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A universal formula for stabilization with bounded controls *

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Abstract: We provide a formula for a stabilizing feedback law using a bounded control, under the assumption that an appropriate control-Lyapunov function is known. Such a feedback, smooth away from the origin and continuous everywhere, is known to exist via Artstein's Theorem. As in the unbounded-control case treated in a previous note, we provide an explicit and 'universal' formula given by an algebraic function of Lie derivatives. In particular, we extend to the bounded case the result that the feedback can be chosen analytic if the Lyapunov function and the vector fields defining the system are analytic.

Keywords: Smooth stabilization; Artstein's theorem.

1. Introduction

This paper represents a continuation of a line of work started in [4]. The goal is to provide explicit feedback control laws that stabilize a given nonlinear system, under the assumption that a 'control Lyapunov function' is known. (For more details on the origin of this problem, and references to previous work, see the above reference, as well as the survey paper [5].)

Consider the following control system on $\mathbb{R}^n$:

$$\dot{x} = f(x) + G(x)u, \tag{1}$$

where all entries of the vector f and the $n \times m$ matrix G are smooth functions on $\mathbb{R}^n$ and $f(0) = 0$. Controls take values in the open unit ball

$$\mathscr{B}_m = \left\{ u \in \mathbb{R}^m \mid \| u \|^2 = u_1^2 + \cdots + u_m^2 < 1 \right\}$$

in some Euclidean space $\mathbb{R}^m$. Suppose that there is a feedback law

$$k : \mathbb{R}^n \to \mathscr{B}_m \tag{2}$$

which stabilizes the system (1), that is, so that the differential equation

$$\dot{x} = f(x) + G(x)k(x) \tag{3}$$

is globally asymptotically stable about $x = 0$, and k is regular enough so that the right hand side of (3) is continuous for $x \neq 0$. Then, converse Lyapunov theorems guarantee the existence of a positive definite (i.e., $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$,) and proper (i.e., $V(x) \to \infty$ as $\| x \| \to \infty$) smooth function

$$V : \mathbb{R}^n \to \mathbb{R}$$

such that

$$\inf_{u \in \mathscr{B}_m} \left\{ a(x) + B(x)u \right\} < 0 \tag{4}$$

for each $x \neq 0$, where we use from now on the notations

$$a(x) := \nabla V(x) f(x)$$

and

$$B(x) = (b_1(x), \ldots, b_m(x)) := \nabla V(x) G(x)$$

(that is, a, $b_1, \ldots, b_m$ are the Lie derivatives of V with respect to the vector fields defining the system). To prove that such a V exists, simply find a Lyapunov function for (3) and now use $u = k(x)$ in the infimum. A proper and positive definite smooth function satisfying (4) will be said to be a *control Lyapunov function* (*clf*) [with respect to the system (1) with controls in $\mathscr{B}_m$].

Moreover, if k is continuous at the origin, then this function has the additional *small control property (scp)* [with respect to (1)]: *For each* $\delta > 0$ *there is an* $\varepsilon > 0$ *such that, if* $x \neq 0$ *satisfies* $\| x \| < \varepsilon$, *then there is some* u *with* $\| u \| < \delta$ *such that* $a(x) + B(x)u < 0$.

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A particular case of a theorem of Artstein (see [1]) provides an elegant converse to these facts, namely: if there is a clf V, then there is a feedback law (2) which globally stabilizes the system, and k is smooth on $\mathbb{R}^n - \{0\}$. If in addition V satisfies the scp, then k can be chosen to be what we call *almost smooth on* $\mathbb{R}^n$, meaning not only smooth away from the origin, but also continuous on all of $\mathbb{R}^n$. Thus Lyapunov functions with scp completely characterize almost-smooth stabilizability. (See also [2,6,7], and references there, for related work.)

The proof in [1] is based on partitions of unity and is highly nonconstructive. In [4] we showed how to obtain a simple formula for k that is explicitly given in terms of a and B, valid in the case of systems with unbounded controls (i.e. u is allowed to take arbitrary values in condition (4), but the feedback law is not guaranteed to take values in $\mathscr{B}_m$). The formula given there is not appropriate for the bounded control case. In this note, we give the following formula for bounded controls:

$$k(x) = \kappa(a(x), \|B(x)\|^2)B(x)' \tag{5}$$

(prime indicates transpose, and $\|B(x)\|$ is the Euclidean norm of the row vector $B(x)$), where κ is the scalar-valued function defined for $(a, b) \in \mathbb{R}^2$, $b \geqslant 0$ by

$$\kappa(a, b) := \begin{cases} -\dfrac{a + \sqrt{a^2 + b^2}}{b(1 + \sqrt{1 + b})} & \text{if } b > 0, \\ 0 & \text{if } b = 0. \end{cases} \tag{6}$$

The new formula is a bit more complicated than the one given in [4], (based on using just $\kappa := -(a + \sqrt{a^2 + b^2})/b$). For instance, in the particular case of scalar-valued controls, and denoting $b = b_1$, the resulting bounded feedback law is $k(x) = \alpha(a(x), b(x))$, where

$$\alpha(a, b) := \begin{cases} -\dfrac{a + \sqrt{a^2 + b^4}}{b(1 + \sqrt{1 + b^2})} & \text{if } b \neq 0, \\ 0 & \text{if } b = 0. \end{cases} \tag{7}$$

Our main result here will be:

Theorem 1. *If V is a control Lyapunov function satisfying the small control property [with respect to the system (1) with controls in $\mathscr{B}_m$] then the feedback (5) is almost smooth on $\mathbb{R}^n$, takes values in $\mathscr{B}_m$, and globally stabilizes the system. Moreover, if the right-hand side of the system is analytic in x and V is analytic, then k is analytic on $\mathbb{R}^n - \{0\}$.*

To close this introduction, note that the case of controls with values in the *closed* unit ball offers no difficulty, as it is true that if (4) holds with such controls then by continuity it also holds with $u \in \mathscr{B}_m$. Thus, the Lyapunov condition holding with the closed constraint set implies the existence of a feedback taking values in the open ball (and hence in particular in its closure).

The next section develops some needed technical facts, and we then prove the main result.

2. An analytic function

Let $\alpha(a, b)$ be as in Equation (7).

Lemma 2.1. *Assume that a, b, δ are three real numbers such that $a < \delta |b|$ and $0 < \delta \leqslant 1$. Then,*

$$|\alpha(a, b)| < \min\{2\delta + |b|, 1\}.$$

Proof. If $b = 0$ then $\alpha(a, b) = 0$ and the result is trivial; hence we assume from now on that $b \neq 0$. Moreover, since $|\alpha(a, -b)| = |\alpha(a, b)|$, we also will assume, without loss of generality, that $b > 0$; thus, $a < \delta b$.

Consider first the case $a \leqslant 0$. From

$$0 \leqslant a + \sqrt{a^2 + b^4} \leqslant a + (|a| + b^2) = b^2$$

we get

$$|\alpha(a, b)| \leqslant \frac{b}{1 + \sqrt{1 + b^2}} < \min\{b, 1\},$$

which is as desired. If instead $a > 0$, then from

$$0 \leqslant a + \sqrt{a^2 + b^4} < \delta b + \sqrt{\delta^2 b^2 + b^4} = b(\delta + \sqrt{\delta^2 + b^2})$$

we have an estimate

$$|\alpha(a, b)| < \frac{\delta + \sqrt{\delta^2 + b^2}}{1 + \sqrt{1 + b^2}}. \tag{8}$$

Note that the right-hand side of (8) is bounded by one, because $\delta \leqslant 1$, and it is also bounded by its numerator

$$\delta + \sqrt{\delta^2 + b^2} < \delta + (\delta + b) = 2\delta + b,$$

as desired. □

Let $\mathscr{D} \subseteq \mathbb{R}^2$ denote the open set

$$\mathscr{D} := \{(a, b) \mid a < |b|,\ a, b \in \mathbb{R}\}. \tag{9}$$

Definition 2.2. A function $\xi : \mathscr{D} \to \mathbb{R}$ will be said to be *K-continuous* if the following property holds: For each $\varepsilon > 0$, there is a $\delta > 0$ such that

$$[\,|b| < \delta \text{ and } a < \delta |b|\,] \Rightarrow |\xi(a, b)| < \varepsilon$$

for all $(a, b) \in \mathscr{D}$.

This condition means that ξ must become small when approaching zero from the left half-plane or when approaching from the right through a curve asymptotic to the vertical axis; see Figure 1 (the 'K' shape justifies the name).

From now on, we consider α, defined in (7), as a function $\mathscr{D} \to \mathbb{R}$. On this set, we also define $\beta(a, b) := \alpha(a, b)/b$ if $b \neq 0$ and $\beta(a, 0) := 0$.

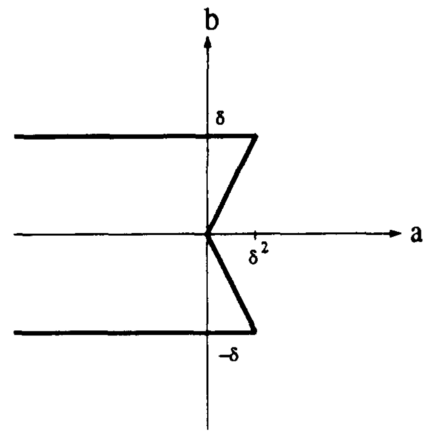

Fig. 1. Region where function must be small.

Lemma 2.3. *The following properties hold*:

(a) β *(and hence also α) is real analytic*;

(b) $\alpha(a, b)$ *is* K-*continuous, in the sense of Definition* 2.2;

(c) $|\alpha(a, b)| < 1$ *for all* $(a, b) \in \mathscr{D}$;

(d) $a + b\alpha(a, b) < 0$ *for all* $(a, b) \in \mathscr{D}$.

Proof. Analyticity follows by the results in [4], since β is obtained by dividing the function

$$-\frac{a + \sqrt{a^2 + b^4}}{b^2}$$

(zero at $b = 0$), shown in that reference – via an implicit-function argument – to be analytic, by the nonzero analytic function $1 + \sqrt{1 + b^2}$. To prove the second property, we can apply Lemma 2.1: If the pair $(a, b) \in \mathscr{D}$ satisfies $|b| < \delta$ and $a < \delta |b|$, where $\delta \leqslant 1$, then the estimate $|\alpha(a, b)| < 3\delta$ holds; this establishes that α is K-continuous. Property (c) follows from the same lemma, applied with $\delta = 1$.

We now prove (d). If $b = 0$ then $a < |b|$ implies that a must be negative, so the property holds. Otherwise, write

$$a + \alpha(a, b)b = \frac{a\sqrt{1 + b^2} - \sqrt{a^2 + b^4}}{1 + \sqrt{1 + b^2}}. \tag{10}$$

When $a \leqslant 0$ this expression is obviously negative. If instead $a > 0$, then $a < |b|$ implies that $a^2 < b^2$, and therefore

$$\sqrt{a^2 + b^4} = \sqrt{a^2 + b^2 b^2} > \sqrt{a^2 + a^2 b^2} = a\sqrt{1 + b^2},$$

and again the expression is negative. □

3. Proof of the main result

Note first that property (4) is equivalent to $a(x) < \| B(x) \|$, and that the existence of an u as in the definition of the small control property is equivalent to the inequality

$$a(x) < \delta \| B(x) \|. \tag{11}$$

In both cases, necessity follows from the Cauchy–Schwartz inequality, and sufficiency follows from the fact that, for any row vector v and number δ, we can write $\delta \| v \| = vu$, where the vector $u := (\delta / \| v \|)v'$ (if $v \neq 0$; otherwise pick $u := 0$) has norm δ.

Assume that V is a clf satisfying the scp. From the above remark, it follows that the pairs $(a(x), \| B(x) \|)$ are in $\mathscr{D}$, for all nonzero states x; we will apply Lemma 2.3 to these pairs.

Observe that, from the definitions (5) and (6), it follows that

$$\| k(x) \| = | \kappa(a(x), \| B(x) \|^2) | \, \| B(x) \| = | \alpha(a(x), \| B(x) \|) |$$

and also

$$a(x) + B(x)k(x) = a(x) + \| B(x) \| \alpha(a(x), \| B(x) \|).$$

Thus, using the same Lyapunov function V, property (d) in the Lemma implies that V has a negative derivative away from the origin, along all trajectories of (3), and k has values in $\mathscr{B}_m$ because of property (c).

Smoothness for $x \neq 0$, and analyticity if the data is analytic, follow from property (a), since

$$k(x) = \beta(a(x), \| B(x) \|) B(x)'.$$

It only remains to prove that k is continuous at the origin. We must show that for each $\varepsilon > 0$ there is some ε' so that, if $0 < \| x \| < \varepsilon'$ then

$$\| k(x) \| = | \alpha(a(x), \| B(x) \|) | < \varepsilon.$$

This follows from the fact that α is K-continuous. Indeed, let δ be as in the definition of K-continuous, for the given ε. As V is positive definite, its gradient vanishes at the origin, so $B(0) = 0$, and by continuity of B there is some ε' so that $\| x \| < \varepsilon'$ implies $| a(x) | < \delta$ and $\| B(x) \| < \delta$. By the scp, we can choose ε' so that also $a(x) < \delta \| B(x) \|$ (cf. Equation (11)). This completes the proof of the result.

4. Remarks

To close, we compare here certain properties of the solution for bounded controls given in this paper with the solution in the unbounded case obtained in [4].

As a first example, take the trivial case of the system $\dot{x} = u$ with $n = m = 1$. The control law in [4], when using the clf $V(x) = \frac{1}{2}x^2$, is precisely the obvious feedback control $k(x) = -x$. When applying instead the formula in this paper, we get instead

$$k(x) = -\frac{x}{1 + \sqrt{1 + x^2}}.$$

As another example, again with $n = m = 1$, consider the system with equations

$$\dot{x} = \frac{x^2}{1 + x^2} + u.$$

Take again $V(x) = \frac{1}{2}x^2$. As

$$\left| \frac{a}{b} \right| = \frac{x^2}{1 + x^2}$$

is small for x near zero, and is always less than one, V is a clf with respect to controls in $(-1, 1)$, and it has the scp. We are interested in the expression $a + bk$ for the derivative of V along the trajectories of the closed-loop system. For the control law in [4], this derivative is

$$-\sqrt{a^2 + b^4} = -\frac{x^2}{1 + x^2} \sqrt{x^4 + 3x^2 + 1}, \qquad (12)$$

while for the law in this paper it is (from Equation (10)):

$$\frac{x^2 \left(x + x^3 - \sqrt{1 + 3x^2 + x^4} \sqrt{1 + x^2} \right)}{(1 + x^2)^{3/2} \left(1 + \sqrt{1 + x^2} \right)}. \qquad (13)$$

Observe that the limit of the expression in (12) is $-\infty$ as $|x| \to \infty$, while the limit in (13) is zero when $x \to +\infty$. This means that an input perturbation (or equivalently, an error in implementing the control law) will be tolerated in the first case (in the sense that trajectories will remain bounded; see [3]), but for large x the second control law will be destabilized even by very small controls.

The fact that the control law given in the unbounded case is always 'robust' in this sense, under weak assumptions, can be proved as follows. (Note that in the last section of [4] it is shown how to modify the control law, using the results in [3], to achieve this extra behavior; what follows shows that the modification was, after all, not necessary.)

Consider a feedback law $u = k(x) + v$, where v is an unknown input and

$$k(x) = -\left(a + \sqrt{a^2 + b^4} \right) / b,$$

and we are considering for simplicity just scalar-input systems (but n is arbitrary). Calculate the derivative of V along the trajectories of the closed-loop system:

$$\begin{aligned} -\sqrt{a^2+b^4}+bv &\leqslant -\tfrac{1}{2}(\,|\,a\,|+b^2)+bv \\ &= -\tfrac{1}{4}(2\,|\,a\,|+b^2)+\left(bv-\tfrac{1}{4}b^2\right) \\ &\leqslant -\tfrac{1}{4}(2\,|\,a\,|+b^2)+v^2, \end{aligned}$$

where the last inequality holds because $bv-\frac{1}{4}b^2$ has a maximum value of v^2 when seen as a function of b. Thus, if one assumes, as in the last section of [4], that a^2+b^2 is proper, then it follows that for each number Q there exists some q so that, if the state x has norm larger than q and the disturbance input v is always less than Q in magnitude, then the above derivative is negative. This implies bounded-input bounded-state stability, as in [3].

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