TIME-OPTIMAL CONTROL FOR UNDERWATER VEHICLES

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Abstract: This paper addresses time-optimal control problems for a special class of controlled mechanical systems, underwater vehicles. Lie algebras associated to mechanical systems enjoy certain very special properties, which, together with the maximum principle, allow the deduction of information regarding the structure of singular extremals, and in particular of time-optimal trajectories. We apply the general theory to a model of an underwater vehicle and illustrate our results with some simulations. We consider the fully actuated and the underactuated situations.

Keywords: Time-Optimal Control, Controlled Mechanical Systems, Underwater Vehicle

1. INTRODUCTION

We consider the time-optimal problem for controlled mechanical systems, studying both the fully actuated and underactuated cases. Fully actuated systems were studied in (Sontag and Sussmann, 1986; Sontag, 1989), with the 2-link manipulator as a motivating example. As in that work, we base our study on the Pontryagin maximum principle, which gives necessary conditions for trajectories of control systems to be optimal with respect to a criterion such as energy or time. One may associate, to any given controlled mechanical system, a set of vector fields describing a control system. Then, from the special form of the Lie algebra generated by these vector fields, one can extract from the maximum principle information on the structure of the optimal trajectories. More precisely, according to the maximum principle, a time-optimal trajectory can be lifted to the cotangent bundle of its phase space as a trajectory of a constrained Hamiltonian system. This trajectory, combined with the corresponding control, is called an extremal. When the pointwise constraints in the maximum principle are nontrivial, one has nonsingular trajectories. These lead to boundary-valued controls, determined by the signs of the associated switching functions. However, it is well-known that an optimal trajectory may well be singular; that is, switching functions may vanish identically along the trajectory. The characterization of such trajectories, which is the question addressed in this paper, is in general a highly nontrivial problem.

Our study is motivated by the problem of motion planning for underwater vehicles. The optimality of a trajectory with respect to a given criterion is in many ways interesting. For underwater vehicles, it is not only worthwhile to minimize the energy expended to realize the desired motion, but one is often also concerned with the continuous power consumption of all devices on board, such

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as sensors and computers. Because of this last quantity ("hotel load"), the amount of time used to travel between two desired configurations becomes a minimization criteria for our performance requirements. In this paper we deal with the time-optimal problem, although ideally a combination of the energy and the time should be considered. This will be analyzed in future work.

2. TIME OPTIMALITY FOR MECHANICAL SYSTEMS

Let us consider the *Lagrangian*

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}^t M(q)\dot{q} - V(q)$$

defined on the phase space, where the symmetric positive definite $n \times n-$matrix $M(q)$ is called the *inertia matrix* and $V(q)$ is the *potential energy.* We assume M and V to be smooth with respect to q. The equations of motion are given by

$$Q(q(t))u(t) = M(q(t))\ddot{q}(t) + N(q(t),\dot{q}(t)) \qquad (1)$$

where $N(q,\dot{q}) = C(q,\dot{q})\dot{q} - \frac{\partial V}{\partial q}^t(q)$ is a vector of dimension n quadratic in $\dot{q}$, $Q(q)$ is a smooth $n \times m$-matrix of rank m and the *control* $u : [0,T] \to \mathcal{U} \subset I\!R^m$ is a measurable bounded function. A system described by equation (1) will be referred to as a *controlled mechanical system.* When $m = n$, the system is fully actuated. It is underactuated if $m < n$. We will assume the domain of control $\mathcal{U}$ is defined by the following constraints: $\alpha_i \leq u_i \leq \beta_i, i = 1,\cdots,m$. Such a system can be rewritten as a *control system* of dimension $2n$ as follows. Let us introduce new variables $x = (y_1, y_2) \in I\!R^{2n}$ by $y_1 = \psi(q)$, $y_2 = P(q)\dot{q}$ where $q \mapsto \psi(q)$ is a smooth diffeomorphism and $P(q)$ is a smooth $n \times n$-invertible matrix. As we will see with the example on underwater vehicles, it is sometimes convenient to consider such changes of variables, instead of simply $x = (q,\dot{q})^t$, to have simpler formulas. Then, from equation (1) we have

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t) \qquad (2)$$

where $f(x)$ is a smooth vector field called the drift and

$$G(x) = \begin{pmatrix} 0 \\ PM^{-1}Q \end{pmatrix}_{2n\times m} \qquad (3)$$

where the argument of the matrices is $\psi^{-1}(y_1)$. Let us denote by $(g_i(x))_{i=1,\cdots,m}$ the columns of $G(x)$; they are smooth functions of the n first state variables.

The goal is to determine the *time-optimal* trajectories: given $x_0, x_1 \in I\!R^{2n}$, find an admissible control $u(.)$ such that the corresponding trajectory steers the system (2) from x_0 to x_1 in the *shortest time.*

¿From the maximum principle, see (Pontryagin *et al.*, 1962), if the control $u : [0,T] \to \mathcal{U}$ and the corresponding trajectory $x(.)$ solution of (2) defined on the same interval are time-optimal, then there exists an absolutely continuous vector $\lambda : [0,T] \to I\!R^{2n}$, $\lambda(t) \neq 0$, such that for $j = 1,\cdots,2n$

$$\dot{x}_j = \frac{\partial H}{\partial \lambda_j}, \qquad \dot{\lambda}_j = -\frac{\partial H}{\partial x_j} \qquad (4)$$

holds almost everywhere, where $H(\lambda,x,u) = \lambda^t f(x) + \sum_{i=1}^m \lambda^t g_i(x)u_i$ is the Hamiltonian function, and the following maximum condition is satisfied a.e.:

$$H(\lambda(t),x(t),u(t)) = \max_{v\in\mathcal{U}} H(\lambda(t),x(t),v) = \lambda_0,$$

$\lambda_0 \leq 0$. The vector $\lambda(.)$ is called the *adjoint* vector. A triple (x,λ,u) which satisfies the maximum principle, in the sense just stated, is called an *extremal.* When the constant λ_0 is zero, the extremal is called *abnormal.* If there exists a nonempty interval $[t_1,t_2]$ such that $\phi_i(t) = \lambda(t)^t g_i(x(t)) = 0$ is identically zero, the corresponding extremal is called u_i*-singular* on $[t_1,t_2]$. If $\phi_i(t) \neq 0$ for almost all $t \in [t_1,t_2]$, the extremal is called u_i*-nonsingular* on $[t_1,t_2]$. The maximum principle implies that on a u_i-nonsingular extremal the component u_i of the control is bang-bang, which means that it takes its values in $\{\alpha_i,\beta_i\}$ for almost every $t \in [t_1,t_2]$. A u_i-nonsingular extremal is called u_i*-regular* if u_i has a finite number of switches. An extremal is said to be *nonsingular* if it is u_i-nonsingular for all i (resp. *regular*); otherwise, it is called *singular.* It is *totally singular* if it is u_i-singular for all i.

The structure of the time-optimal trajectories is related to the Lie algebra generated by the vector fields $f, g_1,\cdots,g_m$ describing the control system. Hence, our results involve conditions on the Lie brackets of these vector fields. The first result states that under certain Lie bracket conditions, an extremal cannot be totally singular. Let us define $I = \{1,\cdots,m\}$.

Theorem 1. Assume the extremal (x,λ,u) is u_i-singular for all $i \neq k$. If along the extremal there exists a nonempty subset $J \subset I$, $k \notin J$, such that

(1) If $l \in J$, then $[g_j,[f,g_l]] \in Span\{g_1,\cdots g_m\}$ for all $j \in I$;
(2) $Span\{g_j,[f,g_j],ad_f^2 g_l\}_{j\in I;l\in J} = I\!R^{2n}$

then it cannot be u_k-singular. More precisely, the component u_k of the control is bang-bang with a finite number of switches. Moreover, if we have for $j,s \in I, s \neq k$ and $l \in J$, that

$$Span\{g_j, [f, g_s], ad_f^2 g_l\}_{jsl} = I\!R^{2n}$$

along the extremal, then u_k is constant.

We remark that in the fully actuated case ($m = n$), both hypotheses on the Lie brackets in the theorem are automatically satisfied. Indeed, because we are dealing with Lagrangian mechanical system and because $\frac{\partial \psi}{\partial \theta}$ and $P(\theta)$ are invertible, the vector fields $\{g_i, [f, g_i]\}_{i=1,\cdots,n}$ constitute a frame for $I\!R^{2n}$. As a consequence, for a fully actuated controlled mechanical system, an extremal can never be totally singular. Moreover, the first assumption is true with $J = I\backslash\{k\}$, but this property is relevant in the fully actuated case only for the second part of the theorem. In (Chyba *et al.*, 2000), the theorem is extended and we consider u_i-singular extremals for no more than l controls with $l < m - 1$.

Notice that if the assumptions in Theorem 1 are satisfied, we have local accessibility from any point on the extremal. In particular, for a fully actuated system, this is true for any point $x \in I\!R^{2n}$.

In (Sontag and Sussmann, 1986), the authors give an algorithm to compute the $n-1$-singular components of the control in the fully actuated situation. The computations in the underactuated case are similar. Let us briefly review the algorithm because it will be useful for the study of underwater vehicles. Assume the extremal to be u_i-singular for $i \neq k$ on the interval $[t_1, t_2]$. It is equivalent to say that the functions $\phi_i = \lambda^t g_i(x)$, $i \neq k$, are identically zero along the extremal on this time interval. Computing the derivative of ϕ_i with respect to time and using the property $[g_j, g_i] = 0$ for all i, j, we obtain that $\lambda^t [f, g_i](x) = 0$, $i \neq k$, for all $t \in [t_1, t_2]$. This is an absolutely continuous function, so we can then compute the second derivative of ϕ_i and we have

$$\lambda^t ad_f^2 g_i(x) + \sum_{j=1}^{m} \lambda^t [g_j, [f, g_i]](x) u_j = 0$$

for almost every t, $i \neq k$. If the matrix $(\lambda^t [g_j, [f, g_i]](x))_{i,j \neq k}$ is invertible along the extremal, the $m-1$ singular components of the control are determined by the equations above. We remark that if assumption 1 of Theorem 1 is satisfied for all $l \neq k$ (which is the case for a fully actuated system) there exist smooth functions $\alpha_{ij}^s(x)$ such that $[g_j, [f, g_i]](x) = \sum_{s=1}^m \alpha_{ij}^s(x) g_s(x)$ for $i \neq k$. Assume $\lambda^t g_k(x)$ not vanishing. Then u_k is a constant and using the fact that the switching functions ϕ_i vanish for $i \neq k$, the equations reduce to

$$-\frac{\lambda^t ad_f^2 g_i(x) + \alpha_{ik}^k \lambda^t g_k(x) u_k}{\lambda^t g_k(x)} = \sum_{j \neq k} \alpha_{ij}^k u_j \qquad (5)$$

and the remaining $m-1$ components of the control are completely determined if $\det(\alpha_{ij}^k) \neq 0$. From Theorem 1, the points in the phase space corresponding to a switch of the nonsingular component of the control u_k are contained in the complement S_k^c of the set

$$S_k = \{x;\ Span\{g_i, [f, g_l], ad_f^2 g_j\} = I\!R^{2n}\}$$

where $i, l \in I; l \neq k; j \in J$.

The following slight modification of Theorem 1 will be useful for our applications in the underactuated case. It provides a gives result in cases when the assumption 2 is not satisfied.

Theorem 2. Assume the extremal (x, λ, u) is u_i-singular for all $i \neq k$. If along the extremal there exist nonempty subsets $J_1, J_2 \subset I$, $k \notin J_1$ and $J_2 \subset J_1$ such that

(1) $l \in J_1, r \in J_2$, we have

$$[g_j, [f, g_l]] \in Span\{g_r, [f, g_r]\}_{r \in I},$$

$$[g_j, ad_f^2 g_v] \in Span\{g_r, [f, g_r], ad_f^2 g_l\}_{r \in I, l \in J_1}$$

for all $j \in I$;

(2) $Span\{g_j, [f, g_j], ad_f^2 g_l, ad_f^3 g_v\}_{jlv} = I\!R^{2n}$ for $j \in I, l \in J_1, v \in J_2$

then the component u_k of the control is bang-bang with a finite number of switches.

Both situations, fully actuated and underactuated, will be illustrated on the example of underwater vehicles.

3. APPLICATION TO UNDERWATER VEHICLES

The dynamics of underwater vehicles can be described with the equations of motion as follows, see (Leonard, 1997) for more details. The position and orientation of the underwater vehicle are identified with the group of rigid-body motions in $I\!R^3$: $SE(3) = \{(R, b);\ R \in SO(3), b \in I\!R^3\}$. If we define $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ and $v = (v_1, v_2, v_3)$ to be the angular and translational velocity of the vehicle in body coordinates, then the kinematic equations are

$$\dot{R} = R\hat{\Omega}, \qquad \hat{\Omega} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \qquad (6)$$

$$\dot{b} = Rv. \qquad (7)$$

We begin by assuming the vehicle is submerged in an infinitely large volume of incompressible, irrotational and inviscid fluid at rest at infinity. Let us denote by Π and P the angular and linear components of the impulse (roughly equivalent to momentum) of the body-fluid system with respect to the body-frame. The Kirchhoff equations of motion for a rigid body in such an ideal fluid are given by

$$\dot{\Pi} = \Pi \times \Omega + P \times v + \tau, \;\; \dot{P} = P \times \Omega + \mathcal{F} \qquad (8)$$

where τ and $\mathcal{F}$ are external torque and force vectors. τ and $\mathcal{F}$ can be used to include gravity, buoyancy and control forces as well as viscous forces such as lift and drag. Notice that Π and P can be computed from the total kinetic energy T of the body fluid system: $T = \frac{1}{2}(\Omega^t J \Omega + 2\Omega^t D v + v^t M v)$, where M and J are respectively the body-fluid mass and inertia matrices. In (Leonard, 1997), the Hamiltonian structure of the dynamics of the underwater vehicle is described.

In this paper, we consider a neutrally buoyant, uniformly distributed, ellipsoidal vehicle moving in the vertical inertial plane and we neglect viscous effects. We denote by (x, z) the absolute position of the vehicle where x is the horizontal position and z the vertical position. θ describes its orientation in this plane so that $q = (x, z, \theta)$. Given our assumptions, M and J are diagonal and $D = 0$ so that for our vehicle restricted to the plane $T = \frac{1}{2}(I\Omega^2 + m_1 v_1^2 + m_3 v_3^2)$ where I is the body-fluid moment of inertia in the plane and m_1, m_3 are body-fluid mass terms in the body horizontal and vertical directions, respectively. We assume that $m_1 \neq m_3$, i.e., the planar vehicle is not a circle. We choose the state vector to be $w = (x, z, \theta, v_1, v_3, \Omega)$. Here Ω is the scalar angular rate in the plane. The equations of motion are

$$\begin{aligned}
\dot{x} &= \cos\theta v_1 + \sin\theta v_3 \\
\dot{z} &= \cos\theta v_3 - \sin\theta v_1 \\
\dot{\theta} &= \Omega \qquad\qquad\qquad\qquad (9) \\
\dot{v}_1 &= -v_3 \Omega \frac{m_3}{m_1} \\
\dot{v}_3 &= v_1 \Omega \frac{m_1}{m_3} \\
\dot{\Omega} &= v_1 v_3 \frac{m_3 - m_1}{I}
\end{aligned}$$

With respect to the notations of Section 2 we have $P^{-1}(q) = Q(q)$ where $Q(q)$ is a rotation matrix $\begin{pmatrix} R(\theta) & 0 \\ 0 & 1 \end{pmatrix}$, $R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. This formulation of the problem leads to simpler computations and to nice geometric interpretation of the results.

3.1 Fully actuated case

We first consider the fully actuated case: the control vector is $u = (u_1, u_2, u_3)$ where u_1 is a force in the body 1-axis, u_2 is a force in the body 3-axis and u_3 is a pure torque in the plane. Accordingly, with the drift vector field f given by the planar equations of motion (9), the input vector fields g_i are given by $g_1 = (0, 0, 0, \frac{1}{m_1}, 0, 0)^t$, $g_2 = (0, 0, 0, 0, \frac{1}{m_3}, 0)^t$, $g_3 = (0, 0, 0, 0, 0, \frac{1}{I})^t$. The Lie brackets of f and these vector fields are given by

$$\begin{aligned}
[f, g_1] &= \frac{1}{m_1}(-\cos\theta, \sin\theta, 0, 0, -\Omega\frac{m_1}{m_3}, -v_3\alpha)^t, \\
[f, g_2] &= \frac{1}{m_3}(-\sin\theta, -\cos\theta, 0, \Omega\frac{m_3}{m_1}, 0, -v_1\alpha)^t, \\
[f, g_3] &= \frac{1}{I}(0, 0, -1, v_3\frac{m_3}{m_1}, -v_1\frac{m_1}{m_3}, 0)^t
\end{aligned}$$

where $\alpha = \frac{m_3 - m_1}{I}$. The assumptions of Theorem 1 are satisfied. We can conclude that there is no totally singular extremal.

Let us study the extremals with 2 components of the control being singular. In order to determine the values of these singular components, we will use the algorithm described in Section 2, in particular equation (5). First, we have to compute the Lie brackets of length 3. It is easy to verify that we have $[g_j, [f, g_j]] = 0$ for $j = 1, 2, 3$, that the following relations are satisfied

$$[g_2, [f, g_1]] = [g_1, [f, g_2]] = -\frac{m_3 - m_1}{m_1 m_3} g_3, \qquad (10)$$

$$[g_3, [f, g_1]] = [g_1, [f, g_3]] = -\frac{1}{I} g_2, \qquad (11)$$

$$[g_3, [f, g_2]] = [g_2, [f, g_3]] = \frac{1}{I} g_1, \qquad (12)$$

and that the Lie brackets of the form $ad_f^2 g_i$ are

$$\begin{aligned}
ad_f^2 g_1 \;=\;& (\sin\theta\Omega\beta, \cos\theta\Omega\beta, v_3\frac{\alpha}{m_1}, -\frac{\Omega^2}{m_1} \\
& -v_3^2\frac{m_3\alpha}{m_1^2}, 0, 0)^t \\
ad_f^2 g_2 \;=\;& (-\cos\theta\Omega\beta, \sin\theta\Omega\beta, v_1\frac{\alpha}{m_3}, 0, \\
& -\frac{\Omega^2}{m_3} + v_1^2\frac{m_1\alpha}{m_3^2}, 0)^t \\
ad_f^2 g_3 \;=\;& \frac{1}{I}(\sin\theta v_1\gamma_1 - \cos\theta v_3\gamma_2, \sin\theta v_3\gamma_2 \\
& + \cos\theta v_1\gamma_1, 0, 0, 0, (v_1^2\frac{m_1}{m_3} - v_3^2\frac{m_3}{m_1})\alpha)^t
\end{aligned}$$

where $\beta = \frac{1}{m_1} + \frac{1}{m_3}$, $\gamma_1 = \frac{m_1}{m_3} - 1$, $\gamma_2 = \frac{m_3}{m_1} - 1$ and α is as above.

3.1.1. u_1, u_2-singular extremals We start with the u_1, u_2-singular extremals. From Theorem 1, the switches of u_3 are located in the set of points $S_3^c = A_1 \cap A_2$ where

$$A_i = \{w;\; \det(g_1, g_2, g_3, [f, g_1], [f, g_2], ad_f^2 g_i) = 0\},$$

$i = 1, 2$. Computing these sets, we find $A_1 = \{w;\; \frac{m_1 - m_3}{I m_1^3 m_3^2} v_3 = 0\}$, $A_2 = \{w;\; \frac{m_1 - m_3}{I m_1^2 m_3^3} v_1 = 0\}$, and

$$S_3^c = \{w;\; v_1 = v_3 = 0\}.$$

By using the Hamiltonian equations (4), we can prove that along a u_1, u_2-singular extremal there is at most one switch and there is one only if $v_1 \equiv v_3 \equiv 0$ along the extremal. This corresponds to a pure rotating motion for the underwater vehicle. Intuitively, this motion is a good candidate for optimality, this will be discussed in (Chyba *et al.*, 2000).The switching time is determined by the initial and final configurations of the phase space.

Let us determine the singular controls u_1 and u_2 when v_1 and v_3 are not both identically zero (it is equivalent to say that the velocities cannot vanish at the same time along the extremal). By Theorem 1, u_3 is constant. Using the algorithm described in Section 2, the equations (10),(11),(12) and the fact that $\lambda^t g_i = 0$, $i = 1, 2$, we have

$$u_1 = \frac{\lambda^t ad_f^2 g_2}{\lambda^t g_3}\frac{m_1 m_3}{m_3 - m_1} = \Omega v_3 (1 + \frac{m_3}{m_1}) + v_1 \frac{\lambda_3}{\lambda_6},$$
$$u_2 = \frac{\lambda^t ad_f^2 g_1}{\lambda^t g_3}\frac{m_1 m_3}{m_3 - m_1} = \Omega v_1 (\frac{m_1}{m_3} - 1) + v_3 \frac{\lambda_3}{\lambda_6}.$$

We remark that these equations are well defined as $\lambda^t g_3 \neq 0$ and the matrix

$$(\alpha_{ij}^3) = \begin{pmatrix} 0 & \dfrac{m_1 - m_3}{m_1 m_3} \\ \dfrac{m_1 - m_3}{m_1 m_3} & 0 \end{pmatrix}$$

is always invertible. If we denote by $\lambda = (\lambda_1, \cdots, \lambda_6)^t$ the adjoint vector, the differential Hamiltonian system corresponding to the u_1, u_2-singular extremals without switches on u_3 can be reduced from 12 to 8 equations. We can prove that

$$\begin{aligned} \dot{v}_1 &= \Omega v_3 + v_1 \frac{\lambda_3}{\lambda_6} \\ \dot{v}_3 &= -\Omega v_1 + v_3 \frac{\lambda_3}{\lambda_6} \\ \dot{\Omega} &= v_1 v_3 \frac{m_3 - m_1}{I} + \frac{u_3}{I} \\ \dot{\lambda}_3 &= \frac{m_1 - m_3}{I} \lambda_6 (v_1^2 + v_3^2) \\ \dot{\lambda}_6 &= \lambda_3 \end{aligned}$$

where u_3 is a constant, $(\dot{x}, \dot{z}, \dot{\theta})$ being given by the equations of motion (9).

Simulations. To perform numerical simulations, we consider an ellipsoidal underwater vehicle with body 1-axis length $L_1 = 0.457$ m and body 3-axis length $L_3 = 0.305$ m. The corresponding mass and inertia terms for the vehicle in water are $m_1 = 13.2$kg, $m_3 = 25.6$kg and $I = 0.12$kg-m^2. In Figure 1 we represent the motion described by the projection of a u_1, u_2-singular extremal in the x, z-plane as well as the motion of the angle θ. Figures 2 and 3 show respectively the corresponding velocities and the values of the singular controls (along this extremal, the function λ_6 is strictly positive and we assume the bound on u_3 is 0.8 N-m). Remark that because the velocities v_1 and v_3 are not identically zero, the underwater vehicle does not move tangentially or perpendicularly to the trajectory. The velocity v_3 vanishes exactly at the point where the trajectory in the x, z-plane has a vertical tangent. At this point the underwater vehicle has made a rotation of $\frac{\pi}{2}$ from its intial configuration $(x, z, \theta) = (0, 0, 0)$.

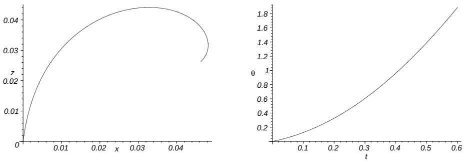

Fig. 1. projection in the x, z-plane; angle

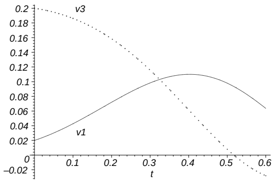

Fig. 2. translational velocities v_1, v_3

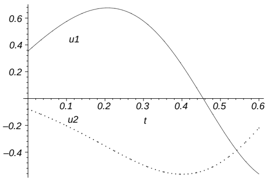

Fig. 3. components u_1, u_2 of the control

3.1.2. u_1, u_3-singular extremals In this case, the switches of the control u_2 are located in $S_2^c = B_1 \cap B_3$ where

$$B_i = \{w;\ \det(g_1, g_2, g_3, [f, g_1], [f, g_3], ad_f^2 g_i) = 0\},$$

$i = 1$ or 3. We have $B_1 = \{w;\ \frac{\Omega}{I^2 m_1^2 m_3}(\frac{1}{m_1} + \frac{1}{m_3}) = 0\}$, $B_3 = \{w;\ \frac{v_1}{I^2 m_1^2 m_3}(\frac{m_1}{m_3} - 1) = 0\}$ and $S_2^c = \{w;\ v_1 = \Omega = 0\}$. If Ω and v_1 are identically zero, there is at most one switch and the motion corresponds to a translation in the direction of the vertical axis of the body frame coordinates. If the extremal belongs to the set S_2, then u_2 is constant, $\lambda^t g_2 \neq 0$, and the singular components of the control u_1 and u_3 are given by $u_1 = \frac{\lambda^t ad_f^2 g_3}{\lambda^t g_2} I$ and $u_3 = \frac{\lambda^t ad_f^2 g_1}{\lambda^t g_2} I$.

3.1.3. u_2, u_3-singular extremals This situation is completely analogous to the preceding one. The switches of u_1 belongs to $S_1^c = C_2 \cap C_3$, $C_i = \{w;\ \det(g_1, g_2, g_3, [f, g_2], [f, g_3], ad_f^2 g_i) = 0\}$, $i = 2, 3$ with $C_2 = \{w;\ \frac{\Omega}{I^2 m_1 m_3^2}(\frac{1}{m_1} + \frac{1}{m_3}) = 0\}$, $C_3 = \{w;\ \frac{v_3}{I^2 m_1 m_3^2}(\frac{m_3}{m_1} - 1) = 0\}$ and $S_1^c =$

$\{w;\ v_3 = \Omega = 0\}$. When $\Omega \equiv v_3 \equiv 0$, then the motion has at most one switch and is a translation in the direction of the horizontal axis of the body frame coordinates. The singular controls u_2, u_3 for an extremal in the set S_1 are given by $u_2 = -\frac{\lambda^t ad_f^2 g_3}{\lambda^t g_1} I$ and $u_3 = -\frac{\lambda^t ad_f^2 g_2}{\lambda^t g_1} I$.

3.2 Underactuated case

Let us consider the case when the control is 2-dimensional $u = (u_2, u_3)$ and $g_2 = (0, 0, 0, -\frac{\sin\theta}{m_1}, \frac{\cos\theta}{m_3}, 0)^t$, $g_3 = (0, 0, 0, 0, 0, \frac{1}{I})^t$. In this case u_2 corresponds to a force in the *inertial* vertical direction and u_3 is again a pure torque in the plane. This choice of control is motivated by a buoyancy driven underwater glider. The Lie brackets involving only the vector fields f and g_3 are given in Section 3.1. We also have

$$[f, g_2] = (\frac{\delta \sin 2\theta}{2}, -\frac{\sin^2\theta}{m_1} - \frac{\cos^2\theta}{m_3}, 0, 0, 0, h)^t$$

where $\delta = (\frac{1}{m_1} - \frac{1}{m_3})$, $\alpha = \frac{m_3 - m_1}{I}$ and $h = \alpha(\frac{v_3 \sin\theta}{m_1} - \frac{v_1 \cos\theta}{m_3})$. The Lie brackets of length 3 are given by

$$[g_2, [f, g_2]] = \frac{\alpha I \sin 2\theta}{m_1 m_3} g_3,$$
$$[g_3, [f, g_2]] = [g_i, [f, g_3]] = 0, \quad i = 2, 3$$

and

$$ad_f^2 g_2 = (\delta \cos 2\theta\Omega, -\delta \sin 2\theta\Omega, -h, \frac{v_3 m_3}{m_1} h, -\frac{v_1 m_1}{m_3} h, \tilde{h})^t$$

where $\tilde{h} = 2\Omega\alpha(\frac{v_1 \sin\theta}{m_3} + \frac{v_3 \cos\theta}{m_1})$. Assumption 1 of Theorem 1 is satisfied with $J = \{2, 3\}$ but the vector fields $\{g_i, [f, g_i], ad_f^2 g_i\}_{i=2,3}$ are linearly dependent everywhere. Therefore we cannot use Theorem 1 to draw conclusions. However, if we assume $\Omega \neq 0$ we have

$$[g_2, ad_f^2 g_3] = h[f, g_3] - \frac{I}{\Omega} ad_f^2 g_2.$$

As $[g_3, ad_f^2 g_3] = 0$, assumption 1 of Theorem 2 is then satisfied. By computing the vector field $ad_f^3 g_3$ we can prove that if $w \notin U$ where $U = \{w;\ \Omega = 0 \text{ and/or } v_1 = v_3 = 0\}$, then assumption 2 of Theorem 2 is verified. As a consequence, if the extremal belongs to the complement of the set U, it cannot be totally singular. More precisely, if one control is singular the other one must be bang-bang with a finite number of switches. More details and the case when the extremal does not belong to U^c everywhere are analysed in (Chyba *et al.*, 2000).

4. CONCLUSION

In this paper, we study the singular extremals of a controlled mechanical system for the time-optimal problem. Characterizing singular extremals is a first step in the study of optimal strategies. Under Lie-algebraic assumptions, we proved that along $m-1$-singular extremals there can only be a finite number of switches. Although one might expect the existence in general of only finitely many switches along optimal motions, many optimal control problems lead to accumulation points of switches, see for instance (Sussmann, 1997), a behavior which is often referred to as the "Fuller phenomenon". Indeed, the maximum principle gives no direct information on the way to connect the pieces of an extremal or a bound on the number of switches, along an optimal trajectory for an optimal control. In (Sussmann, 1979), the author gives conditions for scalar control systems, under which a bang-bang theorem with bounds on the number of switches can be proved. This result can be applied to the underwater vehicle in the underactuated situation with only one control; see (Chyba *et al.*, 2000) for more details.

Optimizing a cost function is clearly interesting by itself. Furthermore, in dealing with the motion planning problem in the presence of obstacles, steering methods based on optimality guarantee the convergence of general motion planning schemes, see (Laumond *et al.*, 1990; Sekhavat and Laumond, 1996).

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