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Control-Lyapunov Universal Formulas for Restricted Inputs

Yuandan Lin*
Department of Mathematics
Florida Atlantic University
Boca Raton, FL 33431
FAX:+1 407-367-2436
yuandan@polya.math.fau.edu

Eduardo D. Sontag†
Department of Mathematics
Rutgers University
New Brunswick, NJ 08903
FAX: +1 908-445-5530
sontag@control.rutgers.edu

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Abstract

We deal with the question of obtaining explicit feedback control laws that stabilize a nonlinear system, under the assumption that a "control Lyapunov function" is known. In previous work, the case of unbounded controls was considered. Here we obtain results for bounded and/or positive controls. We also provide some simple preliminary remarks regarding a set stability version of the problem and a version for systems subject to disturbances.

1 Introduction

A widespread technique in nonlinear control relies upon the use of "energy" functions V which can be made to decrease pointwise by means of instantaneous controls; given such a V, the control applied at each instant is one that forces a decrease. Functions V with this property are generically called *control-Lyapunov functions* ("clf's"), in analogy to the Lyapunov functions classical in dynamical systems when no control is available. The use of clf's is standard in engineering; see for instance the many examples in section 3.6 of the textbook [12]. In the early 1980s, Artstein ([1]) and one of the authors ([13]) produced independent and mathematically complementary theoretical justifications of the clf approach (the former work assumed more regularity; the latter required less smoothness but applied more generally). More recently, the work [14] provided a systematic methodology for the use of clf's, resulting in "universal formulas" which permit a direct computation of the appropriate control, with no need to search for a control that makes the clf decrease at each point of the state space. Universal formulas produce the necessary control directly from the derivative of the clf and the data defining the system. The purpose of this paper is to generalize universal formulas to cases where inputs are constrained in magnitude or in sign.

Motivations for Universal Formulas

As mentioned above, the use of clf's is standard in practice. In this context, universal formulas are a natural mathematical object to study, and their use avoids a pointwise minimization. But also, since clf's are as a general rule easier to obtain than the feedback laws themselves

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—after all, in order to prove that a given feedback law stabilizes, one typically has to exhibit a suitable Lyapunov function anyway— these techniques provide in principle an extremely powerful approach to nonlinear stabilization. The availability of universal formulas allows the search for a feedback law to be confined to just *one scalar function* and can be used as the basis of numerical approaches. An excellent illustration of this principle can be found in the recent paper [10], which employed the universal formula given in [14] as the main component of a "neural network" controller. Other applications of universal formulas can be found in [5], which employs such formulas in disturbance attenuation problems, the work in [17] on global stabilization with continously differentiable feedback for nonlinear affine systems, the work in [7] on generalizations to discontinuous stabilization, and the research in [4] on stochastic clf's. Also, the book [2] has a chapter devoted to universal clf formulas and their applications. As it is often the case in applications that controls are constrained, it can be reasonably expected that the functions given in this paper will be of similar or even more interest.

Definitions

This paper concerns systems evolving on $\mathbb{R}^n$ and affine on controls:

$$\dot{x} = f(x) + G(x)u \,, \tag{1}$$

where all entries of the vector f and the $n \times m$ matrix G are smooth functions on $\mathbb{R}^n$, and $f(0) = 0$. We assume that controls are restricted to take values in some subset of $\mathbb{R}^m$,

$$u \in \mathcal{U} \subseteq \mathbb{R}^m \,.$$

For the preliminary discussion, we do not need to impose any structure on the set $\mathcal{U}$.

Assume that there is some feedback law

$$k : \mathbb{R}^n \longrightarrow \mathcal{U} \subseteq \mathbb{R}^m \tag{2}$$

which is smooth (differentiability is enough) on $\mathbb{R}^n \setminus \{0\}$ and which stabilizes the system (1), in the sense that the origin $x = 0$ is a globally asymptotically stable solution of the differential equation

$$\dot{x} = f(x) + G(x)\,k(x) \,, \tag{3}$$

then classical Converse Lyapunov Theorems due to Massera, Kurzweil, and others (see e.g. [11, 6]) establish the existence of a positive definite and proper (i.e., $V(x) \to \infty$ as $|x| \to \infty$) smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

$$\inf_{u \in \mathcal{U}} \{\, a(x) + B(x)u \,\} < 0, \;\; \forall x \neq 0 \,. \tag{4}$$

(Here and throughout the paper, we use the notations:

$$a(x) \;\stackrel{\text{def}}{=}\; \nabla V(x)\, f(x) \tag{5}$$

and

$$B(x) = (b_1(x), \ldots, b_m(x)) \;\stackrel{\text{def}}{=}\; \nabla V(x)\, G(x) \tag{6}$$

for Lie derivatives.) To prove this fact, simply find a Lyapunov function for (3) and now use $u = k(x)$ in the infimum.

This motivates the following definition (see e.g. [14, 8], as well as [3, 15, 16] for related work).

Definition 1.1 A proper and positive definite smooth function

$$V : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$$

is said to be a *control Lyapunov function (clf)* [with respect to controls taking values in $\mathcal{U}$] if (4) holds. The function V is said to satisfy the *small control property (scp)* if for any $\varepsilon > 0$, there is a $\delta > 0$ such that, if $x \neq 0$ satisfies $|x| < \delta$, then there is some $u \in \mathcal{U}$ with $|u| < \varepsilon$ such that $a(x) + B(x)u < 0$. □

If the above mentioned feedback $k(x)$ is also continuous at the origin, then the Lyapunov function V obtained from the converse theorems has the scp. Thus existence of stabilizing feedback with regularity implies existence of a clf (possibly with scp). There is a converse as well. For various choices of control-value sets $\mathcal{U}$, a theorem of Artstein (see [1]) guarantees that if there is a clf V, then there is a feedback law

$$u = k(x)$$

which globally stabilizes the system (1), and k is smooth on $\mathbb{R}^n \backslash \{0\}$. If in addition V satisfies the scp, then k can be chosen to be what we call *almost smooth* on $\mathbb{R}^n$, meaning not only smooth away from the origin, but also continuous on all of $\mathbb{R}^n$. Thus Lyapunov functions with scp completely characterize almost-smooth stabilizability.

The proof in [1] is based on partitions of unity and is therefore nonconstructive. In [14] a "universal" formula (the term will be defined later; it means roughly an explicit formula for obtaining $k(x)$ from $a(x)$ and $B(x)$) was given for the case when the control set $\mathcal{U}$ is the whole of $\mathbb{R}^m$. The formula was

$$u = k(x) = \kappa\left(a(x),\, |B(x)|^2\right)\, B(x)'\,, \tag{7}$$

where

$$\kappa(a,\, b) \;\stackrel{\text{def}}{=}\; \begin{cases} -\dfrac{a + \sqrt{a^2 + b^2}}{b}\,, & \text{if } b \neq 0, \\ 0\,, & \text{if } b = 0. \end{cases} \tag{8}$$

This was generalized to the case when $\mathcal{U}$ is the unit ball of $\mathbb{R}^m$ in the paper [8]. The formula obtained in that case was as follows: The feedback law $u = k(x)$ has the same form as in (7), except that the function κ is now defined instead to be:

$$\kappa(a, b) \;\stackrel{\text{def}}{=}\; \begin{cases} -\dfrac{a + \sqrt{a^2 + b^2}}{b(1 + \sqrt{1 + b})}\,, & \text{if } b \neq 0, \\ 0\,, & \text{if } b = 0. \end{cases} \tag{9}$$

As Artstein's Theorem is valid on rather general control-value sets $\mathcal{U}$, it is natural to ask if such "universal" formulas exist for other $\mathcal{U}$ as well. That is the focus of this work. Specifically, we provide universal formulas for the particular cases in which $\mathcal{U} = (0, \infty)$ and $\mathcal{U} = (0, 1)$, which correspond up to rescalings to the possible instances of scalar positive controls. Also, we provide some preliminary results on clf's for systems with disturbances, as well as on clf's for the problem of stabilization with respect to non-equilibrium compact attractors.

2 Universal Formulas

For any subset $\mathcal{U} \subseteq \mathbb{R}^m$, consider the open set

$$\mathcal{D}_{\mathcal{U}} \stackrel{\text{def}}{=} \{ (a,\, B) \in \mathbb{R} \times \mathbb{R}^m \mid \exists u \in \mathcal{U}, a + \langle B,\ u \rangle < 0 \} \ , \tag{10}$$

where we use B to denote the row vector $(b_1,\, b_2,\, \ldots,\, b_m)$.

For instance,

$$D_{\mathcal{U}} \ = \ \{(a,\, b) \mid a < |b|,\ a,\, b \in \mathbb{R}\} \ ,$$

when the control is one-dimensional and satisfies $-1 < u < 1$, i.e., $m = 1$ and $\mathcal{U} = (-1,\, 1)$.

Then the definition of clf is equivalent to the requirement: for the given V and any $x \neq 0$, it must hold that

$$(a(x),\, b_1(x),\, \ldots,\, b_m(x)) \in \mathcal{D}_{\mathcal{U}} \ .$$

In other words, $\mathcal{D}_{\mathcal{U}}$ is the largest possible subset of $\mathbb{R}^{m+1}$ where the $(m+1)$-tuples

$$(a(x),\, b_1(x),\, \ldots,\, b_m(x)) \ , \ x \neq 0 \,,$$

can lie. Note that for any given $\mathcal{U}$, $\mathcal{D}_{\mathcal{U}}$ does not contain any points of the form $(a,\, 0,\, \ldots,\, 0)$ for all $a \geq 0$, so in particular the origin is not in $\mathcal{D}_{\mathcal{U}}$.

Definition 2.1 Let $\mathcal{U}$ be a subset of $\mathbb{R}^m$. A *universal* stabilizing formula relative to $\mathcal{U}$ is a real-analytic function

$$\alpha = \alpha_{\mathcal{U}} \ : \ \mathcal{D}_{\mathcal{U}} \subseteq \mathbb{R} \times \mathbb{R}^m \longrightarrow \mathcal{U} \subseteq \mathbb{R}^m$$

such that

- for any $(a,\, B) \in \mathcal{D}_{\mathcal{U}}$, $\ a + B\,\alpha(a,\, B) < 0$;
- for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\mathcal{D}_{\mathcal{U}} \bigcap \Big[a < \delta\,|B|\,,\ |a| < \delta,\ |B| < \delta \Big] \quad \Longrightarrow \quad |\alpha(a, B)| < \varepsilon \ . \tag{11}$$

(The region where α must be small is shown in Figure 1.) □

One could also define universality in weaker senses, for instance requiring just smoothness of α. However, we impose analyticity in order to disallow solutions involving tricks such as partitions of unity.

Proposition 2.2 Let α be a universal stabilizing formula relative to $\mathcal{U} \subseteq \mathbb{R}^m$. Then for each system (1) and clf V,

$$k(x) \ \stackrel{\text{def}}{=} \ \alpha(a(x),\, B(x))$$

is smooth on $\mathbb{R}^n \backslash \{0\}$ and globally stabilizes the system. If in addition V satisfies the scp, then

$$\alpha(a(x),\, B(x))$$

is also continuous in the origin, i.e., $k(\cdot)$ is in fact almost smooth on $\mathbb{R}^n$. Moreover, if the right-hand side of the system is analytic in x and V is analytic, then k is analytic on $\mathbb{R}^n \backslash \{0\}$.

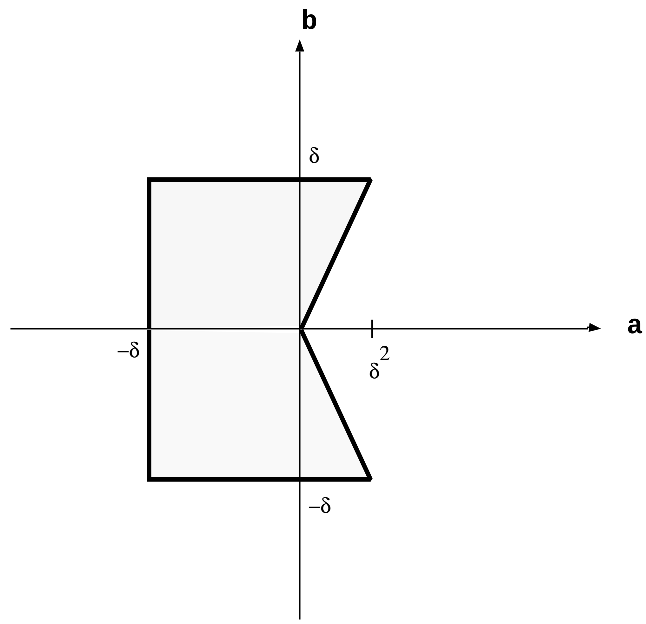

Figure 1: *Region where function must be small.*

Proof. From the fact that

$$(a(x),\, B(x)) \in \mathcal{D}_{\mathcal{U}},\ \ \forall x \neq 0\ ,$$

it is immediate that

$$a(x) + \langle B(x),\ \alpha(a(x),\, B(x))\rangle < 0,\ \ \forall x \neq 0\ .$$

So the same function V is also a Lyapunov function for the closed-loop system obtained when using $u = k(x)$ as a feedback. Since $a(x)$ and $B(x)$ are smooth functions, we also get that $\alpha(a(x),\, B(x))$ is smooth for $x \in \mathbb{R}^n \backslash \{0\}$. In addition, if the right-hand side of the system and V are real-analytic, then both $a(x)$ and $B(x)$ are real-analytic. We again obtain that $\alpha(a(x),\, B(x))$ is real-analytic for $x \in \mathbb{R}^n \backslash \{0\}$. Now we only need to show that if a clf V satisfies the scp, then $\alpha(a(x),\, B(x))$ is small in magnitude if x is near 0.

Take any $\varepsilon > 0$. From the definition of universal stabilizing formula, there exists $\delta > 0$ such that,

$$\Big[\ (a,\, B) \in \mathcal{D}_{\mathcal{U}}\ ,\ a < \delta\,|B|\ ,\ |a| < \delta\ \text{ and }\ |B| < \delta\ \Big]\ \Longrightarrow\ |\,\alpha(a,\, B)\,| < \varepsilon\,.$$

As V is positive definite, its gradient vanishes at the origin, so $a(0) = 0$ and $B(0) = 0$, and by continuity, there exists some $\delta_1 > 0$ such that

$$|x| < \delta_1 \ \ \Longrightarrow\ \ |a(x)| < \delta\ \text{ and }\ |B(x)| < \delta\,.$$

By the scp we can choose δ_1 so that also $a(x) < \delta\,|B(x)|$. Combining with the above discussion, we know that

$$|\alpha(a(x),\, B(x))| < \varepsilon\,,\ \ \text{whenever}\ \ |x| < \delta_1\,.$$

This completes the proof of the proposition. ∎

In what follows, we provide various universal formulas for different choices of the control value set $\mathcal{U}$.

3 Positive Controls

We first illustrate, by means of counterexamples, that the use of the previously known formulas can lead to wrong results if there are positivity constraints.

Unbounded Case

Let us first take the control value set to be $\mathcal{U} = (0, \infty)$, the positive unbounded case. Consider the following system:

$$\dot{x} = x^2 - u \tag{12}$$

with $n = m = 1$. Take the clf $V(x) = \frac{1}{2}\, x^2$. Then for any $x < 0$,

$$\inf_{u>0} (a(x) + ub(x)) \;=\; \inf_{u>0} (x^3 - ux) \;=\; x^3 \;<\; 0\,;$$

if instead $x > 0$,

$$\inf_{u>0} (a(x) + ub(x)) \;=\; \inf_{u>0} (x^3 - ux) \;=\; -\infty < 0\,.$$

Also

$$\left|\frac{a(x)}{b(x)}\right| = x^2$$

is small when x is near zero. So V is a clf with respect to controls in $\mathcal{U} = (0, \infty)$, and it has the scp. But the control law given by (7)–(8) is

$$k(x) \;=\; -\,\frac{x^3 + \sqrt{x^6 + x^4}}{-x} \;=\; x^2 + x\sqrt{x^2 + 1}\,,$$

which fails to be positive if $x < 0$. This shows that more work is required in finding a universal formula in the unbounded positive case.

Bounded Case

Now we give an example for the positive bounded control case. Consider the following one-dimensional system:

$$\dot{x} = \frac{x^2}{1 + x^2} - u \tag{13}$$

with the control value set $\mathcal{U} = (0, 1)$. Take again the clf $V(x) = \frac{1}{2}\, x^2$. Then it follows that

$$\inf_{u\in(0,\,1)} (a(x) + ub(x)) \;=\; \inf_{u\in(0,\,1)} \left(\frac{x^3}{1 + x^2} - ux\right)\,.$$

If $x < 0$, then

$$\inf_{u\in(0,\,1)} \left(\frac{x^3}{1 + x^2} - ux\right) \;\le\; \frac{x^3}{1 + x^2} \;<\; 0\,;$$

if instead $x > 0$, then

$$\begin{aligned}\inf_{u\in(0,\,1)} \left(\frac{x^3}{1 + x^2} - ux\right) \;&\le\; \frac{x^3}{1 + x^2} - \left(\frac{\frac{1}{2} + x^2}{1 + x^2}\right) x \\ &=\; -\,\frac{x}{2(1 + x^2)} \;<\; 0\,.\end{aligned}$$

Also

$$\left|\frac{a(x)}{b(x)}\right| = \frac{x^2}{1+x^2}$$

is small when x is near zero. So V is a clf with respect to controls in $\mathcal{U} = (0,\, 1)$, and it has the scp. But the control law given by (7)–(9) is

$$\begin{aligned} k(x) &= -\frac{\dfrac{x^3}{1+x^2} + \sqrt{\left(\dfrac{x^3}{1+x^2}\right)^2 + x^4}}{-x\left(1+\sqrt{1+x^2}\right)} \\ &= \frac{x^2 + x\sqrt{x^2+(1+x^2)^2}}{(1+x^2)(1+\sqrt{1+x^2})}\,. \end{aligned}$$

It fails to be positive (in particular, in $(0,\, 1)$), since

$$x^2 + x\sqrt{x^2+(1+x^2)^2} \;<\; 0 \;\text{ if } x < 0\,.$$

The Region of Interest

In this section, we assume that the control in (1) is one-dimensional. Again, if V is a clf for (1), we denote

$$a(x) \stackrel{\text{def}}{=} \nabla V(x) f(x)$$

and

$$b(x) \stackrel{\text{def}}{=} \nabla V(x) g(x)\,.$$

Let $\mathcal{D}$ be the open subset of $\mathbb{R}^2$ obtained by deleting the closed positive half x-axis,

$$\mathcal{D} \stackrel{\text{def}}{=} \mathbb{R}^2 \setminus \{\, (x,\, 0) \mid x \geq 0 \,\}\,.$$

The polar coordinates $(r,\, \theta)$ of elements of $\mathcal{D}$, if we restrict $\theta \in (0,\, 2\pi)$, are real-analytic functions of $(x,\, y) \in \mathcal{D}$. Note that for any control value set $\mathcal{U}$, $\mathcal{D}_{\mathcal{U}} \subseteq \mathcal{D}$.

Let

$$\mathcal{U}_1 \stackrel{\text{def}}{=} (0,\, 1)$$

and

$$\mathcal{U}_2 \stackrel{\text{def}}{=} (0, \infty)\,.$$

From the definition given in (10),

$$\mathcal{D}_{\mathcal{U}_1} = \left\{ (r\cos\theta,\, r\sin\theta) \;\middle|\; r > 0,\; \frac{\pi}{2} < \theta < \frac{7\pi}{4} \right\}\,, \tag{14}$$

and

$$\mathcal{D}_{\mathcal{U}_2} = \left\{ (r\cos\theta,\, r\sin\theta) \;\middle|\; r > 0,\; \frac{\pi}{2} < \theta < 2\pi \right\}\,. \tag{15}$$

The two regions are shown in Figure 2.

For any $r > 0$ and any $\theta \in \mathbb{R}$, define

$$\chi(r,\, \theta) \stackrel{\text{def}}{=} \left(\frac{2}{\pi}\arctan\left(\frac{\theta}{r}\right) + 1\right)\theta\,, \tag{16}$$

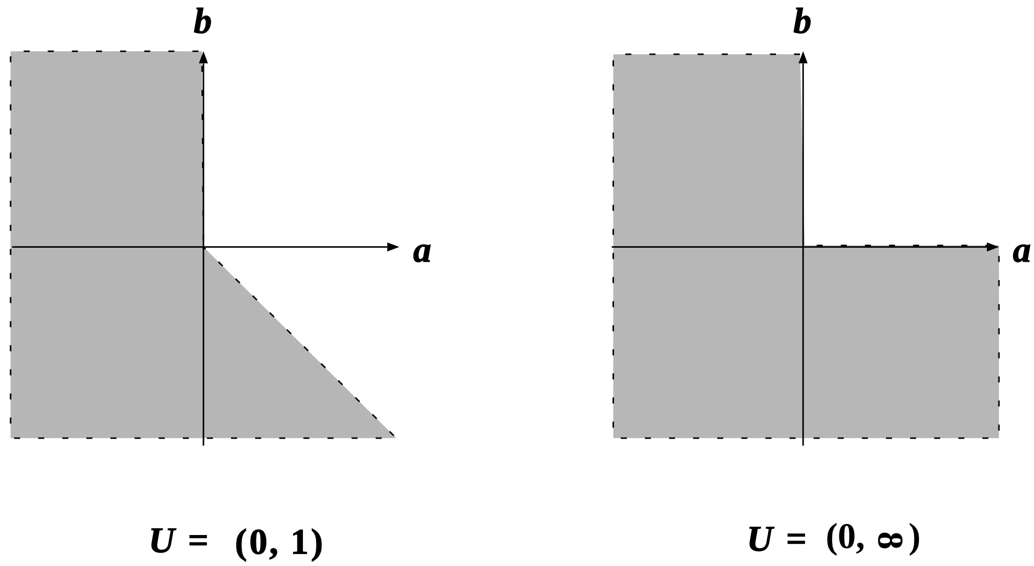

Figure 2: *The two regions* $\mathcal{D}_{(0,1)}$ *and* $\mathcal{D}_{(0,\infty)}$.

and

$$\zeta(r,\,\theta) \stackrel{\text{def}}{=} \chi(r,\,\theta) - \chi(r,\,-\pi)\,. \tag{17}$$

For $r > 0$ and $\theta \in [-\pi,\, \frac{\pi}{4}]$, let

$$k_1(r,\,\theta) \stackrel{\text{def}}{=} \frac{\zeta(r,\,\theta)}{\zeta(r,\,\frac{\pi}{4})} = \frac{\chi(r,\,\theta) - \chi(r,\,-\pi)}{\chi(r,\,\frac{\pi}{4}) - \chi(r,\,-\pi)}\,. \tag{18}$$

For $r > 0$ and $\theta \in (-\pi,\, \frac{\pi}{2})$, let

$$\begin{aligned} k_2(r,\,\theta) &\stackrel{\text{def}}{=} \frac{3\pi}{2(\pi - 2\theta)}\,\zeta(r,\,\theta) \\ &= \frac{3\pi}{2(\pi - 2\theta)}\,(\chi(r,\,\theta) - \chi(r,\,-\pi))\,. \end{aligned} \tag{19}$$

Now we are ready to state and prove our main results on positive controls.

3.1 Statements of Results for Positive Controls

Theorem 1 *Let* $\mathcal{U}_1 = (0,\ 1)$*. Then*

$$k_1\left(r,\ \theta - \frac{3}{2}\pi\right) \tag{20}$$

is a universal stabilizing formula relative to $\mathcal{U}_1$*, where the function* k_1 *is defined by (18), and* r *and* θ *are the polar coordinates of* $(a,\,b) = (r\cos\theta,\, r\sin\theta) \in \mathcal{D}_{\mathcal{U}_1}$.

Proof. The real-analyticity of $k_1(r,\,\theta - \frac{3\pi}{2})$ for $(a,\,b) \in \mathcal{D}_{\mathcal{U}_1}$ is clear from Item 1 of Proposition 3.4 given in the next section and also the fact that r and θ are real-analytic functions of

$(a,\, b) \in \mathcal{D}_{\mathcal{U}_1}$. Because of Item 2 of the same Proposition, k_1 maps any point $(a,\, b) \in \mathcal{D}_{\mathcal{U}_1}$ to $\mathcal{U}_1$. Also

$$a + k_1 \cdot b = r \cdot \left(\cos\theta + k_1(r,\, \theta - \frac{3\pi}{2}) \sin\theta \right) < 0$$

by the Item 3. So the only thing that remains to be shown is that it satisfies (11).

Given any $\varepsilon > 0$. Fix $(a,\, b) = (r\cos\theta,\, r\sin\theta) \in \mathcal{D}_{\mathcal{U}_1}$, and assume that

$$|a| = r|\cos\theta| < \delta, \;\; |b| = r|\sin\theta| < \delta, \;\text{ and }\; \cos\theta < \delta|\sin\theta| \tag{21}$$

for some $\delta > 0$ (which will be chosen later).

If $\frac{\pi}{2} < \theta \le \frac{3\pi}{2}$, then from Item 3 of Proposition 3.4,

$$\lim_{r \to 0^+} k_1 \left(r,\, \theta - \frac{3\pi}{2} \right) = 0 \;,$$

uniformly in θ; hence there exists $\delta_1 > 0$, such that if $0 < |r| < \delta_1$,

$$k_1 \left(r,\, \theta - \frac{3\pi}{2} \right) < \varepsilon \;.$$

If $\frac{3\pi}{2} < \theta < \frac{7\pi}{4}$, then also from Item 3 of Proposition 3.4,

$$\lim_{r \to 0^+} k_1 \left(r,\, \theta - \frac{3\pi}{2} \right) = \frac{\pi}{4} \left(\theta - \frac{3\pi}{2} \right) \;,$$

hence there exists $\delta_2 > 0$, such that if $0 < |r| < \delta_2$,

$$k_1 \left(r,\, \theta - \frac{3\pi}{2} \right) < \frac{\pi}{4} \left(\theta - \frac{3\pi}{2} \right) + \frac{\varepsilon}{2} \;.$$

Based on (21), we have

$$\theta - \frac{3\pi}{2} < \tan \left(\theta - \frac{3\pi}{2} \right) = \frac{\cos\theta}{|\sin\theta|} < \delta \;,$$

and so

$$k_1 \left(r,\, \theta - \frac{3\pi}{2} \right) < \frac{\pi}{4}\, \delta + \frac{\varepsilon}{2} \;.$$

Combine the results above, if we let

$$\delta \overset{\text{def}}{=} \min \left\{ \delta_1,\, \delta_2,\, \frac{2\varepsilon}{\pi} \right\} \;,$$

then if $(a,\, b) \in \mathcal{D}_{\mathcal{U}_1}$ and satisfies (21), then indeed

$$k_1 \left(r,\, \theta - \frac{3\pi}{2} \right) < \varepsilon \;.$$

This completes the proof of the Theorem. ∎

Remark 3.1 From the proof of Theorem 1, we can see that in order to find a universal formula for $\mathcal{U}_1 = (0,\, 1)$, we only need to find a real-analytic function

$$k :\; \mathcal{D}_{(0,\,1)} \longrightarrow (0,\, 1)$$

such that

- $\cos\theta + k(r,\,\theta)\sin\theta < 0$, i.e.,

$$k(r,\,\theta) < -\cot\theta,\ \forall\theta \in (\frac{\pi}{2},\,\pi),\ \forall r > 0\,,$$

  and

$$k(r,\,\theta) > -\cot\theta,\ \forall\theta \in (\pi,\,\frac{7\pi}{4}),\ \forall r > 0\,;$$

- $\forall\theta \in (\frac{\pi}{2},\,\frac{3\pi}{2})$, the limit

$$\lim_{r\to 0^+} k(r,\,\theta) = 0\,.$$

In other words, we need a real-analytic function as shown in Figure 3. □

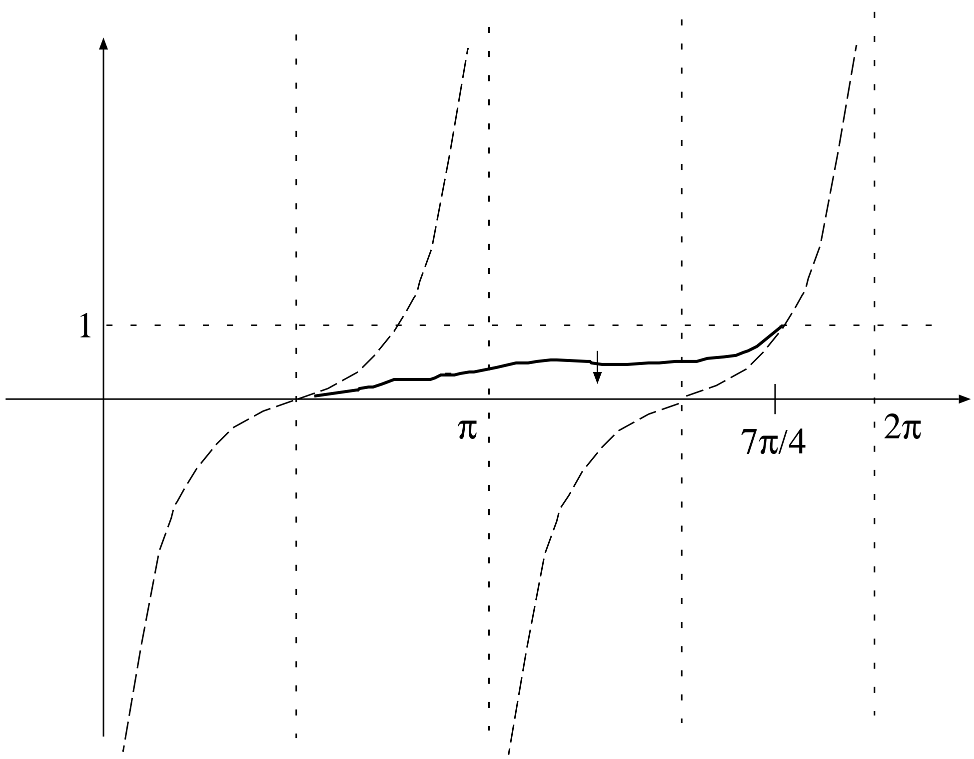

Figure 3: *Graph for $k(r,\,\theta)$ when $\mathcal{U} = (0,\,1)$.*

Similarly to the argument above, we need a function as shown in Figure 4 for the case $\mathcal{U}_2 = (0,\,\infty)$. Proceeding as in the proof for Theorem 1, we can get the following universal formula for this case.

Theorem 2 *Let $\mathcal{U}_2 = (0,\,\infty)$. Then*

$$k_2\left(r,\ \theta - \frac{3}{2}\pi\right) \tag{22}$$

is a universal stabilizing formula relative to $\mathcal{U}_2$, where the function k_2 is defined by (19), and r and θ are the polar coordinates of $(a,\,b) = (r\cos\theta,\,r\sin\theta) \in \mathcal{D}_{\mathcal{U}_2}$. ∎

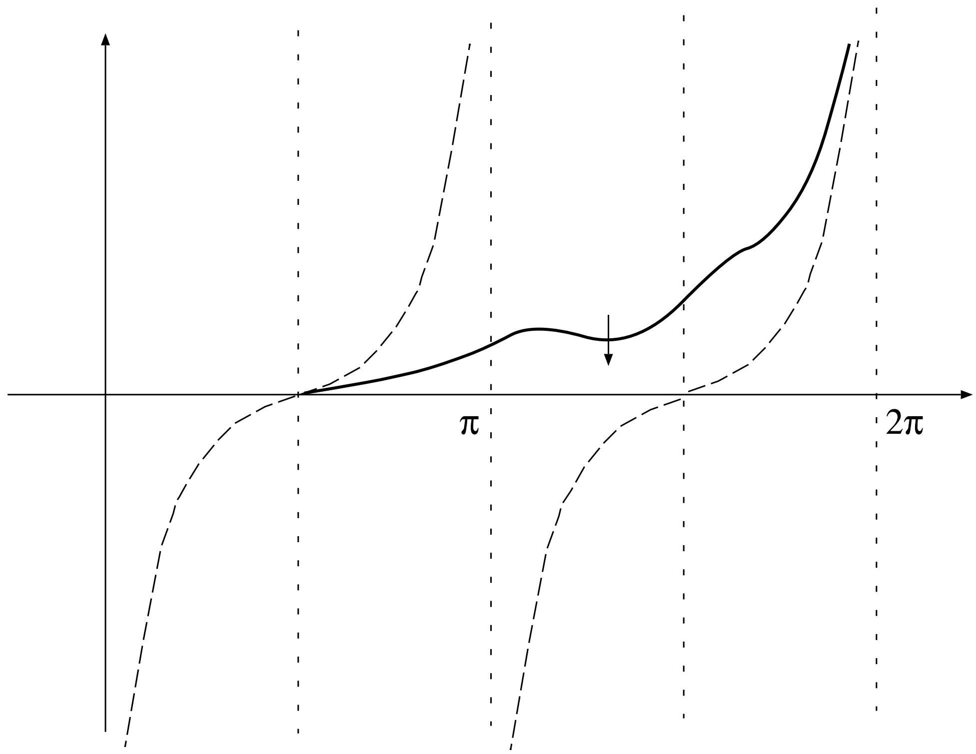

Figure 4: *Graph for $k(r, \theta)$ when $\mathcal{U} = (0, \infty)$.*

3.2 Proofs of Technical Results Needed for Positive Controls

Lemma 3.2 For any fixed $r > 0$, χ defined by (16) is a strictly increasing function of θ.

Proof. For any fixed $r > 0$, let $\xi \overset{\text{def}}{=} \theta/r$, then we have

$$\frac{\partial \chi}{\partial \theta} \;=\; \frac{1}{r}\frac{\partial \chi}{\partial \xi} \;=\; \frac{2}{\pi}\arctan(\xi) + 1 + \xi\,\frac{2}{\pi}\,\frac{1}{1+\xi^2}\,. \tag{23}$$

Since

$$\frac{1}{r}\frac{\partial^2 \chi}{\partial \xi^2} \;=\; \frac{4}{\pi}\frac{1}{(1+\xi^2)^2} \;>\; 0\,, \tag{24}$$

we have

$$\frac{1}{r}\frac{\partial \chi}{\partial \xi} \;>\; \lim_{\xi \to -\infty}\left(\frac{2}{\pi}\arctan(\xi) + 1 + \xi\,\frac{2}{\pi}\,\frac{1}{1+\xi^2}\right) \;=\; 0\,, \tag{25}$$

i.e., $\dfrac{\partial \chi}{\partial \theta} > 0$. Hence for any fixed $r > 0$, χ is a strictly increasing function of θ. ∎

Lemma 3.3 For any $r > 0$ and $\theta \in \mathbb{R}$, let

$$\eta(r, \theta) \;\overset{\text{def}}{=}\; \zeta(r, \theta) - \zeta(r, \frac{\pi}{4}) \cdot \frac{4}{5\pi}\,(\theta + \pi)\,, \tag{26}$$

where ζ is defined in (17). Then:

- for any fixed $r > 0$, $\zeta(r, \theta)$ is an increasing function of $\theta \in \mathbb{R}$; and

- for any fixed $r > 0$ and $\theta \in (-\pi, -\frac{\pi}{2})$, $\zeta(r, \theta) > 0$ and $\eta(r, \theta) < 0$.

Proof. The increasing and positive character of $\zeta(r, \theta)$ follows from Lemma 3.2. Fix any $\theta \in (-\pi, -\frac{\pi}{2})$. Then

$$\frac{\partial \eta}{\partial r} = -\alpha(r, \theta)\,(\theta + \pi)(4\theta - \pi)\,\left[\pi(3\theta + \pi) + 4r^2\right],$$

where

$$\alpha(r, \theta) \stackrel{\text{def}}{=} \frac{2r^2}{\pi}\frac{1}{(r^2 + \theta^2)(r^2 + \pi^2)(16r^2 + \pi^2)} > 0 .$$

Hence for any fixed $\theta \in (-\pi, -\frac{\pi}{2})$, $\frac{\partial \eta}{\partial r} < 0$ when

$$0 < r < r_0 \stackrel{\text{def}}{=} \frac{\sqrt{-\pi(3\theta + \pi)}}{2},$$

and $\frac{\partial \eta}{\partial r} > 0$ when $r > r_0$. Therefore we have

$$\eta(r, \theta) < \max\left(\lim_{r\to 0^+} \eta(r, \theta),\ \lim_{r\to +\infty} \eta(r, \theta)\right), \tag{27}$$

for any $r \in (0, \infty)$ and $\theta \in (-\pi, -\frac{\pi}{2})$.

Note that

$$\lim_{r\to 0^+} \chi(r, \theta) = \begin{cases} 0, & \text{if } \theta \le 0, \\ 2\theta, & \text{if } \theta > 0, \end{cases}$$

uniformly for bounded θ, so it follows by its definition that

$$\lim_{r\to 0^+} \zeta(r, \theta) = \max\{2\theta,\ 0\}, \tag{28}$$

uniformly for bounded θ. Therefore, for any fixed $\theta \in (-\pi, -\frac{\pi}{2})$,

$$\lim_{r\to 0^+} \eta(r,\theta) = -\frac{2}{5}\,(\theta + \pi) < 0 . \tag{29}$$

Since for any $\theta \in \mathbb{R}$,

$$\lim_{r\to +\infty} \zeta(r, \theta) = \theta - (-\pi) = \theta + \pi,$$

it follows that

$$\lim_{r\to +\infty} \eta(r, \theta) = (\theta + \pi) - \left(\frac{\pi}{4} + \pi\right) \cdot \frac{4}{5\pi}\,(\theta + \pi) = 0 . \tag{30}$$

Combining (27), (29) and (30), we get

$$\eta(r, \theta) < 0, \quad \forall r > 0, \quad \forall \theta \in \left(-\pi, -\frac{\pi}{2}\right),$$

and this establishes the lemma. ∎

Proposition 3.4 The function k_1 defined by (18) has the following properties:

1. $k_1(r,\theta)$ is a real-analytic function of $r \in (0,\infty)$ and $\theta \in \left(-\pi, \frac{\pi}{4}\right)$;

2. $0 < k_1(r,\theta) < 1$, for any $r \in (0,\infty)$ and $\theta \in \left(-\pi, \frac{\pi}{4}\right)$;

3. $\lim_{r\to 0^+} k_1(r,\theta) = \begin{cases} 0\,, & \text{if } -\pi < \theta \le 0\,, \\ \frac{4\theta}{\pi}\,, & \text{if } 0 < \theta < \frac{\pi}{4}\,, \end{cases}$

   (the limit is uniformly on θ);

4. $\cos\theta + k_1\left(r, \theta - \frac{3\pi}{2}\right)\sin\theta < 0$, for any $r \in (0,\infty)$ and $\theta \in \left(\frac{\pi}{2}, \frac{7\pi}{4}\right)$, i.e.,
   $\sin\theta - k_1(r,\theta)\cos\theta < 0$, for any $r \in (0,\infty)$ and $\theta \in \left(-\pi, \frac{\pi}{4}\right)$.

Proof. Analyticity follows from the fact that arctan is analytic, Item 2 follows from Lemma 3.3, and Item 3 follows from (28).

We only need to prove Item 4. We want for any $r \in (0,\infty)$ and $\theta \in (-\pi, \frac{\pi}{4})$,

$$\sin\theta - k_1(r,\theta)\cos\theta < 0\,. \tag{31}$$

Because of Item 2, (31) is automatically satisfied if $\theta \in [-\frac{\pi}{2}, 0]$. Hence we only need to consider (31) for $r > 0$ and

$$\theta \in \left(-\pi, -\frac{\pi}{2}\right) \bigcup \left(0, \frac{\pi}{4}\right).$$

Or, equivalently, we need to prove

$$\tan\theta - k_1(r,\theta) > 0, \quad \forall r \in (0,\infty),\ \forall \theta \in \left(-\pi, -\frac{\pi}{2}\right) \tag{32}$$

and

$$k_1(r,\theta) - \tan\theta > 0, \quad \forall r \in (0,\infty),\ \forall \theta \in \left(0, \frac{\pi}{4}\right)\,. \tag{33}$$

Now we assume that $\theta \in (-\pi, -\frac{\pi}{2})$. From Lemma 3.3, as

$$\eta(r,\theta) = \zeta(r,\theta) - \zeta\left(r, \frac{\pi}{4}\right) \cdot \frac{4}{5\pi}(\theta + \pi) < 0$$

and the fact that

$$\zeta\left(r, \frac{\pi}{4}\right) > 0\,,\ \forall r > 0\,,$$

we know

$$k_1(r,\theta) = \frac{\zeta(r,\theta)}{\zeta(r,\frac{\pi}{4})} < \frac{4}{5\pi}(\theta+\pi)\,. \tag{34}$$

Since

$$\begin{aligned} \frac{d}{d\theta}\left(\tan\theta - \frac{4}{5\pi}(\theta+\pi)\right) &= \frac{1}{\cos^2\theta} - \frac{4}{5\pi} \\ &> \frac{1}{\cos^2\theta} - 1 \ > \ 0\,, \end{aligned}$$

it follows that for any $\theta \in (-\pi, -\frac{\pi}{2})$,

$$\tan\theta - \frac{4}{5\pi}(\theta + \pi) > \lim_{\theta \to (-\pi)^+} \left(\tan\theta - \frac{4}{5\pi}(\theta + \pi)\right) = 0\,. \tag{35}$$

Combining (34) and (35), we get (32).

Now we prove (33). Let

$$\delta(r,\,\theta) \stackrel{\text{def}}{=} \zeta(r,\,\theta) - \zeta\left(r,\,\frac{\pi}{4}\right) \cdot \frac{4\theta}{\pi} \tag{36}$$

be defined for all $r > 0$ and $\theta \in (0,\,\frac{\pi}{4})$, where ζ is the function defined in (17). For any $\theta \in (0,\,\frac{\pi}{4})$ and $r > 0$, since

$$\frac{\partial \delta}{\partial r} \;=\; -\frac{2r^2}{\pi}\,(4\theta - \pi)\Upsilon(r,\,\theta) \;>\; 0\,,$$

where

$$\Upsilon(r,\theta) \;\stackrel{\text{def}}{=}\; \frac{15\pi\theta^2 + 4\theta r^2 + 4\pi^2\theta + 16\pi r^2 + \pi^3}{(r^2+\theta^2)(r^2+\pi^2)(16r^2+\pi^2)} \;>\; 0\,,$$

it follows that

$$\delta(r,\,\theta) \;>\; \lim_{r \to 0^+} \delta(r,\,\theta) \;=\; 0\,,$$

and hence

$$k_1(r,\theta) \;=\; \frac{\zeta(r,\theta)}{\zeta(r,\frac{\pi}{4})} \;>\; \frac{4\theta}{\pi} \tag{37}$$

for any $r > 0$ and $\theta \in (0,\,\frac{\pi}{4})$. Since $\tan\theta$ is a strictly convex function on $\theta \in [0,\,\frac{\pi}{2})$,

$$\tan\theta < \frac{4}{\pi}\theta\,,\;\; \forall\theta \in (0,\,\frac{\pi}{4})\,. \tag{38}$$

Combining the two inequalities in (37) and (38), we get (33). ∎

Proposition 3.5 The function k_2 defined by (19) satisfies the following properties:

1. $k_2(r,\,\theta)$ is a real analytic function of $r \in (0,\,\infty)$ and $\theta \in \left(-\pi,\,\frac{\pi}{2}\right)$;

2. $k_2(r,\,\theta) > 0$, for any $r \in (0,\,\infty)$ and $\theta \in \left(-\pi,\,\frac{\pi}{2}\right)$;

3. $\lim_{r \to 0^+} k_2(r,\theta) = \begin{cases} 0\,, & \text{if } -\pi < \theta \le 0\,, \\ \dfrac{3\pi\theta}{\pi - 2\theta}\,, & \text{if } 0 < \theta < \dfrac{\pi}{2}\,; \end{cases}$

4. $k_2(r,\,\theta) < \tan\theta$, for any $r > 0$ and $\theta \in \left(-\pi,\,-\frac{\pi}{2}\right)$; and $k_2(r,\,\theta) > \tan\theta$, for any $r > 0$ and $\theta \in \left(0,\,\frac{\pi}{2}\right)$.

Proof. Analyticity is clear, and Items 2 and 3 follow from Lemma 3.3 and (28) respectively; we now only prove Item 4.

Since for any fixed $\theta \in (-\pi, \dfrac{\pi}{2})$,

$$\frac{\partial k_2}{\partial r} = \frac{3r^2}{\pi - 2\theta} \frac{(\pi - \theta)(\pi + \theta)}{(r^2 + \theta^2)(r^2 + \pi^2)} > 0\,,$$

we have

$$k_2(r,\,\theta) \;<\; \lim_{r \to +\infty} k_2(r,\,\theta) \;=\; \frac{3\pi(\theta + \pi)}{2(\pi - 2\theta)}$$

and

$$k_2(r,\,\theta) \;>\; \lim_{r \to 0^+} k_2(r,\,\theta) \;=\; \begin{cases} 0\,, & \text{if } -\pi < \theta \le 0\,, \\ \dfrac{3\pi\theta}{\pi - 2\theta}\,, & \text{if } 0 < \theta < \dfrac{\pi}{2}\,. \end{cases}$$

Now in order to prove Item 4, it is enough to show

$$\frac{3\pi(\theta + \pi)}{2(\pi - 2\theta)} - \tan\theta \le 0\,,\ \forall \theta \in (-\pi, -\frac{\pi}{2}) \tag{39}$$

and

$$\frac{3\pi\theta}{\pi - 2\theta} - \tan\theta \ge 0\,,\ \forall \theta \in (0, \frac{\pi}{2})\,. \tag{40}$$

As

$$\sin\theta \;<\; \theta \;<\; \tan\theta\,,\ \forall \theta \in \left(0,\, \frac{\pi}{2}\right)\,,$$

in order to prove (40), it is enough to show that

$$\frac{3\pi\theta}{\pi - 2\theta} - \frac{\theta}{\cos\theta} > 0\,,\ \forall \theta \in (0, \frac{\pi}{2})\,.$$

or, equivalently,

$$\rho(\theta) \overset{\text{def}}{=} \cos\theta - \frac{\pi - 2\theta}{3\pi} \;<\; 0\,,\ \forall \theta \in \left(0,\, \frac{\pi}{2}\right)\,. \tag{41}$$

We have

$$\rho'(\theta) = -\sin\theta + \frac{2}{3\pi}\,,$$

it then follows that

$$\rho(\theta) \;>\; \min\left\{\rho(0),\ \rho\left(\frac{\pi}{2}\right)\right\} \;=\; 0\,,\ \forall \theta \in \left(0, \frac{\pi}{2}\right)\,.$$

This completes the proof of (41).

Now the proof of (39). It is enough to show that

$$\frac{3\pi\varphi}{6\pi - 4\varphi} - \tan\varphi \;\le\; 0\,,\ \forall \varphi \in \left(0,\, \frac{\pi}{2}\right)\,. \tag{42}$$

Note that for any $\varphi \in (0,\, \frac{\pi}{2})$,

$$\frac{3\pi\varphi}{6\pi - 4\varphi} \;=\; \frac{\varphi}{2} \cdot \frac{1}{1 - \dfrac{2\varphi}{3\pi}} \;<\; \frac{\varphi}{2} \cdot \frac{1}{1 - \dfrac{\pi}{3\pi}} \;=\; \frac{3\varphi}{4} \;<\; \varphi \;<\; \tan\varphi\,,$$

which proves (42). ∎

3.3 Examples

Let us now return to the examples introduced in the beginning of this section and get the actual stabilizing control laws from our universal formulas. The system of our first example, that in equation (12), was

$$\dot{x} = x^2 - u\,,\ \ u \in (0, \infty)\,,$$

and we still use the obvious clf $V(x) = \frac{1}{2}x^2$. Since $(a(x), b(x)) = (x^3, -x) = (r\cos\theta, r\sin\theta)$, the almost smooth stabilizing law $u = k(x)$ is given, for any $x \neq 0$, by (22) as

$$\begin{aligned} u &= \frac{3\pi}{4(2\pi - \theta)}\left[\chi\left(r,\, \theta - \frac{3}{2}\pi\right) - \chi(r,\, -\pi)\right] \\ &= \frac{3\pi}{4(2\pi - \theta)}\left[\theta - \frac{1}{2}\pi + \frac{2\theta - 3\pi}{\pi}\arctan\left(\frac{2\theta - 3\pi}{2r}\right) - 2\arctan\left(\frac{\pi}{r}\right)\right], \end{aligned} \tag{43}$$

where

$$r \;=\; \sqrt{a^2(x) + b^2(x)} \;=\; |x|\,\sqrt{x^2+1} \;>\; 0\,,$$

and

$$\theta = \begin{cases} 2\pi - \arctan\left(\dfrac{1}{x^2}\right), & \text{if } x > 0, \\ \pi - \arctan\left(\dfrac{1}{x^2}\right), & \text{if } x < 0. \end{cases}$$

The second system discussed, in equation (13), involved the system

$$\dot{x} = \frac{x^2}{1+x^2} - u\,, \quad u \in (0, 1)\,,$$

again with the clf $V(x) = \frac{1}{2}x^2$. As

$$(a(x), b(x)) = \left(\frac{x^3}{1+x^2},\, -x\right) = (r\cos\theta, r\sin\theta)\ ,$$

the almost smooth stabilizing law $u = k(x)$ is given, for any $x \neq 0$, by (20) as

$$\begin{aligned} u &= \frac{\chi(r,\, \theta - \frac{3}{2}\pi) - \chi(r,\, -\pi)}{\chi(r,\, \frac{\pi}{4}) - \chi(r,\, -\pi)} \\ &= \frac{\dfrac{5\pi}{4} + \dfrac{1}{2}\arctan\left(\dfrac{\pi}{4r}\right) - 2\arctan\left(\dfrac{\pi}{r}\right)}{\theta - \dfrac{1}{2}\pi + \dfrac{2\theta - 3\pi}{\pi}\arctan\left(\dfrac{2\theta - 3\pi}{2r}\right) - 2\arctan\left(\dfrac{\pi}{r}\right)}, \end{aligned} \tag{44}$$

where

$$r \;=\; \sqrt{a^2(x) + b^2(x)} \;=\; \frac{|x|}{1+x^2}\sqrt{2x^4 + 2x^2 + 1} \;>\; 0\,,$$

and

$$\theta = \begin{cases} 2\pi - \arctan\left(\dfrac{1+x^2}{x^2}\right), & \text{if } x > 0, \\ \pi - \arctan\left(\dfrac{1+x^2}{x^2}\right), & \text{if } x < 0. \end{cases}$$

4 Universal Formulas for Set Stabilization

In this section, we establish some results regarding the existence of universal formulas for stabilization with respect to compact invariant sets which do not consist of just an equilibrium point. The basic definitions are presented for arbitrary closed invariant sets, but results are only given in the compact case. (An Appendix introduces terminology and basic facts concerning uniform global asymptotic stability with respect to sets.)

Consider systems affine on controls evolving on $\mathbb{R}^n$:

$$\dot{x} = f(x) + G(x)u\,, \tag{45}$$

where all entries of the vector f and the $n \times m$ matrix G are smooth functions on $\mathbb{R}^n$, and $\mathcal{A} \subseteq \mathbb{R}^n$ is a nonempty closed set. As before, controls take values in some subset of $\mathbb{R}^m$, $u(t) \in \mathcal{U} \subseteq \mathbb{R}^m$. Again, we use the notations (5) and (6).

Definition 4.1 A *control Lyapunov function* for the system (45) with respect to $\mathcal{A}$ and controls in $\mathcal{U}$ is a smooth function $V : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ satisfying:

1. there exist two $\mathcal{K}_\infty$-functions α_1 and α_2 such that for any $\xi \in \mathbb{R}^n$,

$$\alpha_1(|\xi|_{\mathcal{A}}) \leq V(\xi) \leq \alpha_2(|\xi|_{\mathcal{A}})\,; \tag{46}$$

2. there exists a continuous, positive definite function α_3 such that

$$\inf_{u \in \mathcal{U}} \{a(\xi) + B(\xi)u\} \leq -\alpha_3(|\xi|_{\mathcal{A}})\,,\ \forall \xi \in \mathbb{R}^n \backslash \mathcal{A}\,. \tag{47}$$

The function V is said to satisfy the *small control property* if for any $\varepsilon > 0$, there is a $\delta > 0$ such that, if $\xi \notin \mathcal{A}$ satisfies $|\xi|_{\mathcal{A}} < \delta$, then there is some $u \in \mathcal{U}$ with $|u| < \varepsilon$ such that

$$a(\xi) + B(\xi)u \leq -\frac{1}{2}\alpha_3(|\xi|_{\mathcal{A}})\,. \qquad \square$$

First of all, we note that if the set $\mathcal{A}$ is not compact, then all of the universal formulas given earlier might fail, as the following example shows.

Example 4.2 Consider the following system on $\mathbb{R}^2$:

$$\begin{aligned} \dot{x}_1 &= -\frac{u}{1 + x_2^2}\,, \\ \dot{x}_2 &= 0\,, \end{aligned}$$

with $\mathcal{A} \overset{\text{def}}{=} \{(x_1, x_2) \mid x_1 = 0\}$, and controls taking values in $\mathbb{R}$. Let $V(x_1, x_2) = \frac{1}{2}x_1^2$; then it follows that

$$\inf_{u \in \mathbb{R}} \{a(x) + ub(x)\} \;=\; \inf_{u \in \mathbb{R}} \left\{ -\frac{x_1\, u}{1 + x_2^2} \right\} \;=\; -\infty\,, \quad \forall x_1 \neq 0\,.$$

Applying the formula (7)–(8), we get

$$u = -b(x) = \frac{x_1}{1 + x_2^2}\,.$$

So the closed-loop system is:

$$\dot{x}_1 = -\frac{x_1}{(1+x_2^2)^2}\,, \tag{48}$$

$$\dot{x}_2 = 0\,. \tag{49}$$

But this system is not UGAS with respect to $\mathcal{A}$: Otherwise, if it were UGAS with respect to $\mathcal{A}$, it would follow that for $x_1(0) = 1$ and any $x_{2,0} \stackrel{\text{def}}{=} x_2(0)$, there exists some $T > 0$, such that

$$|x_1(t)| < e^{-1}\,, \quad \text{whenever } t \geq T\,. \tag{50}$$

Solving the differential equation with $x_1(0) = 1$, we get

$$x_1(t) = \exp\left(-\frac{t}{(1+x_{2,0}^2)^2}\right)\,.$$

Choose $x_{2,0}$ large enough so that

$$\frac{T}{(1+x_{2,0}^2)^2} \leq 1\,.$$

Then it follows that

$$|x_1(T)| = \exp\left(-\frac{T}{(1+x_{2,0}^2)^2}\right) \geq e^{-1}\,,$$

contradicting (50). This shows that the system (48)–(49) is not UGAS with respect to $\mathcal{A}$, and hence the formula defined by (7)–(8) is not "universal" in the set case. □

Now let $\mathcal{A}$ be a *compact* subset of $\mathbb{R}^n$. Introduce the following notations:

- $\mathcal{U}_1 \stackrel{\text{def}}{=} \mathbb{R}^m$;
- $\mathcal{U}_2 \stackrel{\text{def}}{=} \mathcal{B}_m$, the open unit ball in $\mathbb{R}^m$;
- $\mathcal{U}_3 \stackrel{\text{def}}{=} \{s \mid s \in (0,\, 1)\} \subseteq \mathbb{R}$; and
- $\mathcal{U}_4 \stackrel{\text{def}}{=} \{s \mid s \in (0,\, \infty)\} \subseteq \mathbb{R}$.

Let k_1 be the function defined by (7)–(8), k_2 be the function defined by (7)–(9), k_3 be the function defined by (20), (18) and (16), and k_4 be the function defined by (22), (19) and (16).

Theorem 3 *Let i be any number in the set $\{1,\,2,\,3,\,4\}$. If V is a control Lyapunov function satisfying the small control property with respect to $\mathcal{A}$ and controls in $\mathcal{U}_i$, then the control law $u = k_i(x)$ is smooth on $\mathbb{R}^n \backslash \mathcal{A}$, continuous on $\mathbb{R}^n$, and it stabilizes the system (45) in the following sense: there exists a $\mathcal{KL}$-function β_i such that every trajectory of the closed-loop system $x(t)$ satisfies*

$$|x(t)|_{\mathcal{A}} \leq \beta_i(|x(0)|_{\mathcal{A}}\,,\, t)\,,$$

for all t. Moreover, if the right-hand side of the system is analytic in x and V is analytic, then k_i is analytic on $\mathbb{R}^n \backslash \mathcal{A}$.

The proof is virtually the same as in the case $\mathcal{A} = \{0\}$. We omit the details here.

5 Systems with Disturbances

In this section we study generalizations of our results to systems subject to disturbances. By this we mean systems which evolve on $\mathbb{R}^n$ and are described by equations of the following type:

$$\dot{x}(t) \;=\; f(x(t), d(t)) \;+\; G(x(t), d(t))\, u(t)\,. \tag{51}$$

The functions d, to be thought of as "disturbances" acting on the system, are measurable functions taking values in an arbitrary but fixed compact subset $\mathcal{D}$ of $\mathbb{R}^l$. Controls take values in some subset $\mathcal{U} \subseteq \mathbb{R}^m$. We assume that all entries of f and G are smooth functions (i.e., smooth on a neighborhood of $\mathbb{R}^n \times \mathcal{D} \subseteq \mathbb{R}^n \times \mathbb{R}^l$). Furthermore, $f(0, \lambda) = 0$ for all λ.

Suppose, for system (51), that there exists a smooth function

$$k:\; \mathbb{R}^n \times \mathcal{D} \to \mathcal{U}$$

such that the closed-loop system

$$\dot{x} = f(x,\, d) + G(x,\, d)k(x,\, d)$$

is robustly uniformly globally asymptotically stable, in the sense reviewed in the Appendix. Then, just as for systems not subject to disturbances, a converse Lyapunov theorem (specifically, one may use the one in [9] and reviewed in the Appendix) implies that there exists a smooth, proper and positive definite function $V(x)$ such that

$$\inf_{u\in\mathcal{U}} \{a(x,\, \lambda) + B(x,\, \lambda)u\} < 0, \quad \forall\; x \neq 0, \quad \forall\; \lambda \in \mathcal{D}, \tag{52}$$

where $a(x,\, \lambda) \;=\; DV(x)f(x, \lambda)$ and $B(x, \lambda) \;=\; DV(x)G(x, \lambda)$. This motivates the following definition.

Definition 5.1 A proper and positive definite smooth function $V(x)$ is said to be a *uniform control Lyapunov function* (uclf) with respect to the control value set $\mathcal{U}$ if (52) holds.

The function V is said to satisfy the *uniform small control property* (uscp) if for any $\varepsilon > 0$, there exists a δ such that if $x \neq 0$ satisfies $|x| < \delta$, then there is some $u \in \mathcal{U}$ with $|u| < \varepsilon$ such that $a(x,\, \lambda) + B(x, \lambda)u < 0$ for all λ. □

Note that Definition 2.1 still applies in the case when a and B are dependent of λ. Similar to Proposition 2.2, we have the following:

Proposition 5.2 If α is a universal stabilizing formula given by the uclf V for system (51) relative to $\mathcal{U} \subseteq \mathbb{R}^m$, then the feedback law defined by

$$k(x,\, \lambda) \;=\; \alpha(a(x,\, \lambda),\, B(x,\, \lambda))$$

is smooth on $\{(x,\, \lambda) :\; x \neq 0\}$ and robustly uniformly globally stabilizes system (51).

If in addition V satisfies the uscp, then k is continuous everywhere. Moreover, if the right-hand side of the system is analytic in $(x,\, \lambda)$ and V is analytic, then k is analytic on the set where $x \neq 0$. □

The proof of the result basically follows the same steps as the proof of Proposition 2.2. The only difference is that, when showing that V is a Lyapunov function for the closed-loop system

–see the appendix for the precise definition of this concept for systems with disturbances– one needs to notice that the fact that

$$a(x,\, \lambda) + \langle B(x,\, \lambda),\ \alpha(a(x, \lambda), B(x, \lambda))\rangle < 0\,,\ \forall x \neq 0$$

implies that

$$\sup_{\lambda \in \mathcal{D}} \{a(x,\, \lambda) + \langle B(x,\, \lambda),\ \alpha(a(x,\, \lambda), B(x,\, \lambda))\rangle\} < 0\,,\ \forall x \neq 0\,,$$

as $\mathcal{D}$ is compact. Then by Theorem 5, one concludes the robust uniform stability of the system.

Remark 5.3 Note that if would weaken the definition of control Lyapunov function to allow a dependency on λ, then the above universal formulas may fail to robustly stabilize the system. This is because the Lyapunov function of the closed-loop system would be dependent on λ, which does not guarantee robust stability. □

Let $\mathcal{U}_i$ and k_i ($i = 1, 2, 3, 4$) be defined as before. With the same proof as in the case of systems without disturbances, we conclude the following:

Theorem 4 *Let i be any number in the set $\{1,\, 2,\, 3,\, 4\}$. If V is a uniform control Lyapunov function satisfying the uniform small control property with respect to the control value set $\mathcal{U}_i$, then the control law $u = k_i(x, \lambda)$ is smooth on $\{(x, \lambda) : x \neq 0\}$, continuous everywhere, and it robustly stabilizes system (51). Furthermore, if the right-hand side of the system is analytic and V is analytic, then k_i is analytic on $\{(x, \lambda) : x \neq 0\}$.* ∎

A special but interesting class of systems is that consisting of systems for which there is no disturbance in the control channel, i.e., those described by equations of the following type:

$$\dot{x}(t) = f(x(t),\, d(t)) + G(x(t))u(t). \tag{53}$$

Observe that that for such systems, "B" in the formulas does not depend on λ. One can then replace $a(x, \lambda)$ by $\hat{a}(x)$, where

$$\hat{a}(x) = \max_{\lambda \in \mathcal{D}} \{a(x,\, \lambda)\}$$

in all the universal formulas. This is because under the compactness assumption of $\mathcal{D}$, (52) implies that

$$\sup_{u \in \mathcal{U}} \{\hat{a}(x) + B(x)u\} < 0, \quad \forall x \neq 0.$$

Further, using an approximation, one can assume that $\hat{a}(x)$ is smooth (just pick a slighly larger function so that the inequalities still hold). Hence, we conclude the following:

Proposition 5.4 Let i be any number in $\{1,\, 2,\, 3,\, 4\}$. Let V be a uniform control Lyapunov function for system (53) satisfying the uniform small control property with the control value set $\mathcal{U}_i$. Then the control feedback law $u = k_i(x)$ defined as before, but with a replaced by $\hat{a}$, is smooth on $\mathbb{R}^n \setminus \{0\}$, continuous everywhere, and robustly stabilizes the system. □

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A A Converse Lyapunov Theorem

We briefly review some needed definitions and a converse Lyapunov theorems for systems with disturbances and stability with respect to sets.

Recall that a function $\gamma : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is a *$\mathcal{K}$-function* if it is continuous, strictly increasing and $\gamma(0) = 0$; it is a *$\mathcal{K}_\infty$-function* if it is a $\mathcal{K}$-function and also $\gamma(s) \to \infty$ as $s \to \infty$; and it is *positive definite* if $\gamma(s) > 0$ for all $s > 0$, and $\gamma(0) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is a *$\mathcal{KL}$-function* if for each fixed $t \geq 0$ the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function and for each fixed $s \geq 0$ the function $\beta(s, \cdot)$ is decreasing to zero as $t \to \infty$.

Consider the following system evolving on $\mathbb{R}^n$:

$$\dot{x}(t) = f(x(t),\, d(t))\,, \tag{54}$$

The functions d, called "disturbances," are measurable and take values in a compact subset $\mathcal{D}$ of $\mathbb{R}^m$ for some m; $f : \mathbb{R}^n \times \mathcal{D} \to \mathbb{R}^n$ is assumed to be continuous and locally Lipschitz on x uniformly on d (that is, for each compact subset K of $\mathbb{R}^n$ there is some constant c so that $|f(x, \lambda) - f(z, \lambda)| \leq c\,|x - z|$ for all $x, z \in K$ and all $\lambda \in \mathcal{D}$). Let $\mathcal{M}_\mathcal{D}$ be the set of all measurable functions from $\mathbb{R}$ to $\mathcal{D}$. For each $d \in \mathcal{M}_\mathcal{D}$, let $x(t,\, \xi,\, d)$ be the trajectory starting from ξ with the disturbance d. (Sometimes we will need to consider disturbances d that are functions defined only on some interval $I \subseteq \mathbb{R}$. In those cases, by abuse of notation, $x(t,\, \xi,\, d)$ will still be used, but only times $t \in I$ will be considered.)

We say that a closed set $\mathcal{A}$ is an *invariant set* for (54) if for each $\xi \in \mathcal{A}$, it holds that

$$x(t,\, \xi,\, d) \in \mathcal{A}, \quad \forall t \geq 0, \quad \forall d \in \mathcal{M}_\mathcal{D}.$$

For each nonempty subset $\mathcal{A}$ of $\mathbb{R}^n$, and each $\xi \in \mathbb{R}^n$, we let

$$|\xi|_\mathcal{A} = d(\xi,\, \mathcal{A}) \stackrel{\text{def}}{=} \inf_{\eta \in \mathcal{A}} d(\xi,\, \eta).$$

Let $\mathcal{A}$ be a closed, invariant set for (54). We assume the mild technical condition $\sup_{\xi \in \mathbb{R}^n} \{|\xi|_\mathcal{A}\} = \infty$.

Definition A.1 System (54) is *(absolutely) uniformly globally asymptotically stable* (UGAS) if the following two properties hold:

1. *Uniform Stability.* There exists a $\mathcal{K}_\infty$-function $\delta(\cdot)$ such that for any $\varepsilon \geq 0$,
$$|x(t,\, \xi,\, d)|_\mathcal{A} \;\leq\; \varepsilon \;\text{ for all } d \in \mathcal{M}_\mathcal{D}, \;\text{ whenever }\; |\xi|_\mathcal{A} \leq \delta(\varepsilon) \text{ and } t \geq 0\,. \tag{55}$$
2. *Uniform Attraction.* For any $r, \varepsilon > 0$, there is a $T > 0$, such that for every $d \in \mathcal{M}_\mathcal{D}$,
$$|x(t,\, \xi,\, d)|_\mathcal{A} \;<\; \varepsilon \tag{56}$$
whenever $|\xi|_\mathcal{A} < r$ and $t \geq T$. □

Note that this definition also applies to systems without disturbances or the case when the invariant set is a single equilibrium.

It can be shown (see e.g. [9]) that the UGAS property can be equivalently characterized by an estimate of the form $|x(t, \xi, d)|_\mathcal{A} \leq \beta(|\xi|_\mathcal{A}\,, t)$ for some $\mathcal{KL}$-function β.

Definition A.2 A *smooth function* $V : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is a *Lyapunov function* for system (54) with respect to $\mathcal{A}$ if there exist two $\mathcal{K}_\infty$-functions α_1, α_2 as well as a $\mathcal{K}$-function α_3 such that

1. $\alpha_1(|\xi|_\mathcal{A}) \;\leq V(\xi) \;\leq\; \alpha_2(|\xi|_\mathcal{A})$ for all $\xi \in \mathbb{R}^n$;
2. $\dfrac{\partial V(\xi)}{\partial x} \cdot f(\xi, \lambda) \leq -\alpha_3(|\xi|_\mathcal{A})\,.$

□

The following is Theorem 2 in [9]:

Theorem 5 *Then system (54) is* UGAS *with respect to a compact, invariant set $\mathcal{A}$ if and only if it admits a smooth Lyapunov function.* ∎