that are obtained as concatenations u at time T_0 of arbitrary controls w with analytic ones v. Thus in order to conclude (18) we need to know that (1) holds at time $t = T_0$ for all such u. The assumption is somewhat different: the equation is only known to hold for all $\mathscr{C}^k$ pairs. However, all derivatives $u^{(l)}(t)$ and $y^{(l)}(t)$ exist and are continuous at $t \to T_0^+$ for the i/o pairs corresponding to these concatenated controls u, and further any u of that form can be approximated in $\mathscr{U}_T$ by smooth controls which coincide with u for all $t \geq T_0$. Thus (1) must hold also for the concatenated controls. The proof is then completed as before. □

6. An example

Consider the system with $M = \mathbb{R}$ and

$$\dot{x} = u, \qquad y = \tfrac{1}{2}x^2. \tag{20}$$

Since $y' = xu$ and $y'' = u^2 = xu'$, there results the second order affine equation

$$uy'' - u'y' = u^3.$$

Since $y''' = 3uu' + xu''$, there is also the output-linear third order equation

$$u^2y''' - 3uu'y'' + \left[3(u')^2 - uu''\right]y' = 0.$$

It is trivial here to give a bilinear immersion:

$$\dot{x}_1 = ux_3, \qquad \dot{x}_2 = x_1u, \qquad \dot{x}_3 = 0,$$

$$y = x_2, \qquad \theta(x) := \left(x, \tfrac{1}{2}x^2, 1\right).$$

We wish to use this example to emphasize that one must allow for 'singular' equations. (This is closely related to the fact that the output (20) has a singularity at the origin, see [7].) We claim that in fact *there exists no integer k and no function* $R:\mathbb{R}^{2k}\to\mathbb{R}$ *such that*

$$y^{(k)}(t)=R\big(y(t),\dots,y^{(k-1)}(t),\ u(t),\dots,u^{(k-1)}(t)\big)$$

for all i/o pairs. Further, this happens even if we impose the fixed initial state $x(0)=0$. To prove the claim, it is sufficient to provide for any given integer k a time $\tau>0$ and two smooth i/o pairs (u_1, y_1) and (u_2, y_2) corresponding to $x(0)=0$ and defined on an interval containing $[0,\tau]$, such that $u_1^{(i)}(\tau)=u_2^{(i)}(\tau)$ and $y_1^{(i)}(\tau)=y_2^{(i)}(\tau)$ for all $i<k$ but $y_1^{(k)}(\tau)\neq y_2^{(k)}(\tau)$. These pairs can be obtained as follows, for an arbitrary $\tau>0$. First let $u:[0,\frac{1}{2}\tau]\to\mathbb{R}$ be a smooth function all whose derivatives at $\frac{1}{2}\tau$ as well as its first $k-2$ derivatives at 0 are equal to zero, and which is so that

$$u^{(k-1)}(0)=-1 \quad\text{and}\quad \rho:=\int_0^{\tau/2}u(s)\,\mathrm{d}s\neq 0.$$

Now let

$$u_1(t)=\begin{cases}u(t) & \text{if } t\in[0,\frac{1}{2}\tau],\\ u(\tau-t) & \text{if } t\in[\frac{1}{2}\tau,\tau],\end{cases}$$

$$u_2(t)=\begin{cases}-3u(t) & \text{if } t\in[0,\frac{1}{2}\tau],\\ u(\tau-t) & \text{if } t\in[\frac{1}{2}\tau,\tau].\end{cases}$$

The corresponding state trajectories starting at $x(0)=0$ are such that $x_1(\tau)=\rho$ and $x_2(\tau)=-\rho$. In particular, $y_1(\tau)=y_2(\tau)$. For $l\geq 1$, the derivatives of the outputs have the form

$$y_i^{(l)}(t)=P_l\big(u_i(t),\dots,u_i^{(l-2)}(t)\big)+u_i^{(l-1)}(t)x_i(t)$$

for polynomials P_l which vanish at the origin, so all derivatives of order $l\leq k-1$ coincide at τ, but $y_1^{(k)}(\tau)=\rho\neq-\rho=y_2^{(k)}(\tau)$. Thus this i/o pair provides the desired counterexample.

Finally, we give an example which illustrates the fact mentioned in the introduction, that an equation (1) may not be satisfied by any bilinear system, and for that matter, there may not exist any i/o operator whose i/o pairs satisfy this equation. This serves to emphasize that our results require the knowledge that there is some such operator, even if its precise form is not known. Consider for this the equation

$$u(t)y'(t)=1. \tag{21}$$

Assume that there would exist some operator F_c and some $T>0$ so that F_c is defined for small enough inputs on $[0,T]$. Let $u_\delta(t)\equiv\delta$ and $y_\delta:=F_c[u_\delta]$. Then, there is a constant k (namely, $\langle c,\varepsilon\rangle$ in the previous notations) such that

$$y_\delta(0)=k$$

for all δ. From this and (21) we conclude that

$$y_\delta(T)=k+T/\delta.$$

This diverges as $\delta\to 0$, contradicting continuity on controls and the fact that y_0 should be defined. If one were to restrict attention just to controls that never vanish, then there would of course be a realization. This suggests a 'local' study, for controls whose values are restricted; see [7,2] for example.

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