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On characterizations of the input-to-state stability property

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Abstract

We show that the well-known Lyapunov sufficient condition for "input-to-state stability" (ISS) is also necessary, settling positively an open question raised by several authors during the past few years. Additional characterizations of the ISS property, including one in terms of nonlinear stability margins, are also provided.

Keywords: Nonlinear stability; Input/state stability; Lyapunov function techniques

1. Introduction

In practice, control systems are very often affected by noise, expressed for instance as perturbations on controls and errors on observations. Thus, it is desirable for a system not only to be stable, but also to display the so-called "input/state" stability (ISS) properties. Intuitively, this means that the behavior of the system should remain bounded when its inputs are bounded, and should tend to equilibrium when inputs tend to zero. These notions are closely related to the topic of stability under perturbations (total stability), studied in the classical dynamical systems literature.

In the later 1980s, one of the co-authors introduced a particular precise definition of *input/state stability*, and established a few basic results; see for instance [4–6]. These results then were applied in different areas, including observer design and new small-gain theorems; see for instance [1, 3, 7, 8].

One of the main observations used in [4] as well as in subsequent papers has been the fact that a system is input/state stable if it admits an "ISS-Lyapunov function". This motivates checking the ISS property by investigating the ISS-Lyapunov functions for the given system. In this work, we show that it is also necessary for a input/state stable system to admit an ISS-Lyapunov function. The proof is based on a reduction to a question about systems with disturbances and relies on a new converse Lyapunov theorem for such systems proved in [2].

In the process of proving the main result, a number of other natural characterizations became available, including one in terms of nonlinear stability margins. The main theorem of this paper will state the equivalence of all these new notions, which will probably be of interest in themselves.

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2. Input to state stability

In this section we introduce the property of input/state stability. Consider the following general nonlinear system:

$$\dot{x} = f(x, u). \tag{1}$$

Here $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable and satisfies $f(0,0) = 0$.

Controls or *inputs* are measurable locally essentially bounded functions $u: \mathbb{R}_{\geqslant 0} \to \mathbb{R}^m$. The set of all such functions, endowed with the (essential) supremum norm $\|u\| = \sup\{|u(t)|, t \geqslant 0\} \leqslant \infty$, is denoted by L_∞^m. (Everywhere, $|\cdot|$ denotes the usual Euclidean norm.) For each $\xi \in \mathbb{R}^n$ and each $u \in L_\infty^m$, we denote by $x(t, \xi, u)$ the trajectory of system (1) with initial state $x(0) = \xi$ and the input u. This is defined on some maximal interval $[0, T_{\xi,u})$, with $T_{\xi,u} \leqslant +\infty$. We recall that a function $\gamma: \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a *$\mathscr{K}$-function* if it is continuous, strictly increasing and $\gamma(0) = 0$; it is a *$\mathscr{K}_\infty$-function* if it is a $\mathscr{K}$-function and also $\gamma(s) \to \infty$ as $s \to \infty$ and it is a *positive-definite* function if $\gamma(s) > 0$ for all $s > 0$, and $\gamma(0) = 0$. A function $\beta: \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a *$\mathscr{KL}$-function* if for each fixed $t \geqslant 0$ the function $\beta(\cdot, t)$ is a $\mathscr{K}$-function, and for each fixed $s \geqslant 0$ it is decreasing to zero as $t \to \infty$.

Definition 2.1. System (1) is *(globally) input/state stable* if there exist a $\mathscr{KL}$-function $\beta: \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ and a $\mathscr{K}$-function γ such that, for each input $u \in L_\infty^m$ and each $\xi \in \mathbb{R}^n$, it holds that

$$|x(t, \xi, u)| \leqslant \beta(|\xi|, t) + \gamma(\|u\|) \tag{2}$$

for each $t \geqslant 0$.

Note that, by causality, the same definition would result if one would replace $\|u\|$ by $\|u_t\|$ in (2), where u_t is the truncation of u at t; i.e., $u_t(s) = u(s)$ if $s \leqslant t$, and $u(s) = 0$ if $s > t$.

The definition is intended as a nonlinear generalization of the bound $|x(t)| \leqslant |\xi| \mathrm{e}^{-\alpha t} + c\|u\|$ which holds for linear systems $\dot{x} = Ax + Bu$ when the matrix A is asymptotically stable. Using this definition, it was proved in [4, 6] that a system can be stabilized by a smooth feedback if and only if it is feedback equivalent to an ISS system. Thus, the definition appears to be natural, and the further characterizations given later confirm this fact.

Definition 2.2. A smooth function $V: \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ is called an *ISS-Lyapunov function* for system (1) if there exist $\mathscr{K}_\infty$-functions α_1, α_2, and $\mathscr{K}$-functions α_3 and χ, such that

$$\alpha_1(|\xi|) \leqslant V(\xi) \leqslant \alpha_2(|\xi|) \tag{3}$$

for any $\xi \in \mathbb{R}^n$ and

$$\nabla V(\xi) \cdot f(\xi, \mu) \leqslant -\alpha_3(|\xi|) \tag{4}$$

for any $\xi \in \mathbb{R}^n$ and any $\mu \in \mathbb{R}^m$ so that $|\xi| \geqslant \chi(|\mu|)$.

Observe that if V is an ISS-Lyapunov function for (1), then V is a Lyapunov function, in the usual sense, for the autonomous system $\dot{x} = f(x, 0)$ obtained when no controls are applied.

Note that the first inequality in Eq. (3) states that V is positive-definite (because $\alpha_1(r)$ is nonzero for $r \neq 0$) and proper, that is, "radially unbounded" (because $\alpha_1(r)$ increases to infinity). The second inequality in Eq. (3) is redundant, since the existence of a function α_2 is an immediate consequence of continuity of V, but it is useful to have the function α_2 explicitly for proofs, and when dealing with more general stability considerations (such as stability with respect to invariant sets, or time-varying systems). Also, without loss of generality, one can assume that $\alpha_3 \in \mathscr{K}_\infty$ (cf. [2, Remark 4.1]). Finally, observe that Eq. (4) states that the derivative $\dot{V}$ along trajectories is negative-definite for large enough x, given any control magnitude.

In [4] as well as in the literature since the publication of that paper, the fact that a system is ISS has always been established by showing that there is an ISS-Lyapunov function. The main result of this paper will be that the converse holds as well:

System (1) *is ISS if and only if it admits an ISS-Lyapunov function.*

There turns out to be also an interesting connection between the ISS and the robust stability.

By a (possibly time-varying) *feedback law* for system (1) we will mean any (at least measurable) function $k:\mathbb{R}_{\geqslant 0}\times\mathbb{R}^n\to\mathbb{R}^n$ for which the differential equation corresponding to using k as a feedback,

$$\dot{x}=f(x,k(t,x)), \tag{5}$$

is well posed; that is, for each initial state $x(0)$ there is an absolutely continuous solution, defined at least for small times, and any two such solutions coincide on their interval of existence. (For instance this will happen if $k(t,\xi)$ is continuous in ξ and measurable locally essentially bounded in t, and is locally Lipschitz in ξ uniformly with respect to t on finite intervals.)

Let ρ be any $\mathscr{K}_\infty$-function. A feedback law will be said to be *bounded by* ρ if for each $\xi\in\mathbb{R}^n$, $|k(t,\xi)|\leqslant\rho(|\xi|)$ holds for almost all (recall that k is assumed to be only measurable) $t\geqslant 0$.

We will say that system (1) is *robustly stable* if there exist a $\mathscr{K}_\infty$-function ρ (called a *stability margin*) and a $\mathscr{KL}$-function β such that, for every feedback law bounded by ρ, it holds that

$$|x(t)|\leqslant\beta(|x(0)|,t)\quad\forall t\geqslant 0 \tag{6}$$

for every solution of the corresponding system (5).

Remark 2.3. Note that the ISS is not a notion of local stability with respect to "small" perturbations. It is a global notion, and perturbations can be arbitrarily large (since the function ρ is in class $\mathscr{K}_\infty$). In some sense, this is analogous to exponential stability for linear systems, where a perturbation of the spectrum preserves global asymptotic stability.

The main result in this context will be:

System (1) *is ISS if and only if it is robustly stable.*

After introducing several other notions, we will prove a theorem which will imply the above two claims as well as the equivalence with several other control systems properties.

2.1. Restatements of the definition of ISS

In this section, we introduce various notions and establish their equivalence with input/state stability.

Remark 2.4. A smooth function V is an ISS-Lyapunov function for (1) if and only if there exist $\alpha_i\in\mathscr{K}_\infty$ $(1\leqslant i\leqslant 4)$ such that (3) holds, and

$$\nabla V(\xi)f(\xi,\mu)\leqslant-\alpha_3(|\xi|)+\alpha_4(|\mu|). \tag{7}$$

This provides a "dissipation" type of characterization for the ISS property. Clearly (7) implies (4). Assume now that (4) holds with some $\alpha_3\in\mathscr{K}_\infty$ and $\chi\in\mathscr{K}$. Let $\alpha_4(r)=\max\{0,\hat{\alpha}_4(r)\}$ where $\hat{\alpha}_4(r)=\max\{\nabla V(\xi)f(\xi,\mu)+\alpha_3(\chi(|\mu|)):\ |\mu|\leqslant r,\ |\xi|\leqslant\chi(r)\}$. Then α_4 is continuous and $\alpha_4(0)=0$, and one can assume that α_4 is a $\mathscr{K}_\infty$-function (majorize it by one if it is not). Note then that (7) holds because $\alpha_4(r)\geqslant\sup_{|\mu|=r}\nabla V(\xi)f(\xi,\mu)+\alpha_3(|\xi|)$ (consider the two separate cases $|\xi|\geqslant\chi(|\mu|)$ and $|\xi|\leqslant\chi(|\mu|)$).

Remark 2.5. Noticing that $\beta + \gamma \leqslant \max\{2\beta, 2\gamma\}$, one sees that an equivalent form of decay estimation for the ISS systems can be given as follows: system (1) is ISS if and only if there exist $\beta \in \mathscr{KL}$ and $\gamma \in \mathscr{K}$ such that

$$|x(t,\xi,u)| \leqslant \max\{\beta(|\xi|,t), \gamma(\|u\|)\}, \quad \forall \xi \in \mathbb{R}^n,\ \forall u,\ \forall t \geqslant 0. \tag{8}$$

The next lemma relates the ISS to the input/output stability property introduced in [4].

Lemma 2.6. *System* (1) *is ISS if and only if there exist* $\mathscr{KL}$*-functions* β_0, β_1 *and a* $\mathscr{K}$*-function* γ *such that, for any* $\xi \in \mathbb{R}^n$ *and any input* u, *it holds that*

$$|x(t,\xi,u)| \leqslant \beta_0(|\xi|,t) + \beta_1(\|u_T\|, t-T) + \gamma(\|u^T\|) \tag{9}$$

for any $0 \leqslant T \leqslant t$, *where* u^T *is defined by* $u^T = u - u_T$.

Proof. Clearly, (9) implies (2). We now assume that there exist a $\mathscr{KL}$-function β and a $\mathscr{K}$-function γ such that (2) holds. Then one has the following inequality:

$$\begin{aligned}|x(t,\xi,u)| &\leqslant \beta(|x(T,\xi,u)|, t-T) + \gamma(\|u^T\|)\\ &\leqslant \beta(\beta(|\xi|,T) + \gamma(\|u_T\|), t-T) + \gamma(\|u^T\|)\end{aligned}$$

for any $\xi \in \mathbb{R}^n$, any u and any $0 \leqslant T \leqslant t$. Since $\beta(r+s,\tau) \leqslant \beta(2r,\tau) + \beta(2s,\tau)$ holds for any $r,s,\tau \geqslant 0$, it follows that

$$|x(t,\xi,u)| \leqslant \beta(2\beta(|\xi|,T), t-T) + \beta(2\gamma(\|u_T\|), t-T) + \gamma(\|u^T\|). \tag{10}$$

Now define β_0 by letting $\beta_0(s,t) = \sup\{\beta(2\beta(s,\tau), t-\tau)\colon 0 \leqslant \tau \leqslant t\} + s/(1+t)$. Then $\beta_0(0,t) = 0$ for any $t \geqslant 0$; for each fixed t, $\beta_0(s,t)$ is continuous and strictly increasing in s. Thus, $\beta_0(\cdot,t) \in \mathscr{K}$ for each $t \geqslant 0$. It is not hard to see that for each fixed $s \geqslant 0$, $\beta_0(s,t)$ is decreasing in t. To show that $\beta_0(s,t) \to 0$ as $t \to \infty$ for each fixed s, just notice that

$$\beta(2\beta(s,T), t-T) \leqslant \max\{\beta(2\beta(s,t/2),0), \beta(2\beta(s,0),t/2)\}$$

holds for any $s \geqslant 0$, $0 \leqslant T \leqslant t$, and both of the functions in the parentheses are $\mathscr{KL}$-functions.

Let β_1 be defined by $\beta_1(s,t) = \beta(2\gamma(s),t)$. Then $\beta_1 \in \mathscr{KL}$, and it follows from (10) that (9) holds for any $\xi \in \mathbb{R}^n$, any u, and any $0 \leqslant T \leqslant t$. □

The ISS property can also be described without using class $\mathscr{KL}$-functions, in a manner analogous to the standard definition of global asymptotic stability for systems with no controls.

Lemma 2.7. *System* (1) *is ISS if and only if the following two properties hold*:

1. *For each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$|x(t,\xi,u)| \leqslant \varepsilon \quad \forall t \geqslant 0 \tag{11}$$

for all inputs u *and initial states* ξ *with* $|\xi| \leqslant \delta$ *and* $\|u\| \leqslant \delta$.

2. *There exists a* $\mathscr{K}$*-function* γ *such that, for any* $r, \varepsilon > 0$, *there is a* $T > 0$ *so that for every input* u:

$$|x(t,\xi,u)| \leqslant \varepsilon + \gamma(\|u\|), \tag{12}$$

whenever $|\xi| \leqslant r$ *and* $t \geqslant T$.

This will be proved below.

Remark 2.8. In Lemma 2.7, Property 1 can be replaced by

1′. There exist a $\mathscr{K}_\infty$-function $\delta(\cdot)$ and a $\mathscr{K}$-function $\tilde{\gamma}$ such that

$$|x(t,\xi,u)| \leqslant \varepsilon + \tilde{\gamma}(\|u\|), \quad \forall t \geqslant 0, \quad \forall u, \text{ whenever } |\xi| \leqslant \delta(\varepsilon) \text{ and } t \geqslant 0. \tag{13}$$

Clearly Property 1′ implies Property 1. Next we show that Property 1 implies Property 1′. Pick $r, s > 0$, and consider the trajectories $x(t, \xi, u)$ with $|\xi| \leqslant s$ and $\|u\| \leqslant r$. Applying Property 2 with $\varepsilon = 1$, one knows that there exists a $T > 0$ such that

$$|x(t, \xi, u)| \leqslant 1 + \gamma(r) \tag{14}$$

for all $t \geqslant T$. Note that Property 2 also implies that the system is forward complete, that is, $T_{\xi,u} = \infty$ for all $\xi \in \mathbb{R}^n$ and all u. By Proposition 5.1 in [2], there is an $L > 0$ such that $|x(t, \xi, u)| \leqslant L$ for all $0 \leqslant t \leqslant T$, all $|\xi| \leqslant s$ and all $\|u\| \leqslant r$. Combining this with (14), one concludes that

$$|x(t, \xi, u)| \leqslant C, \quad \forall t \geqslant 0, \quad |\xi| \leqslant s, \quad \|u\| \leqslant r, \tag{15}$$

where $C = 1 + L + \gamma(r)$. Let $\varphi(r) = \inf\{C \geqslant 0\colon |x(t, \xi, u)| \leqslant C, \ \forall t \geqslant 0, \ \forall |\xi| \leqslant r, \ \forall \|u\| \leqslant r\}$. Note that $\varphi(r) < \infty$ for each $r \geqslant 0$ because of (15). Clearly, φ is nondecreasing, and $|x(t, \xi, u)| \leqslant \varphi(|\xi|) + \varphi(\|u\|)$ for all ξ, all u, and all $t \geqslant 0$. Also, it follows from Property 1 that $\varphi(r) \to 0$ as $r \to 0$. Let $\tilde{\varphi}(r) = 1/r \int_r^{2r} \varphi(s)\,ds$, for $r \geqslant 0$, and let $\tilde{\varphi}(0) = 0$. Then $\tilde{\varphi}$ is continuous, and $\varphi(r) \leqslant \tilde{\varphi}(r)$. Now we define $\tilde{\varphi}(r) = r + \max_{0 \leqslant s \leqslant r} \tilde{\varphi}(s)$. Then $\tilde{\varphi}$ is of class $\mathscr{K}_\infty$, and $|x(t, \xi, u)| \leqslant \tilde{\varphi}(|\xi|) + \tilde{\varphi}(\|u\|)$ holds for all $t \geqslant 0$, all $\xi \in \mathbb{R}^n$ and all inputs u. Let $\delta(r) = \varphi^{-1}(r)$; then $\delta(\cdot)$ is a $\mathscr{K}_\infty$-function. Clearly, with such a choice of δ and $\tilde{\gamma} = \tilde{\varphi}$, it holds that $|x(t, \xi, u)| \leqslant \varepsilon + \tilde{\gamma}(\|u\|)$ for all $|\xi| \leqslant \delta(\varepsilon)$, all u, and all $t \geqslant 0$. □

Lemma 2.9. *Property 2 in Lemma 2.7 is equivalent to the following: There exist a $\mathscr{K}$-function γ and a family of mappings $\{T_r\}_{r>0}$ with the properties*

- *for each fixed $r > 0$, $T_r\colon \mathbb{R}_{>0} \xrightarrow{\text{onto}} \mathbb{R}_{>0}$ is continuous and is strictly decreasing, and in particular, $\lim_{s \to +\infty} T_r(s) = 0$;*
- *for each fixed $\varepsilon > 0$, $T_r(\varepsilon)$ is (strictly) increasing as r increases and $\lim_{r \to \infty} T_r(\varepsilon) = \infty$; such that, for each input u, $|x(t, \xi, u)| \leqslant \varepsilon + \gamma(\|u\|)$ whenever $|\xi| \leqslant r$ and $t \geqslant T_r(\varepsilon)$.*

Proof. Sufficiency is clear. Now we show the necessity part. This is very similar to the proof of Lemma 3.1 in [2], so only a sketch is given. Let γ be given as in Property 2 of the ISS definition. For each $r, \varepsilon > 0$, let $\tilde{T}_{r,\varepsilon} = \inf\{\tau\colon |x(t, \xi, u)| \leqslant \varepsilon + \gamma(\|u\|), \ \forall t \geqslant \tau, \ \forall |\xi| \leqslant r\}$. Note then that $\tilde{T}_{r,\varepsilon} < \infty$ for any $r, \varepsilon > 0$. Further, $\tilde{T}_{r_1,\varepsilon} \leqslant \tilde{T}_{r_2,\varepsilon}$ if $r_1 \leqslant r_2$, and $\tilde{T}_{r,\varepsilon_1} \geqslant \tilde{T}_{r,\varepsilon_2}$ if $\varepsilon_1 \leqslant \varepsilon_2$. Also, Property 1′ implies that, for every fixed r, $T_{r,s} \to 0$ as $s \to \infty$. Now, for each $r > 0$ and $s > 0$, let $\bar{T}_r(s) = (2/s)\int_{s/2}^{s} \tilde{T}_{r,\sigma}\,d\sigma$. Then, for each fixed $r > 0$, $\bar{T}_r$ is a continuous function, and $\bar{T}_r(s) \geqslant \tilde{T}_{r,s}$. Finally, for each $r > 0$ and $s > 0$, we let $T_r(s) = (r/s) + \sup_{\sigma \geqslant s} \bar{T}_r(\sigma)$. One can easily check that the family $\{T_r\}_{r>0}$ satisfies all conditions in the lemma. □

We now return to prove Lemma 2.7.

Proof of Lemma 2.7. The necessity part is clear. Assume now that conditions in Lemma 2.7 hold. Let δ be as in Property 1′, and without loss of generality, we assume that $\tilde{\gamma}$ in Property 1′ and γ in Property 2 are the same function, denoted by γ. Let $\varphi(\cdot)$ be the $\mathscr{K}_\infty$-function δ^{-1}. Then it holds that

$$|x(t, \xi, u)| \leqslant \varphi(|\xi|) + \gamma(\|u\|) \quad \forall t \geqslant 0. \tag{16}$$

Let $\{T_r\}_{r>0}$ be as in Lemma 2.9, and for each $r > 0$, let $\psi_r(s) = T_r^{-1}(s)$ for $s > 0$, and for $s = 0$, we also denote $\psi_r(0) = \infty$. Note then that ψ_r is continuous on $(0, \infty)$ and $\lim_{s \to 0} \psi_r(s) = \infty$ for each $r > 0$. Since $|\xi| \leqslant r, t \geqslant T_r(\varepsilon) \Rightarrow |x(t, \xi, u)| \leqslant \varepsilon + \gamma(\|u\|)$, and $t = T_r(\psi_r(t))$ for $t > 0$, it follows from the above, applied in particular for $t = T_r(\varepsilon)$ that for $t > 0$:

$$|x(t, \xi, u)| \leqslant \psi_r(t) + \gamma(\|u\|) \tag{17}$$

for any u and any $|\xi| \leqslant r$. This formula also holds for $t = 0$ by the definition $\psi_r(0) = \infty$.

Now for any $s \geqslant 0$ and $t \geqslant 0$, let $\bar{\psi}(s,t) = \min\{\inf_{r \geqslant s} \psi_r(t), \varphi(s)\}$. Then by (16) and (17), one has

$$|x(t,\xi,u)| \leqslant \bar{\psi}(|\xi|,t) + \gamma(\|u\|). \tag{18}$$

Pick any function $\tilde{\psi}: \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ with the following properties: (1) for any fixed $t \geqslant 0$, $\tilde{\psi}(\cdot,t)$ is continuous and strictly increasing; (2) for any fixed $s \geqslant 0$, $\tilde{\psi}(s,t)$ decreases to 0 as $t \to \infty$; (3) $\tilde{\psi}(s,t) \geqslant \bar{\psi}(s,t)$. It is an easy exercise to show that such a majorizing function $\tilde{\psi}$ for $\bar{\psi}(s,t)$ always exists; for details, we refer the reader to similar step in the proof of Proposition 2.5 in [2]. Finally, we let $\beta(s,t) = \sqrt{\varphi(s)}\sqrt{\tilde{\psi}(s,t)}$. Then β is of class $\mathcal{KL}$, and moreover,

$$\beta(s,t) \geqslant \min\{\varphi(s), \tilde{\psi}(s,t)\}, \quad \forall s \geqslant 0, \quad \forall t \geqslant 0. \tag{19}$$

Combining (18) and (19), one concludes that $|x(t,\xi,u)| \leqslant \beta(|\xi|,t) + \gamma(\|u\|)$ for all $\xi \in \mathbb{R}^n$, all $t \geqslant 0$ and all inputs u. □

2.2. Robustness with respect to feedback

Lemma 2.10. *If there exists an ISS-Lyapunov function for system* (1), *then the system is robustly stable.*

Proof. Assume that V is an ISS-Lyapunov function. Let α_i $(i = 1,2,3)$ and χ be as in Definition 2.2. Without loss of generality, we assume that $\chi \in \mathcal{K}_\infty$ (otherwise, replace $\chi(r)$ by $\chi(r) + r$). Let $\rho(\tau) = \chi^{-1}(r)$; then $\rho \in \mathcal{K}_\infty$ as well, and $DV(\xi)f(\xi,\mu) \leqslant -\alpha_3(|\xi|) \leqslant -\alpha_3 \circ \alpha_2^{-1}(V(\xi))$ whenever $\mu \leqslant \rho(|\xi|)$, which implies that $DV(\xi)f(\xi,k(\xi,t)) \leqslant -\hat{\alpha}_3(V(\xi))$ for any feedback k bounded by ρ and for almost all t, where $\hat{\alpha}_3 = \alpha_3 \circ \alpha_2^{-1}$. This implies that for any such feedback, and for every solution $x(t)$ of the corresponding system (5), it holds that $(\mathrm{d}/\mathrm{d}t)V(x(t)) \leqslant -\hat{\alpha}_3(V(x(t)))$ for almost all t. A simple comparison principle (see e.g. [2]) implies that there exists some $\mathcal{KL}$-function, $\tilde{\beta}$, which depends only on $\hat{\alpha}_3$ and not on the particular k being used, such that $V(x(t)) \leqslant \tilde{\beta}(V(x(0)),t)$, $\forall t \geqslant 0$, for every solution $x(t)$ of (5) and feedback bounded by ρ, from which it follows that there exists some $\mathcal{KL}$-function β such that (6) holds for every solution of (5) whenever k is a feedback of the form considered. □

Note that the proof of Lemma 2.10 provides a slightly stronger conclusion than stated, namely, robustness holds even with respect to feedback laws that do not result in unique solutions.

A particular case of the above setup is as follows. Fix any smooth function φ. Then, for any $d(t) \in \mathcal{M}_{\mathcal{D}}$ = the set of all measurable functions from $\mathbb{R}_{\geqslant 0}$ to $\mathcal{D} = [-1,1]^m$, the function $k(t,\xi) = d(t)\varphi(\xi)$ is an admissible feedback law. We view the system

$$\dot{x}(t) = f(x(t), d(t)\varphi(x(t))) = g(x(t), d(t)) \tag{20}$$

as a "system with disturbances $d(t)$".

For such systems, there is a natural definition of *uniform global asymptotic stability* (UGAS): this means the system must be *uniformly stable* (that is, for some $\mathcal{K}_\infty$-function $\delta(\cdot)$, and for each $\varepsilon \geqslant 0$, the estimate $|x(t,\xi,d)| \leqslant \varepsilon$ holds for all $d \in \mathcal{M}_{\mathcal{D}}$, $|\xi| \leqslant \delta(\varepsilon)$, and $t \geqslant 0$) and *uniformly attractive* (that is, for each $r, \varepsilon > 0$, there is a $T > 0$ such that, $|x(t,\xi,d)| < \varepsilon$ for every $d \in \mathcal{M}_{\mathcal{D}}$, $|\xi| < r$ and $t \geqslant T$). This definition, when restricted to the current context, is exactly the same as saying that an estimate like the one in Eq. (6) must hold, for every solution, when using any $d(t) \in \mathcal{M}_{\mathcal{D}}$, where β does not depend on the particular d.

We say that system (1) is *weakly robustly stable* if there exists a smooth function φ, satisfying $\psi(|\xi|) \leqslant \varphi(\xi)$ for some $\mathcal{K}_\infty$-function ψ (that is, φ is positive-definite and proper or "radially unbounded"), so that the corresponding system (20) is UGAS.

Note that for any $\mathcal{K}_\infty$-function ρ, there exist a smooth function $\varphi: \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ and a $\mathcal{K}_\infty$-function ψ such that $\psi(|\xi|) \leqslant \varphi(\xi) \leqslant \rho(|\xi|)$ for all $\xi \in \mathbb{R}^n$. [*Proof.* Let ρ be a $\mathcal{K}_\infty$-function. Let $\tilde{\rho}(r) = \rho(\sqrt{r})$. Note that $\tilde{\rho}$ is still of class $\mathcal{K}_\infty$. Pick any smooth $\mathcal{K}_\infty$-function σ with $\sigma(r) \leqslant \tilde{\rho}(r)$. Let $\varphi(\xi) = \sigma(|\xi|^2)$. Then φ is the desired function, and we may take $\psi(r) = \sigma(r^2)$.] Given ρ, pick any smooth positive-definite proper function φ in this manner. Now

$d(t)\varphi(\xi)$ is just a particular type of feedback bounded by ρ. Consequently, if there exists an ISS-Lyapunov function for system (1), then (1) is weakly robustly stable. We have then proved the following result.

Lemma 2.11. *If system* (1) *is robustly stable, then it is also weakly robustly stable.*

The following two lemmas provide the key connections between the ISS, robust stability, and existence of the ISS-Lyapunov functions.

Lemma 2.12. *If system* (1) *is ISS, then it is also weakly robustly stable.*

Proof. Assume that (1) is ISS. Then by Remark 2.5, there exist some $\mathcal{KL}$-function β and some $\mathcal{K}$-function γ such that (8) holds for any ξ, any u and all $t \geqslant 0$. Let $\alpha(r) = \beta(r, 0)$. Then α is a $\mathcal{K}$-function. Without loss of generality, one can always assume that $\alpha(r) > r$ for all $r > 0$ (otherwise, one can replace $\alpha(r)$ by $\max\{\alpha(r), \frac{3}{2}r\}$) and thus α is $\mathcal{K}_\infty$. Similarly, one can assume that γ is also $\mathcal{K}_\infty$. It follows that α^{-1} is $\mathcal{K}_\infty$ and $\alpha^{-1}(r) < r$ for all $r > 0$. Now let $\sigma(r)$ be a $\mathcal{K}_\infty$-function satisfying $\sigma(r) < \gamma^{-1}(\frac{1}{4}\alpha^{-1}(r))$ for all $r > 0$. For instance, one can simply let $\sigma(r)$ be $\frac{1}{2}\gamma^{-1}(\frac{1}{4}\alpha^{-1}(r))$. Arguing as in the proof of Lemma 2.11, there exist a smooth function φ and a $\mathcal{K}_\infty$-function ψ such that $\psi(|\xi|) \leqslant \varphi(\xi) \leqslant \sigma(|\xi|)$ for all ξ. Now for the fixed function φ, consider system (20). In what follows we show that with the φ chosen above, system (20) is UGAS. Let $x_\varphi(t, \xi, d)$ denote the solution of (20) with initial state ξ and disturbance d. To prove the desired conclusion, we first show that

$$\gamma(|d(t)\varphi(x_\varphi(t, \xi, d))|) \leqslant \tfrac{1}{2}|\xi| \quad \text{a.e.} \quad t \geqslant 0 \tag{21}$$

for any $\xi \in \mathbb{R}^n$ and any $d \in \mathcal{M}_\mathcal{D}$. For this it is enough to show, because of the monotonicity of γ, that

$$\gamma(\varphi(x_\varphi(t, \xi, d))) \leqslant \tfrac{1}{2}|\xi| \quad \forall t \geqslant 0. \tag{22}$$

Pick any $\xi \in \mathbb{R}^n \setminus \{0\}$ and $d \in \mathcal{M}_\mathcal{D}$, and use simply $x(t)$ to denote $x_\varphi(t, \xi, d)$. Notice then that $\gamma(\varphi(x(t))) \leqslant \frac{1}{4}|\xi|$ for all t small enough, since $\gamma(\varphi(x(0))) \leqslant \gamma(\sigma(|\xi|)) < \frac{1}{4}\alpha^{-1}(|\xi|) \leqslant \frac{1}{4}|\xi|$. Now let $t_1 = \inf\{t > 0\colon \gamma(\varphi(x(t))) > \frac{1}{2}|\xi|\}$. Assume that $t_1 < \infty$. Then (22) holds for all $t \in [0, t_1)$, from which it follows that $\gamma(|d(t)\varphi(x(t))|) \leqslant \frac{1}{2}\alpha(|\xi|)$, for almost all $t \in [0, t_1)$. By (8), one sees that $|x(t)| \leqslant \beta(|\xi|, 0) \leqslant \alpha(|\xi|)$ for all $0 \leqslant t \leqslant t_1$, which, in turn, implies that $\gamma(|\varphi(x(t_1))|) \leqslant \gamma(\sigma(|x(t_1)|)) \leqslant \frac{1}{4}\alpha^{-1}(|x(t_1)|) \leqslant \frac{1}{4}|\xi|$. This contradicts the definition of t_1 (by continuity, from the definition it must hold that $\gamma(|\varphi(x(t_1))|) \geqslant \frac{1}{2}|\xi|$. Thus, $t_1 = \infty$, and (22) is proved.

Claim. For each $r > 0$ there is some $T_r \geqslant 0$ so that

$$t \geqslant T_r,\ |\xi| \leqslant r \ \Rightarrow\ |x_\varphi(t, \xi, d)| \leqslant \tfrac{1}{2}r. \tag{23}$$

To establish this claim, note that, from (8) and (21), it follows that $|x_\varphi(t, \xi, d)| \leqslant \max\{\beta(|\xi|, t), \frac{1}{2}|\xi|\}$ for all ξ. On the other hand, since $\beta \in \mathcal{KL}$, for each $r > 0$ there exists $T_r > 0$ such that $\beta(r, t) < \frac{1}{2}r$ for all $t \geqslant T_r$. This T_r satisfies the requirements of the claim.

Now pick any $\varepsilon > 0$. Let k be a positive integer such that $2^{-k}r < \varepsilon$. Let $r_1 = r$ and $r_i = \frac{1}{2}r_{i-1}$ for $i \geqslant 2$, and let $\tau = T_{r_1} + T_{r_2} + \cdots + T_{r_k}$. Then for $t \geqslant \tau$, it holds that $|x_\varphi(t, \xi, d)| \leqslant r/2^k < \varepsilon$ for all $|\xi| \leqslant r$, all $d \in \mathcal{M}_\mathcal{D}$ and all $t \geqslant \tau$. This shows that the origin is a uniform attractor for system (20).

To show the uniform stability for the system, notice that (8) and (21) imply that $|x_\varphi(t, \xi, d)| \leqslant \beta(|\xi|, 0)$ for all $t \geqslant 0$, all $\xi \in \mathbb{R}^n$, and all $d \in \mathcal{M}_\mathcal{D}$. We conclude that system (20) is UGAS. □

Lemma 2.13. *If system* (1) *is weakly robustly stable, then there exists an ISS-Lyapunov function for the system.*

Proof. This is an easy consequence of the converse Lyapunov theorem for systems with bounded disturbances proved in [2]. The details are as follows. Assume that system (1) is weakly robustly stable. Then there exists a smooth function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$, such that $\varphi(\xi) \geqslant \psi(|\xi|)$ for some $\mathcal{K}_\infty$-function ψ, for which system (20) is UGAS, as defined earlier. It then follows from the converse Lyapunov result in [2] that there exists a uniform smooth Lyapunov function V for system (20), that is, a smooth V so that, for some positive-definite $\mathcal{K}_\infty$-functions $\alpha_i, i = 1,2,3$, it holds that $\alpha_1(|\xi|) \leqslant V(\xi) \leqslant \alpha_2(|\xi|)$ $\forall \xi \in \mathbb{R}^n$ and

$$\nabla V(\xi)\cdot f(\xi, \boldsymbol{d}\varphi(\xi)) \leqslant -\alpha_3(|\xi|) \quad \forall \xi \in \mathbb{R}^n, \quad \forall |\boldsymbol{d}| \leqslant 1. \tag{24}$$

Note that from (24), it follows that

$$\nabla V(\xi)\cdot f(\xi, v) \leqslant -\alpha_3(|\xi|), \tag{25}$$

whenever $|v| \leqslant \varphi(\xi)$. Since $\psi(|\xi|) \leqslant \varphi(\xi)$, (25) holds whenever $|v| \leqslant \psi(|\xi|)$. Let $\chi(r) = \psi^{-1}(r)$. Clearly, χ is a function as required for Definition 2.2, and thus, V is an ISS-Lyapunov function for (1). □

The next lemma provides a Lyapunov-like criterion for input/state stability. It was introduced and proved in [4]. To make this work self-contained, we provide the statement together with its proof in this paper.

Lemma 2.14. *If system* (1) *admits an ISS-Lyapunov function, then it is ISS.*

Proof. Assume system (1) admits an ISS-Lyapunov function V. Let α_i $(i = 1,2,3)$ and χ be as in Definition 2.2. Fix a point $\xi \in \mathbb{R}^n$ and an input function u. Let $x(t)$ denote the corresponding trajectory $x(t, \xi, u)$. Consider the set: $S = \{\eta\colon V(\eta) \leqslant c\}$, where $c = \alpha_2 \circ \chi(\|u\|)$.

Claim: If there exists $t_0 \geqslant 0$ such that $x(t_0) \in S$, then $x(t) \in S$ for all $t \geqslant t_0$.

Proof. Assume that this is not true. Then there exist some $t > t_0$ and some $\varepsilon > 0$ such that $V(x(t)) > c + \varepsilon$. Let $\tau = \inf\{t \geqslant t_0\colon V(x(t)) \geqslant c + \varepsilon\}$. It then follows that $|x(\tau)| \geqslant \chi(\|u\|)$, which in turn, implies that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=\tau} V(x(t)) \leqslant -\alpha_3(|x(\tau)|) < 0.$$

Thus, $V(x(t)) \geqslant V(x(\tau))$ for some t in (t_0, τ). This contradicts the minimality of τ, and hence, $x(t) \in S$ for all $t \geqslant t_0$, as claimed. □

Proof of Lemma 2.14 (*Continued*). Now let $t_1 = \inf\{t \geqslant 0\colon x(t) \in S\} \leqslant \infty$. Then it follows from the above argument that $V(x(t)) \leqslant \alpha_2 \circ \chi(\|u\|)$ for all $t \geqslant t_1$. This implies that

$$|x(t)| \leqslant \gamma(\|u\|) \quad \forall t \geqslant t_1, \tag{26}$$

where $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \chi$. For $t < t_1, x(t) \notin S$, which implies that $|x(t)| \geqslant \chi(\|u\|)$ for all $t \leqslant t_1$. Consequently, one has

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) = DV(\eta) f(\eta, u)\Big|_{\eta = x(t)} \leqslant -\alpha_3(|x(t)|) \leqslant -\alpha_3 \circ \alpha_2^{-1}(V(x(t))), \text{ a.e.} \quad t \leqslant t_1.$$

First of all, this inequality guarantees that $x(t)$ is defined for all $t \geqslant 0$. Secondly, a standard comparison principle [2] implies that there exists some $\mathcal{KL}$-function $\tilde{\beta}$ which only depends on α_2 and α_3, such that $V(x(t)) \leqslant \tilde{\beta}(V(\xi), t)$ for $t \leqslant t_1$, from which it follows that

$$|x(t)| \leqslant \beta(|\xi|, t) \quad \forall t \leqslant t_1, \tag{27}$$

where $\beta(r, t) = \alpha_1^{-1} \tilde{\beta}(\alpha_2(r), t)$. Combining (26) and (27), one concludes that $|x(t)| \leqslant \beta(|\xi|, t) + \gamma(\|u\|)$ for all $t \geqslant 0$. Noting that in the above proof, the functions β and γ do not depend on each individual initial state ξ or the input u, one concludes that the system is ISS. □

Summarizing all the above, we have our main result.

Theorem 1. *The following properties are equivalent for any system:*

1. *It is ISS.*
2. *It admits an ISS-Lyapunov function.*
3. *It is robustly stable.*
4. *There exist $\mathcal{KL}$-functions β_0, β_1 and a $\mathcal{K}$-function γ so that* (9) *holds.*
5. *There exists a $\mathcal{K}$-function γ, such that for any $\varepsilon > 0$,* (11) *and* (12) *hold for properly chosen δ and T.*
6. *It is weakly robustly stable.*

Proof. We have: $1 \Leftrightarrow 4$ (see Lemma 2.6); $1 \Leftrightarrow 5$ (see Lemma 2.7); $1 \Rightarrow 6$ (see Lemma 2.12); $6 \Rightarrow 2$ (see Lemma 2.13); $2 \Rightarrow 1$ (see Lemma 2.14); $2 \Rightarrow 3$ (see Lemma 2.10); $3 \Rightarrow 6$ (see Lemma 2.11). □

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