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Comments on "Some results on pole-placement and reachability"

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Abstract: We present various comments on a question about systems over rings posed in a recent note by Sharma, proving that a ring R is pole assignable if and only if, for every reachable system (F, G), G contains a rank-one summand of the state space. We also provide a generalization to deal with dynamic feedback.

Keywords: Systems over rings, Feedback, Pole placement.

1. Introduction

In the nice paper [9], it is shown (Corollary 1) that for projective-free rings, every reachable system of dimension 2 is pole assignable iff every such system is such that the image of the 'G' matrix contains a unimodular. It is then asked whether this holds for systems of greater dimension. The statement of the open question is somewhat ambiguous, in that it is unclear whether one wants this to hold as stated above or for *particular* systems. In the discussion after the statement of Theorem 1, the above interpretation is the one used, but when posing the open problem, the statement for particular systems (as in Theorem 2) is probably intended. In any case, the answer to the question in the 'global' sense stated above is yes, and is included in [6]; a slightly weaker version, stated in terms of $F^{-1}G$, had been proved in [5]. (The stronger statement, for particular systems, is almost surely false.) We show here that the answer is still positive even if the ring is not projective free, as appear when considering families of systems parameterized by periodic functions (see [5]). Since we want to deal with arbitrary commutative rings, we must replace "contains a rank-one summand" for "contains a unimodular"; of course for projective-free rings the two are equivalent.

The result to be given has been probably known for the last year to many of those working in this area, since it is the natural step after the paper [5] (where it was actually stated without proof in a short remark), and uses ideas from [2,4,6]. Thus we do not wish to claim (much) originality in presenting it here. However, there are a few non-trivial steps involved in the generalization to arbitrary commutative rings.

The construction is of interest in itself, in that it can be used to develop results on 'dynamic' pole-assignment. This is a 'stabilized' version of the pole-assignment property, and corresponds to the inclusion of memory in the feedback loop (details for instance in [8]). We relate this extended notion to a generalization of the above property about summands.

A remark on our approach; to avoid having to deal with matrix presentation of homomorphisms between non-free modules, and to make the proofs much simpler, we take the 'coordinate-free' approach to definitions of systems used in [5]. Thus a system (see below) is given by a pair (f, B), where f is an endomorphism and B is a submodule; the usual 'B' (or 'G') matrix corresponds to any presentation of this submodule B. A feedback transformation corresponds then to specifying a new endomorphism g (the '$f+gk$' in the usual formalism) such that the image of $f-g$ is in the submodule B. Further, in order to avoid having to introduce characteristic polynomials for endomorphisms of non-free modules, we shall use a definition of pole assignment which is different from the usual one. The new property will refer to triangularization of f under the feedback group, with arbitrary eigenvalues along the diagonal. This is a priori stronger than pole shifting, but is in fact

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equivalent, because of the results provided here. (It clearly implies pole assignment, and it is shown to be equivalent to the 'GCS' property, which in turn is known to be implied by pole assignment.) For an introduction to systems over rings, see the exposition in [1].

2. PA rings

For the entire paper, R will be an arbitrary commutative ring. A *system* (over R) is a pair (f, B), where $f: M \to M$ is an endomorphism of a finitely generated projective module M of (constant) rank n, and B is a finitely generated submodule of M. The module M is the *state-space* of (f, B). The module M is *decomposable* if it splits into rank-1 summands. We shall denote the composition of linear maps f, g just by juxtaposition fg.

Let (f, B) and (g, C) be two systems, with state spaces M and N respectively. Then, (f, B) is *equivalent to* (g, C), denoted $(f, B) \sim (g, C)$, if there is an isomorphism $\tau: N \to M$ such that

$$(\tau^{-1}f\tau - g)(N) \subseteq \tau^{-1}(B) = C.$$

This is an equivalence relation, and in particular for any (f, B) and any g similar to f, (g, C) is equivalent to (f, B) for a suitable C. Fix now an arbitrary (f, B) for which M is decomposable. If $a_1, \ldots, a_n$ are in R, $\mathrm{Ut}(a_1, \ldots, a_n)$ denotes the set of all those endomorphisms of M which, in terms of some decomposition of M into rank one projectives, have a matrix which is upper triangular with $a_1, \ldots, a_n$ in the diagonal. (We are identifying $\mathrm{End}(I)$ with R, for I of rank 1.) The system (f, B) is *(arbitrarily) triangularizable* iff M is decomposable and for every $a_1, \ldots, a_n$ in R there are a $g: M \to M$ in $\mathrm{Ut}(a_1, \ldots, a_n)$ and a C such that $(f, B) \sim (g, C)$. It is well known that a triangularizable (hence, 'pole-assignable' in the usual sense) system is necessarily *reachable*, i.e. M is the smallest f-invariant submodule containing B [3,7]. A *pole assignable* (*PA*) ring R is one for which every reachable system is triangularizable.

As remarked earlier, the definition of PA ring is often given in terms of characteristic polynomials, but will be equivalent to the one given here. Also, it is known that PA rings are such that all finitely generated projectives of constant rank are decomposable [5]. Our definition of PA ring is consistent with the usage in [5]; for non-projective-free rings, however, this is at variance with the older terminology, (e.g. in [3]) as discussed in that reference.

The following situation will arise below. Assume that f, g, θ are endomorphisms of M, B is a submodule of M, and $\theta(M) \subseteq B$. Let $\tau := \theta + 1$. Since

$$(\tau g - g)(M) = (\theta g)(M) \subseteq B,$$

the following two properties are equivalent:

$$(f\tau - \tau g)(M) \subseteq B, \tag{2.1}$$

$$(f\tau - g)(M) \subseteq B. \tag{2.2}$$

When τ happens to be an automorphism such that $\tau B = B$, property (2.1) says precisely that τ establishes an equivalence $(f, B) \sim (g, B)$.

The main lemma is a generalization and simplification of the result in [4] (see also [6]), to cover arbitrary commutative rings and, more importantly, non-free projective modules. When the module M is a direct sum $M_1 \oplus M_2$, we sometimes identify M_1 and M_2 with submodules of M in the canonical way. When necessary to prevent confusion, we display explicitely the inclusion maps ι_1 and ι_2 of M_1 and M_2 respectively,and the corresponding projection maps π_1, π_2.

Lemma. *Assume that $M = M_1 \oplus M_2$, with $M_2 \subseteq B$. Let $\theta \in \mathrm{End}(M)$ be such that $\theta(M) \subseteq B$ and $\theta^2 = 0$. Then, for any $\rho \in \mathrm{End}(M_1)$ and any $\sigma \in \mathrm{End}(M_2)$ such that*

$$(f + f\theta - \rho)(M_1) \subseteq B, \tag{2.3}$$

there is a $g: M \to M$ such that $(f, B) \sim (g, B)$, $g \mid M_1 = \rho$, and the map induced by g on M/M_1 is σ.

Proof. Let $\tau := 1 + \theta$. Then, $\tau(1 - \theta) = 1 - \theta^2 = 1$, so in particular τ is an automorphism. Since $\theta(M) \subseteq B$, for any $b \in B$ it holds that $\tau(b) = \theta(b) + b$, which is in B, and that $b = \tau(1 - \theta)(b) \in \tau B$; thus $\tau B = B$. We let g be the unique mapping with $g \mid M_1 := \rho$ and $g \mid M_2 := \pi_1 f\tau + \sigma$. Since $(\pi_1 f\tau)(M) \subseteq M_1$, g indeed induces σ on M/M_1. To show that (f, B) is equivalent to (g, B) under τ, we need to establish property (2.2). On M_1, this holds by assumption (2.3), and for x in M_2 we have that

$$(f\tau - g)(x) = (f\tau - \pi_1 f\tau)(x) + \sigma(x) = (\pi_2 f\tau)(x) + \sigma(x),$$

and both terms are in M_2, and hence in B as described. □

Corollary 1. *Assume that $M = M_1 \oplus M_2$, with $M_2 \subseteq B$, M_2 decomposable. Suppose that the system*

$$(f_1, B_1) := (\pi_1 f\iota_1, \pi_1(f(M_2) + B))$$

(with state-space M_1) is triangularizable, where π_1, ι_1, are as above. Then (f, B) is also triangularizable.

Proof. Without loss, we assume that M_1 and M_2 (and hence M) have been decomposed into rank one summands. Pick any $a_1, \ldots, a_n$. Let $m :=$ rank of M_1. By assumption, there is an endomorphism

$$\rho' : M_1 \to M_1 \text{ in } \mathrm{Ut}(a_1, \ldots, a_m)$$

and a $C \subseteq M_1$ with $(f_1, B_1) \sim (\rho', C)$. Let τ be as in the definition of equivalence, and denote $\rho := \tau\rho'\tau^{-1}$. Finally, take any

$$\sigma : M_2 \to M_2 \text{ in } \mathrm{Ut}(a_{m+1}, \ldots, a_n).$$

The corollary will follow from the lemma if we show that, for some θ with $\theta(M) \subseteq B$ and $\theta^2 = 0$, equation (2.3) holds, since then the resulting g is similar to an element of $\mathrm{Ut}(a_1, \ldots, a_n)$. We know from the choice of ρ that $(f_1 - \rho)(M_1)$ is contained in $\pi_1(f(M_2) + B)$. Since $\pi_1 = 1 - \pi_2$, the latter is itself contained in $f(M_2) + B$. Consider now the external direct sum $M_2 \oplus B$ and the surjective mapping

$$M_2 \oplus B \to f(M_2) + B$$

given by $(m, b) \mapsto b - f(m)$. Since $f_1 - \rho$ maps M_1 into $f(M_2) + B$, by projectivity of M_1 there is a morphism

$$k = (k_1, k_2) : M_1 \to M_2 \oplus B$$

such that $\pi_1 f - \rho = k_2 - fk_1$. Let $\theta \in \mathrm{End}(M)$ be defined by $\theta := k_1$ on M_1 and $\theta := 0$ on M_2. Since $k : M_1 \to M_2$, it follows that $\theta^2 = 0$ and $\theta(M) \subseteq B$. Further,

$$(\pi_1 f + f\theta - \rho)(M_1) = k_2(M_1) \subseteq B,$$

so equation (2.3) is indeed satisfied. □

Remark. Assume that $M = M_1 \oplus M_2$, with $M_2 \subseteq B$, and that (f, B) is reachable. Then, the above system (f_1, B_1) is also reachable. This can be proved by generalizing the proof in [4], or directly as follows. We claim that, for any positive integer k, and any x in M,

$$f^k(x) = \left[(\pi_1 f\iota_1)^{k-1}(\pi_1 f)\right](x) + \sum_{i=0}^{k-2} \left[(\pi_1 f\iota_1)^i(\pi_1 f)\right](y_{i+1}) + y_0$$

for suitable y_i's in M_2. This is true for $k = 1$, because

$$f(x) = \pi_1(f(x)) + \pi_2(f(x)),$$

and the second term is in M_2. In general,

$$f^{k+1}(x) = f(f^k(x)) = (\pi_1 f)(f^k(x)) + \text{element of } M_2,$$

so the claim follows by induction. Note that if $x = b$ is in B then

$$(\pi_1 f)(b) = (\pi_1 f)(\pi_1(b)) + (\pi_1 f)(y),$$

for some y in M_2, and this is in

$$(\pi_1 f\iota_1)(\pi_1(B)) + \pi_1(f(M_2)).$$

Applying π_1 to both sides, we conclude that $\pi_1(f^k(B))$ is in the smallest f_1-invariant submodule of M_1 which contains $\pi_1(B)$ and $(\pi_1 f)(M_2)$. Since the sum of the $f^k(B)$ is M, the remark is established.

Following [2], we shall say that a finitely generated submodule B of a (finitely generated, of constant rank) projective M is *good* if there is some f in $\mathrm{End}(M)$ such that (f, B) is reachable. A *GCS ring* will be one for which every good submodule B of a module M contains a rank 1 summand of M.

Theorem 1. *The ring R is a ring PA ring if and only if it is a GCS ring.*

Proof. The necessity was proved in [5]. Assume now that R is GCS and that (f, B) is reachable. We prove the result by induction on the rank n of M. When $n = 1$, the result is trivial. Assume now that $n > 1$. Then M admits a decomposition $M = M_1 \oplus M_2$, with M_2 a rank 1 summand of M contained in B. The system (f_1, B_1) in the above corollary has rank $n-1$, and is reachable by the preceding remark. It follows by induction that (f_1, B_1) is triangularizable, so by the corollary (f, B) also is. □

A recent paper [10] establishes that a large class of rings of dimension 1 satisfy what is called there the 'BCS property', which trivially implies the GCS property, and are thus PA rings.

3. Dynamic feedback

Let (f, B) and (g, C) be two systems, with $f: M \to M$ and $g: N \to N$. Their *direct sum* is by definition the system

$$(f, B) \oplus (g, C) := (f \oplus g, B \oplus C)$$

supported by the projective $M \oplus N$, where

$$(f \oplus g)(x, y) = (f(x), g(y))$$

and $B \oplus C$ is interpreted as a submodule of $M \oplus N$. This sum is reachable if each component is. Direct sum gives an associative operation on (equivalence classes of) systems. It corresponds to the use of memory elements in the feedback loop, with N being the state-space of the dynamic regulator. (See for instance [8], or the more detailed exposition in [1], for a discussion of the use of dynamic feedback in the control of families of systems and other types of systems over rings.) Again fix a system (f, B) as above. There is an analogue of Corollary 1 here:

Corollary 2. *Assume that $M = M_1 \oplus M_2$, with $M_2 \subseteq M_2$ decomposable. With the notations in Corollary 1, suppose that there is a system (g, C) such that $(f_1, B_1) \oplus (g, C)$ is triangularizable. Then, $(f, B) \oplus (g, C)$ is also triangularizable.*

Proof. Consider the following system $(\tilde{f}, \tilde{B})$. We let

$$\tilde{M} := \tilde{M}_1 \oplus \tilde{M}_2,$$

where $\tilde{M}_1 = M_1 \oplus N$ and where $\tilde{M}_2 = M_2$ (it is better to use the tilde notation for M_2 in order to avoid confusion when this is seen as a submodule of $\tilde{M}$). Consider the isomorphism

$$\tau: M_1 \oplus M_2 \oplus N \to M_1 \oplus N \oplus M_2$$

corresponding to the reordering of coordinates. Define then

$$\tilde{f} := \tau(f \oplus g)\tau^{-1}, \qquad \tilde{B} := \tau(B \oplus C).$$

Thus, $(f, B) \oplus (g, C)$ is equivalent to $(\tilde{f}, \tilde{B})$ by definition, and it is enough to prove that the latter is triangularizable. We apply Corollary 1 to this system. Let

$$\tilde{f}_1 = \tilde{\pi}_1 \tilde{f} \tilde{\iota}_1, \qquad \tilde{B}_1 = \tilde{\pi}_1(\tilde{f}(\tilde{M}_2) + \tilde{B}).$$

A calculation shows that then

$$(\tilde{f}_1, \tilde{B}_1) = (f_1, B_1) \oplus (g, C).$$

Thus $(\tilde{f}_1, \tilde{B}_1)$ is triangularizable, and Corollary 1 applies. □

We shall say that (f, B) is *k-triangularizable*, where k is a nonnegative integer, iff there is some (g, C) such that its state space N has rank k and $(f, B) \oplus (g, C)$ is triangularizable. Thus 0-triangularizability is the same as the property we had earlier. It is not hard to establish that if the system is k-triangularizable for any single k, then it must be reachable (see [8]).

Although we do not use this, it is worth remarking that k-linearizability is equivalent to $(f, B) \oplus (0, N)$ being triangularizable for some N of rank k. Indeed, if (g, C) is as above, then $(f, B) \oplus (g, N)$ is also triangularizable, but this latter sum is equivalent to $(f, B) \oplus (0, N)$.

Let $\alpha: \mathcal{N} \to \mathcal{N}$ be given. We shall say that the ring R is a PA-α ring if the following property holds: if (f, B) is a reachable system with state space of rank n, then it is $\alpha(n)$-triangularizable. It is well known (again see for instance [8]) that *every ring* is a PA-n^2 ring. From a computational complexity point of view, this may be undesirable (memory requirements grow as the square of the dimension of the system). It would be better to find rings which are PA-α rings with some α such that, say, $\alpha = O(n)$. As a very preliminary step in that direction, we present the following approach, suggested by the material in the previous section.

The ring R is a *GCS-k ring* iff the following property holds: whenever B is a good submodule of a (finitely generated, of constant rank) projective M, then there exists a (finitely generated) projective N of rank k such that $B \oplus N$ contains a rank $k+1$ decomposable summand of $M \oplus N$. (So, GCS-0 is the same as *GCS*.)

Theorem 2. *If R is a GCS-k ring then it is a PA-$(n-1)k$ ring.*

Proof. As in Theorem 1, we proceed by induction on the rank n of the state space M of the given reachable system. We need to establish that (f, B) is $(n-1)k$-triangularizable. For $n = 1$, this is again trivial. Assume the result proved for all systems of rank less than n. Let (f', B') be any reachable system with state space M' of rank n. Since B' is good, there is a projective N of rank k and a decomposition.

$$M := M' \oplus N = M_1 \oplus M_2$$

such that M_2 is decomposable of rank $k+1$ contained in $B \oplus N$. Consider now the system

$$(f, B) := (f', B') \oplus (0, N).$$

This is reachable since both factors are. Let (f_1, B_1) be the system induced on M_1 as in the above corollary. Note that M_1 has rank $(n+k) - (k+1) = n-1$, and that (f_1, B_1) is reachable by the remark in the previous section. By induction, (f_1, B_1) is then $(n-2)k$-triangularizable. By Corollary 2, (f, B) is then also $(n-2)k-$ triangularizable. It follows that (f', B') is $(n-2)k + k = (n-1)k$-triangularizable, as desired. □

4. Final remarks

One should emphasize, in relation to the search in [9] for conditions for given systems to be triangularizable (or, not equivalently for single systems, 'pole assignable' in the usual sense) that the usual conditions in terms of summands of B are only 'first order', in the following sense. If (f, B) is a pole-assignable system in the sense of [3,9], etc., then it is also true — by an argument almost as that in [3] — that, *for each integer* $k < n$, there is a rank-k summand of M which is contained in the k-th reachability module

$$B + f(B) + \cdots + f^{k-1}(B).$$

(The usual condition is just the case $k = 1$ of this.) This is because one may assume without loss that f is invertible (standard argument) and then may assign the polynomial $z^k(z-1)^{n-k}$. We omit details since we wish to avoid introducing determinants in this short note.

Finally, we point out that the result in Section 3 of [9] follows from the well known 'Hautus reachability conditions' from linear systems theory. These can be applied for rings because reachability can be checked locally. Some details are given in [8].

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