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Input-to-state stability with respect to inputs and their derivatives

David Angeli[1,*,†], Eduardo D. Sontag[2,‡] and Yuan Wang[3,§]

[1] *Dip. Sistemi e Informatica, Università di Firenze, Via di Santa Marta 3, 50139 Firenze, Italy*
[2] *Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA*
[3] *Department of Mathematics, Florida Atlantic University, Boca Raton, FL 33431, USA*

SUMMARY

A new notion of input-to-state stability involving infinity norms of input derivatives up to a finite order k is introduced and characterized. An example shows that this notion of stability is indeed weaker than the usual ISS. Applications to the study of global asymptotic stability of cascaded non-linear systems are discussed. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: input-to-state stability; Lyapunov methods; dissipative systems; cascaded systems

1. INTRODUCTION

A central question in control theory is how to formulate, for general non-linear systems, notions of robustness and stability with respect to exogenous input disturbances. The linear case is by now very well understood, and, at least in a finite-dimensional set-up, most 'reasonable' definitions of 'input-to-state' or 'input–output' stability (provided in this last case that additional reachability and observability assumptions are met) boil down to local asymptotic stability, viz. to the classical condition on the systems poles lying in the complex open left half-plane. However, for non-linear systems the range of possibilities is much broader, and the goal of coming up with an effective classification for many different behaviours that might be labeled as 'stable' together with methods which would allow to establish relationships between such stability notions has attracted a substantial research effort within the past years. In this respect, input to state stability (ISS) and integral ISS, as well as $\mathcal{H}_\infty$ theory, have proven to be powerful tools, used successfully in order to tackle problems both of robustness analysis and control synthesis, [1–5].

*Correspondence to: David Angeli, Dipartimento di Sistemi e Informatica, Università di Firenze, Via di Santa Marta 3, 50139 Firenze, Italy.
† E-mail: angeli@dsi.unifi.it
‡ E-mail: sontag@control.rutgers.edu
§ E-mail: ywang@math.fau.edu

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In the ISS-related literature, a 'disturbance' is a locally essentially bounded measurable function. Such an extremely rich set of possible input perturbations is well suited to model Gaussian and random noises, as well as constant or periodic signals, slow parameters drifts, and so on. If, on one side, this makes the notion of ISS extremely powerful, on the other it is known that ISS might sometimes be too strong a requirement [6]. In the output regulation literature [7], instead, the focus is on 'deterministic' disturbances, i.e. signals that can be generated by a finite dimensional non-linear systems, (usually smooth), when the state is evolving in a neighbourhood of a neutrally stable equilibrium position. This is an extremely interesting class of *persistent* disturbances for which, roughly speaking, the following is true:

$$\|d\|_\infty \quad \text{small} \quad \Rightarrow \|\dot{d}\|_\infty \text{ and derivatives of arbitrary order are also small}$$

Under similar circumstances, for instance when cascading asymptotically stable systems, regarding the 'forcing' system's state as a disturbance typically yields

$$\limsup_{t\to+\infty} |d(t)| = 0 \quad \Rightarrow \quad \limsup_{t\to+\infty} |\dot{d}(t)| = 0$$

Nevertheless, the classical definition of input-to-state stability completely disregards such additional information. Tracking of output references, see Reference [8], is another area where 'derivative' knowledge is usually disregarded (the analysis is often performed only taking into account constant set-points), whereas such information could be exploited to get tighter estimates for the steady-state tracking error due to time-varying, smooth reference signals.

Analogous situations arise when parameters variations are taken into account (i.e. in adaptive control) and we expect the system to have nice and stable behaviour for slow parameters drifts. The study of systems with slowly varying parameters has long been an interesting topic in the literature, see e.g. References [9, 10]. The analysis of such a system is usually carried out by first considering the systems corresponding to 'frozen' parameters. If for all the frozen parameters, the corresponding frozen systems possess certain stability property uniformly, then it is reasonable to expect that the system with slowly varying parameters will possess the same property. See, for instance, Reference [10] for a result of this type. A more general question is how the magnitudes of the time derivatives of the time varying parameters affect the behaviour of the systems.

The main contribution of this note is to show how, in the context of ISS, stability notions can be adjusted in order to take into account robustness with respect to disturbances and their time derivatives. The new notion of D^kISS is defined through an ISS-like estimate which involves the magnitudes of the inputs and their derivatives up to the kth order. We also propose several properties related to the D^kISS notion. All these properties serve to formalize the idea of 'stable' dependence upon the inputs and their time derivatives. They differ in the formulation of the decay estimates which make precise how the magnitudes of derivatives affect the system. We illustrate by means of several interesting examples how these properties differ from each other and from the well known ISS property.

One of our main objectives is to provide equivalent Lyapunov characterizations for these properties. Interestingly enough, one of our Lyapunov formulations already appeared in Reference [9], (see formula (5) in that reference). In this work, we provide a stability property that is equivalent to the existence of this type of Lyapunov functions. As a key step in establishing the Lyapunov formulations, we show how the D^kISS property can be treated as a special case of the input–output-to-state stability property (for detailed discussions on this property, see Reference [11]).

A second objective is to discuss some applications of the newly introduced notions to the analysis of cascaded non-linear systems. The well known result that cascading preserves the ISS property is generalized to the D^kISS property.

The paper is organized as follows: Section 2 provides the basic definitions. Sections 3–5 contain the Lyapunov characterizations of the D^kISS property and some other related properties. Sections 6 and 7 are devoted to the study of cascaded systems. Sections 8 and 9 provide discussions on the relation between the newly introduced stability notions and the well established ISS property.

2. BASIC DEFINITIONS

Consider non-linear systems of the following form:

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ for each $t \geqslant 0$. The function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz continuous. Thus, for any measurable, locally essentially bounded function $u(t) : \mathbb{R} \to \mathbb{R}^m$, and any initial condition $\xi \in \mathbb{R}^n$, there exists a unique solution $x(t, \xi, u)$ of (1) satisfying the initial condition $x(0, \xi, u) = \xi$, defined on some maximal interval $(T^-_{\xi,u}, T^+_{\xi,u})$.

Recall that system (1) is input-to-state stable (ISS for short) if there exist $\gamma \in \mathcal{K}_\infty$* and $\beta \in \mathcal{KL}$ so that the following holds:

$$|x(t, \xi, u)| \leqslant \beta(|\xi|, t) + \gamma(\|u_{[0,t)}\|_\infty) \tag{2}$$

for all $t \geqslant 0$, all $\xi \in \mathbb{R}^n$, and all input signals u, where for any interval I, u_I denotes the restriction of u to I, and where $\|v\|$ denotes the usual L^m_∞-norm (possibly infinite) of v. Usually one can think of u as an exogenous disturbance entering the system. Note that if (2) holds for any trajectory on any interval where the trajectory is defined, then the system is automatically forward complete.

We denote by $W^{k,\infty}(J)$, for any integer $k \geqslant 1$ and any interval J, the space of all functions $u : J \to \mathbb{R}^m$ for which the $(k-1)$st derivative $u^{(k-1)}$ exists and is locally Lipschitz. For $k = 0$, we define $W^{0,\infty}(J)$ as the set of locally essentially bounded $u : J \to \mathbb{R}^m$. When $J = [0, +\infty)$, we omit J and write simply $W^{k,\infty}$. (Since absolutely continuous functions have essentially bounded derivatives if and only if they are Lipschitz, the definition of $W^{k,\infty}(J)$, for positive k, amounts to asking that the $(k-1)$st derivative $u^{(k-1)}$ exists and is absolutely continuous, and hence, its derivative, that is, $u^{(k)}$, is locally essentially bounded. Thus $W^{k,\infty}(J)$ is a standard Sobolev space, justifying our notation.)

Definition 2.1

System (1) is said to be kth derivative input-to-state stable (D^kISS) if there exist some $\mathcal{KL}$-function β, and some $\mathcal{K}$-functions $\gamma_0, \gamma_1, \ldots, \gamma_k$ such that, for every input $u \in W^{k,\infty}$, the

*A function $F : S \to \mathbb{R}$ is *positive definite* if $F(x) > 0$, $\forall x \in S$, $x \neq 0$ and $F(0) = 0$. A function $\gamma : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is of *class* $\mathcal{K}$ if it is continuous, positive definite, and strictly increasing. It is of *class* $\mathcal{K}_\infty$ if it is also unbounded. Finally, $\beta : R_{\geqslant 0} \times R_{\geqslant 0} \to R_{\geqslant 0}$ is of class $\mathcal{KL}$ if for each fixed $t \geqslant 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s > 0$, $\beta(s, t)$ decreases to 0 as $t \to \infty$. An important fact concerning $\mathcal{K}_\infty$ functions which will often be used in the following sections is the so-called 'weak triangular inequality' $\gamma(a + b) \leqslant \gamma(2a) + \gamma(2b)$ for all $a, b \geqslant 0$.

following holds:

$$|x(t,\xi,u)| \leqslant \beta(|\xi|,t) + \gamma_0(\|u\|) + \gamma_1(\|\dot{u}\|) + \cdots + \gamma_k(\|u^{(k)}\|) \tag{3}$$

for all $t \geqslant 0$.

As with ISS, we remark that if estimate (3) was instead only required to hold on the maximal interval of definition of the solution $x(t,\xi,u)$, then $|x(t,\xi,u)|$ is uniformly bounded on any subinterval of the maximal interval. Hence, the solution must be globally defined if $u \in W^{k,\infty}$, and the same definition results.

We say simply that the system is DISS when it is D^1ISS and, of course, ISS is the same as D^kISS for $k = 0$.

It is also clear that a system is D^kISS if and only if there exist some $\beta \in \mathcal{KL}$ and some $\gamma \in \mathcal{K}$ such that

$$|x(t,\xi,u)| \leqslant \beta(|\xi|,t) + \gamma(\|u\|^{[k]}) \tag{4}$$

for all $t \geqslant 0$, where $\|u\|^{[k]} = \max_{0 \leqslant i \leqslant k}\{\|u^{(i)}\|\}$.

Lemma 2.2
System (1) is D^kISS if and only if property (4) holds for all smooth input functions (with the same β, γ).

Proof
One implication is trivial. To prove the non-trivial implication, assume for some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, estimate (4) holds for all smooth input functions. By causality, one may replace $\|u\|^{[k]}$ by $\|u_{[0,t)}\|^{[k]}$ in (4).

Let $u \in W^{k,\infty}$. Fix $T > 0$ such that $x(t,\xi,u)$ is defined on $[0,T]$. Note that $u^{(k)}$ is essentially bounded on $[0,T]$. It is a routine approximation fact (reviewed in Corollary A.2 in the appendix) that there exists an equibounded sequence of $\mathcal{C}^\infty$ functions $\{u_j\}$ such that

- $u_j \to u$ pointwise on $[0,T]$; and
- $\limsup_{j\to\infty} \|(u_j)_{[0,T)}\|^{[k]} \leqslant \|u_{[0,T)}\|^{[k]}$.

Applying (4) to the trajectories with the input function u_j, and then taking the limits, we get

$$|x(t,\xi,u)| \leqslant \beta(|\xi|,0) + \gamma(\|u\|^{[k]}_{[0,T^+_{\xi,u})}) \tag{5}$$

Hence, $T^+_{\xi,u} = \infty$, that is, $x(t,\xi,u)$ is defined on $[0,\infty)$. Thus T can be picked arbitrarily, and (5) holds for all $t \geqslant 0$ where $\|u\|^{[k]}_{[0,T^+_{\xi,u})}$ becomes by $\|u\|^{[k]}$. □

3. A LYAPUNOV CHARACTERIZATION OF D^kISS

Fix $k \geqslant 1$. For system (1), consider the auxiliary system

$$\dot{x} = f(x, z_0), \quad \dot{z}_0 = z_1, \ldots, \dot{z}_{k-1} = v \tag{6}$$

Let

$$\hat{x}(t,\xi,\eta,v) := \begin{pmatrix} x(t,\xi,\eta,v) \\ z(t,\eta,v) \end{pmatrix}$$

denote the trajectory of (6) with the initial state $x(0) = \xi, z(0) = \eta$, (note that the z-component of the solution is independent of the choice of ξ.)

Observe that, if property (4) is known to hold for all inputs in $W^{k,\infty}$, then, for the trajectories $(x(t, \xi, z_0, v), z(t, \xi, z_0, v))$ of the auxiliary system, the following property holds:

$$|x(t, \xi, \eta, v)| \leqslant \beta(|\xi|, t) + \tilde{\gamma}_0(\|z\|_{[0,t]}) + \tilde{\gamma}_1(\|v\|) \tag{7}$$

for all measurable, locally essentially bounded inputs v. Given the fact that $|z(t)| \leqslant \|z\|_{[0,t)}$ is always true, we get

$$|\hat{x}(t, \xi, \eta, v)| \leqslant \beta(|(\xi, \eta)|, t) + \hat{\gamma}_0(\|z\|_{[0,t]}) + \hat{\gamma}_1(\|v\|)$$

for some $\hat{\gamma}_0,\ \hat{\gamma}_1 \in \mathcal{K}$. This shows that if (1) is D^kISS, then (6) is input–output-to-state stable, i.e., IOSS, with v as input and $z = (z_0, z_1, \ldots, z_{k-1})$ as outputs (cf. Reference [11]).

On the other hand, if the auxiliary system (6) is IOSS, then there exist some $\beta \in \mathcal{KL}$ and $\gamma_0, \gamma \in \mathcal{K}$ such that

$$|\hat{x}(t, \xi, \eta, v)| \leqslant \beta(|\xi| + |\eta|, t) + \gamma_0(\|z\|_{[0,t)}) + \gamma(\|v\|)$$

for all locally essentially bounded inputs v. Observe that

$$\beta(|\xi| + |\eta|, t) \leqslant \beta(2|\xi|, t) + \beta(2|\eta|, 0) \leqslant \beta(2|\xi|, t) + \beta(2\|z\|_{[0,t)}, 0)$$

It follows that

$$|\hat{x}(t, \xi, \eta, v)| \leqslant \bar{\beta}(|\xi|, t) + \bar{\gamma}_0(\|z\|_{[0,t)}) + \gamma(\|v\|)$$

holds for all locally essentially bounded v, where $\bar{\beta}(s, t) = \beta(2s, t)$, and $\bar{\gamma}_0(s) = \beta(2s, 0) + \gamma_0(s)$. In particular,

$$|x(t, \xi, \eta, v)| \leqslant \bar{\beta}(|\xi|, t) + \bar{\gamma}_0(\|z\|_{[0,t)}) + \gamma(\|v\|)$$

This implies that for any $u \in W^{k,\infty}$, the trajectory of system (1) with initial state ξ satisfies the estimate:

$$|x(t, \xi, u)| \leqslant \beta(|\xi|, t) + \gamma_1(\|u\|^{[k]})$$

where $\gamma_1(s) = \bar{\gamma}_0(s) + \gamma(s)$. We have therefore proved the following result that underlies the proofs of Theorems 1 and 2 to be given later.

Lemma 3.1

Let $k \geqslant 1$. System (1) is D^kISS if and only the associated auxiliary system (6) is IOSS with v as input and $z = (z_0, z_1, \ldots, z_{k-1})$ as output.

By the main result in Reference [11], System (6) is IOSS if and only if it admits an IOSS-Lyapunov function, that is, a smooth function $V : \mathbb{R}^n \times \mathbb{R}^{km} \to \mathbb{R}_{\geqslant 0}$ such that

- for some $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$, it holds that

$$\underline{\alpha}(|(x, z)|) \leqslant V(x, z) \leqslant \bar{\alpha}(|(x, z)|) \quad \forall (x, z)$$

- for some $\alpha, \rho \in \mathcal{K}_\infty$,

$$\frac{\partial V}{\partial x}(x,z) f(x,z) + \frac{\partial V}{\partial z_0}(x,z) z_1 + \cdots + \frac{\partial V}{\partial z_{k-1}}(x,z) v \leqslant -\alpha(V(x,z)) + \rho(|(z,v)|)$$

for all x, z and v.

Interpreting z as the input and its derivatives for system (1), we get the following:

Theorem 1

Let $k \geqslant 1$. System (1) is D^kISS if and only if there exists a smooth function $V : \mathbb{R}^n \times \mathbb{R}^{km} \to \mathbb{R}_{\geqslant 0}$ such that

- there exist some $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ such that for all $(x, \mu^{[k-1]}) \in \mathbb{R}^n \times \mathbb{R}^{km}$, it holds that

$$\underline{\alpha}(|(x, \mu^{[k-1]})|) \leqslant V(x, \mu^{[k-1]}) \leqslant \bar{\alpha}(|(x, \mu^{[k-1]})|) \tag{8}$$

- there exist some $\alpha \in \mathcal{K}_\infty, \rho \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{R}^n$ and all $\mu^{[k]} \in \mathbb{R}^{m(k+1)}$ with $\mu^{[k]} = (\mu_0, \mu_1, \ldots, \mu_k)$, it holds that

$$\begin{aligned} &\frac{\partial V}{\partial x}(x, \mu^{[k-1]}) f(x, \mu_0) + \frac{\partial V}{\partial \mu_0}(x, \mu^{[k-1]}) \mu_1 + \frac{\partial V}{\partial \mu_1}(x, \mu^{[k-1]}) \mu_2 \\ &+ \cdots + \frac{\partial V}{\partial \mu_{k-1}}(x, \mu^{[k-1]}) \mu_k \leqslant -\alpha(V(x, \mu^{[k-1]})) + \rho(|\mu^{[k]}|) \end{aligned} \tag{9}$$

Remark 3.2

Note that inequality (8) implies that

$$\underline{\alpha}(|x|) \leqslant V(x, \mu^{[k-1]}) \leqslant \bar{\alpha}(|(x, \mu^{[k-1]})|) \tag{10}$$

Suppose a system (1) admits a Lyapunov function V satisfying (9) and (10). Then it can be seen that, along any trajectory $x(t)$ with $u \in W^{k,\infty}$ as the input, (9) yields

$$\frac{d}{dt} V(x(t), u(t), \ldots, u^{(k-1)}(t)) \leqslant -\alpha(V(x(t), u(t), \ldots, u^{(k-1)}(t))) + \rho(\|u\|^{[k]})$$

for almost all $t \geqslant 0$. From this it follows that for some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, it holds that

$$V(x(t), u(t), \ldots, u^{(k-1)}(t)) \leqslant \beta(|V_0|, t) + \gamma(\|u\|^{[k]}) \quad \forall t \geqslant 0$$

where $V_0 = V(x(0), u(0), \ldots, u^{(k-1)}(0))$. Combining this with (10), one sees that system (1) is D^kISS. Hence, an equivalent Lyapunov characterization of D^kISS is the existence of a smooth function V satisfying (9) and (10) for some $\underline{\alpha}, \bar{\alpha}, \alpha \in \mathcal{K}_\infty$ and some $\rho \in \mathcal{K}$.

4. ASYMPTOTIC GAINS

Clearly, if a system is D^kISS, then it is forward complete (for $u \in W^{k,\infty}$) and for some $\gamma_0, \gamma_1, \ldots, \gamma_k \in \mathcal{K}$ it holds that

$$\limsup_{t \to \infty} |x(t, \xi, u)| \leqslant \gamma_0(\|u\|) + \gamma_1(\|\dot{u}\|) + \cdots + \gamma_k(\|u^{(k)}\|). \tag{11}$$

We say that a forward complete system satisfies the *asymptotic gain* (AG) property in $u, \ldots, u^{(k)}$ if, for some $\gamma_0, \ldots, \gamma_k \in \mathcal{K}$, (11) holds for all $\xi \in \mathbb{R}^n$ and all $u \in W^{k,\infty}$.

By Lemma 3.1, the system (1) is D^kISS if and only if the associated auxiliary system (6) is IOSS with v as input and $z = (z_0, z_1, \ldots, z_{k-1})$ as output. Applying the main result in Reference [12] about asymptotic gains for the IOSS property to the auxiliary system (6), one can prove the following:

Theorem 2
For a forward complete system as in (1), the following are equivalent:

1. it is D^kISS;
2. it satisfies the AG property in $u, \ldots, u^{(k)}$, and the corresponding zero-input system

$$\dot{x} = f(x, 0)$$

is (neutrally) stable.

5. RELATED NOTIONS

In this section, we consider two properties related to DISS. We focus specifically on DISS (rather than D^kISS) as it seems to be the most relevant in applications. As a matter of fact, the authors were not able to find any example of a D^2ISS system not being DISS and it is therefore an open question whether or not D^kISS ($k \geq 1$) is equivalent to DISS.

We say that system (1) is ISS in $\dot{u}$ if, for some $\beta \in \mathcal{KL}$ and some $\gamma \in \mathcal{K}$, the following estimate holds for all trajectories with inputs in $W^{1,\infty}$:

$$|x(t, \xi, u)| \leq \beta(|\xi|, t) + \gamma(\|\dot{u}\|) \quad \forall t \geq 0 \tag{12}$$

We say that system (1) is ISS *in constant inputs* if, for some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, the following estimate holds for all trajectories corresponding to constant inputs u:

$$|x(t, \xi, u)| \leq \beta(|\xi|, t) + \gamma(\|u\|) \quad \forall t \geq 0. \tag{13}$$

It is obvious that if a system is ISS in $\dot{u}$, then it is GAS uniformly in all constant inputs, that is, for some $\beta \in \mathcal{KL}$, the following holds for all trajectories with constant inputs:

$$|x(t, \xi, u)| \leq \beta(|\xi|, t) \quad \forall t \geq 0$$

Also note that (ISS in $\dot{u}$) $\Rightarrow$ (DISS). The converse is in general false. This can be seen through the following argument. Suppose DISS implies ISS in $\dot{u}$. Then we would have

$$(\text{ISS}) \Rightarrow (D\text{ISS}) \Rightarrow (\text{ISS in } \dot{u})$$

and hence, (ISS) $\Rightarrow$ (ISS in $\dot{u}$). But this is false, as one can see that the linear system $\dot{x} = -x + u$ is ISS but not ISS in $\dot{u}$.

It is also clear that, for any $k \geq 0$,

$$(D^k\text{ISS}) \Rightarrow (\text{ISS in constant } u)$$

Again, the converse implication is in general false as shown by examples in Section 9. Using similar arguments as in the proof of Lemma 3.1, we get the following:

- System (1) is ISS in $\dot{u}$ if and only if the auxiliary system

$$\dot{x} = f(x, z), \quad \dot{z} = v \tag{14}$$

is state-independent-input-to-output stable, i.e. SIIOS (see Reference [13]) with v as inputs and x as outputs; and

- System (1) is ISS in constant inputs if and only if the auxiliary system

$$\dot{x} = f(x, z), \quad \dot{z} = 0 \tag{15}$$

is output-to-state-stable, i.e. OSS (see Reference [11]) with z as outputs.

Applying Theorem 1.2 of Reference [14] in conjunction with Remark 4.1 in Reference [14] to the SIIOS property for system (14), we get the following:

Proposition 5.1
System (1) is ISS in $\dot{u}$ if and only if there exists a smooth Lyapunov function $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geqslant 0}$ satisfying the following:

- for some $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$,

$$\underline{\alpha}(|x|) \leqslant V(x, \mu_0) \leqslant \bar{\alpha}(|x|) \quad \forall\, x \in \mathbb{R}^n,\ \forall \mu_0 \in \mathbb{R}^m \tag{16}$$

- for some $\chi \in \mathcal{K}_\infty$ and some continuous, positive definite function α,

$$V(x, \mu_0) \geqslant \chi(|\mu_1|) \Rightarrow \frac{\partial V}{\partial x}(x, \mu_0) f(x, \mu_0) + \frac{\partial V}{\partial \mu_0}(x, \mu_0)\mu_1 \leqslant -\alpha(V(x, \mu_0)) \tag{17}$$

for all $x \in \mathbb{R}^n$ and all $\mu_0, \mu_1 \in \mathbb{R}^m$.

Observe that if one restricts the set where the input functions take values to be a *bounded* set $\mathcal{U}$ (as in the case of Reference [9]), then the Lyapunov characterization in Proposition 5.1 is equivalent to the existence of a smooth Lyapunov function V satisfying (16) for some $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ such that for some $\alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$,

$$\frac{\partial V}{\partial x}(x, \mu_0) f(x, \mu_0) + \frac{\partial V}{\partial \mu_0}(x, \mu_0)\mu_1 \leqslant -\alpha(V(x, \mu_0)) + \sigma(|\mu_1|)$$

for all $\xi \in \mathbb{R}^n$, all $\mu_0 \in \mathcal{U}$, and all $\mu_1 \in \mathbb{R}^m$. Such a Lyapunov estimate was used in [9] to analyze the asymptotic behaviour of systems with slowly varying parameters.

Applying Theorem 2 of Reference [11] to the OSS property for system (15), we have the following:

Proposition 5.2
System (1) is ISS with respect to constant inputs if and only if it admits a smooth Lyapunov function $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geqslant 0}$ such that

- for some $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$,

$$\underline{\alpha}(|(x, \mu)|) \leqslant V(x, \mu) \leqslant \bar{\alpha}(|V(x, \mu)|) \quad \forall x \in \mathbb{R}^n,\ \forall \mu \in \mathbb{R}^m \tag{18}$$

- for some $\alpha \in \mathcal{K}_\infty, \sigma \in \mathcal{K}$,

$$\frac{\partial V}{\partial x}(x, \mu) f(x, \mu) \leqslant -\alpha(V(x, \mu)) + \sigma(|\mu|) \tag{19}$$

for all $x \in \mathbb{R}^n, \mu \in \mathbb{R}^m$.

Remark 5.3
It may also be interesting to consider the DISS property with different indexes on different components of the inputs. For instance, for a system

$$\dot{x} = f(x, u, v) \tag{20}$$

with (u, v) as inputs, one may consider the property that for some $\beta \in \mathcal{KL}$, $\gamma_u \in \mathcal{K}$ and $\gamma_v \in \mathcal{K}$, it holds that

$$|x(t, \xi, u, v)| \leqslant \beta(|\xi|, t) + \gamma_u(\|u\|) + \gamma_v(\|v\|) + \gamma_v(\|\dot{v}\|) \tag{21}$$

One can also get a Lyapunov characterization for such a property by using the same argument as in the proof of Theorem 1 with the IOSS results. For instance, a system as in (20) satisfies property (21) if and only if there exists a smooth Lyapunov function V such that

- for some $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$,
$$\underline{\alpha}(|(\xi, v_0)|) \leqslant V(\xi, v_0) \leqslant \bar{\alpha}(|(\xi, v_0)|)$$
- for some $\alpha \in \mathcal{K}_\infty$, some $\rho_u, \rho_v \in \mathcal{K}$, it holds that
$$\frac{\partial V}{\partial x}(x, v_0) f(x, \mu_0, v_0) + \frac{\partial V}{\partial v_0}(x, v_0) v_1 \leqslant -\alpha(V(x, v_0)) + \rho_u(|\mu_0|) + \rho_v(|v_0|) + \rho_v(|v_1|)$$
for all x, μ_0, v_0 and v_1.

6. APPLICATION OF DISS TO THE ANALYSIS OF CASCADE SYSTEMS

An interesting feature of ISS, which makes it particularly useful in feedback design, is that the property is preserved under cascades, (see Reference [15]). Unfortunately, this is not the case for the weaker notion of integral ISS, as remarked in Reference [1] (but, see Reference [16] for related work). Interestingly, however, although DISS is also a weaker property than ISS, it *is* preserved under cascades, as shown in this section.

For a system

$$\dot{x} = f(x, v, u)$$

with (v, u) as inputs, we say that the system is D^kISS in v and D^lISS in u if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that the following holds along any trajectory $x(t, \xi, v, u)$ with initial state ξ, any input (v, u) for which $v \in W^{k,\infty}$ and $u \in W^{l,\infty}$:

$$|x(t, \xi, v, u)| \leqslant \beta(|\xi|, t) + \gamma(\|v\|^{[k]}) + \gamma(\|u\|^{[l]}) \quad \forall t \geqslant 0$$

Lemma 6.1
Consider a cascade system

$$\begin{aligned} \dot{x} &= f(x, z, u) \\ \dot{z} &= g(z, u) \end{aligned} \tag{22}$$

where $x(\cdot)$ and $z(\cdot)$ evolve on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively, the input u takes values in $\mathbb{R}^m$, and where f is locally Lipschitz and g is smooth. Let $k \geqslant 0$. Suppose that the z-subsystem is D^kISS with u as input, and that the x-subsystem D^{k+1}ISS in z and D^kISS in u. Then the cascade system (22) is D^kISS.

Proof
By assumption, there exist $\beta_z \in \mathcal{KL}$ and $\gamma_z \in \mathcal{K}$ such that, along any trajectory $z(t)$ of the z-subsystem with input u, it holds that

$$|z(t)| \leqslant \beta_z(|z(0)|, t) + \gamma_z(\|u\|^{[k]}) \quad \forall t \geqslant 0 \tag{23}$$

and there exist some $\beta_x \in \mathcal{KL}$ and $\gamma_x \in \mathcal{K}$ such that, for any trajectory $x(t, v, u)$ of the system $\dot{x} = f(x, v, u)$,

$$|x(t, \xi, v, u)| \leqslant \beta_x(|x(0)|, t) + \gamma_x(\|v\|^{[k+1]}) + \gamma_x(\|u\|^{[k]}) \quad \forall t \geqslant 0 \tag{24}$$

To prove Lemma 6.1, we need to find a suitable estimate for the x-component of solutions of (22). For this purpose, we define by induction for $1 \leqslant i \leqslant k+1$:

$$g_i(a, b_0, b_1, \ldots, b_{i-1}) = \frac{\partial g_{i-1}}{\partial a} g(a, b_0) + \sum_{j=0}^{i-2} \frac{\partial g_{i-1}}{\partial b_j} b_{j+1}$$

where $g_1(a, b_0) = g(a, b_0)$. It can be seen that $g_i(0, 0, \ldots, 0) = 0$ for all $0 \leqslant i \leqslant k$, hence, there exists some $\sigma_i \in \mathcal{K}$ such that

$$g_i(a, b_0, \ldots, b_{i-1}) \leqslant \sigma_i(|a|) + \sigma_i(|b^{[i-1]}|)$$

Again, by induction, one can show that, along any trajectory $z(t)$ of the z-subsystem of (22) with an input $u \in W^{k,\infty}$, it holds that

$$\frac{\mathrm{d}^i}{\mathrm{d}\, t^i} z(t) = g_i(\eta(t), d(t), \dot{d}(t), \ldots, d^{(i-1)}(t))$$

for all $1 \leqslant i \leqslant k+1$. It then follows that

$$\|z^{[k+1]}\| \leqslant \sigma(\|z\|) + \sigma(\|u^{[k]}\|) \tag{25}$$

for some $\sigma \in \mathcal{K}$. It then follows from (24) and (25) that, for some $\rho \in \mathcal{K}$, it holds that along any trajectory $(x(t), z(t))$ of (22),

$$|x(t)| \leqslant \beta_x(|x(0)|, t) + \rho(\|z\|) + \rho(\|u\|^{[k]}) \quad \forall t \geqslant 0 \tag{26}$$

Applying a standard argument to (26) and (23) as in the proof of the result that a cascade of ISS systems is again ISS, one can show that system (22) is D^kISS.

To be more precise, (26) implies that

$$|x(t)| \leqslant \beta_x\left(|x(t/2)|, \frac{t}{2}\right) + \rho(\|z\|_{[t/2,t)}) + \rho(\|u\|^{[k]}_{[t/2,t)}) \quad \forall t \geqslant 0 \tag{27}$$

along any trajectory of (22). Fix an input u and pick any trajectory $(x(t), z(t))$ of (22) with the input u. Let $x_1 = x(t/2)$. We also have

$$|x_1| \leqslant \beta_x\left(|\xi|, \frac{t}{2}\right) + \rho(\|z\|_{[0,t/2)}) + \rho(\|u\|^{[k]}_{[0,t/2)}) \quad \forall t \geqslant 0$$

Hence, there exist some $\tilde{\beta}_x$, $\tilde{\beta}_z$ and some $\tilde{\rho} \in \mathcal{K}$ (which depend only on β_x, ρ) such that

$$\beta_x\left(|x_1|, \frac{t}{2}\right) \leqslant \tilde{\beta}_x(|\xi|, t) + \tilde{\beta}_z(\|z\|_{[0,t/2)}, t) + \tilde{\rho}(\|u\|^{[k]}_{[0,t/2)}) \tag{28}$$

for all $t \geqslant 0$. By (23), $|z(\tau)| \leqslant \beta_z(|z(0)|, 0) + \gamma_z(\|u\|^{[k]})$ for all $\tau \geqslant 0$, hence,

$$\tilde{\beta}_z(\|z\|_{[0,t/2)}, t) \leqslant \bar{\beta}_z(|z(0)|, t) + \bar{\gamma}_z(\|u\|^{[k]}) \quad \forall t \geqslant 0 \tag{29}$$

for some $\bar{\beta}_z \in \mathcal{KL}$ and some $\bar{\gamma}_z \in \mathcal{K}$. With (29), one sees from (28) that for some $\hat{\beta} \in \mathcal{KL}$ and some $\hat{\gamma}_z \in \mathcal{K}$, it holds that

$$\beta_x\left(|x_1|, \frac{t}{2}\right) \leqslant \tilde{\beta}_x(|\xi|, t) + \hat{\beta}_z(|z(0)|, t) + \hat{\gamma}_u(\|u\|^{[k]}) \tag{30}$$

for all $t \geqslant 0$. Since

$$|z(\tau)| \leqslant \beta_z(|z(0)|, t/2) + \gamma_z(\|u\|^{[k]}) \quad \forall \tau \geqslant t/2$$

it follows that for some $\check{\beta}_z \in \mathcal{KL}$ and some $\check{\gamma}_z \in \mathcal{K}$, it holds that

$$\rho(\|z\|_{[t/2,t)}) \leqslant \check{\beta}_z(|z(0)|, t) + \check{\gamma}_z(\|u\|^{[k]}) \quad \forall t \geqslant 0 \tag{31}$$

Combining (26), (30) and (31), one sees that there exist some $\beta \in \mathcal{KL}$ and some $\gamma \in \mathcal{K}$ such that

$$|x(t)| \leqslant \beta(|x(0)| + |z(0)|, t) + \gamma(\|u\|^{[k]}) \quad \forall t \geqslant 0$$

Note that the choice of β and γ was made independent of the trajectory of (22). Together with (23) this shows that (22) is D^kISS. □

Remark 6.2
Observe from the above proof that to show that system (22) is D^kISS, the assumption that requires g be smooth can be relaxed to requiring that g be $\mathcal{C}^k$ if $k \geqslant 1$, or to requiring that g be locally Lipschitz in the case when $k = 0$.

Applying Lemma 6.1 to the special case of $k = 1$, one gets the following:

Corollary 6.3
Consider a cascade system as in Lemma 6.1, where f and g are $\mathcal{C}^1$ maps. Suppose that the x-subsystem is DISS with (z, u) as inputs that the z-subsystem is DISS with u as inputs, then the cascade system (22) is DISS with u as inputs.

Applying Lemma 6.1 to the following autonomous system:

$$\begin{aligned} \dot{x} &= f(x, z) \\ \dot{z} &= g(z) \end{aligned} \tag{32}$$

where f is locally Lipschitz, and g is smooth, one sees that the system is GAS provided that the z-subsystem is GAS and the x-subsystem is D^kISS with z as inputs for some $k \geqslant 0$.

It is by now a standard result that a system (32) is GAS if the x-subsystem is ISS and the z-subsystem is GAS. Now one sees that the ISS property of the x-subsystem can be relaxed to D^kISS. This result can be further improved by only requiring the D^kISS property hold for small signals produced by the z-subsystems.

For $\delta > 0$, we define the saturation function sat_δ by

$$\text{sat}_\delta(r) = \begin{cases} r & \text{if } |r| < \delta \\ \text{sign}(r)\delta & \text{otherwise} \end{cases} \tag{33}$$

For $z = (z_1, z_2, \ldots, z_m) \in \mathbb{R}^m$, we define $\text{sat}_\delta(z) := (\text{sat}_\delta(z_1), \text{sat}_\delta(z_2), \ldots, \text{sat}_\delta(z_m))$.

Proposition 6.4
A forward complete system as in (32) is GAS provided that for some $\delta > 0$ and for some $k \geqslant 0$, the system

$$\dot{x} = f(x, \mathrm{sat}_\delta(z)) \tag{34}$$

is D^kISS and that the z-subsystem is GAS.

Proof
The local asymptotic stability property of (32) follows directly from the local asymptotic stability property of the x and z subsystems. Thus we only need to show the global attraction property, in particular, the convergence property of $x(t)$ for any trajectory $(x(t), z(t))$ of (32).

First of all, the forward completeness assumption guarantees that $x(t, \xi, z)$ is defined on $[0, \infty)$ for any trajectory of the x-subsystem with initial state ξ and external signal z.

Pick any trajectory $(x(t), z(t))$ of (32). Since the z-subsystem is GAS, there is some $T > 0$ such that $|z(t)| \leqslant \delta$ for all $t \geqslant T$. Consequently, $(x(t), z(t))$ is also a trajectory of (34) with the z-subsystem for all $t \geqslant T$. Since system (34) cascaded with the z-subsystem is GAS, it follows that $x(t)$ converges to 0. □

7. AN ISS RELATED INTERPRETATION OF D^kISS

Definition 7.1
A *smoothly invertible* ISS *filter* is an ISS system

$$\dot{w} = g(w, d) \tag{35}$$

with $w(t), d(t) \in \mathbb{R}^m$, where $g : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is a smooth map for which there exists a smooth map $G : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ such that $G(v_0, g(v_0, v_1)) = v_1$ and $g(v_0, G(v_0, v_1)) = v_1$ for all $v_0,\ v_1$.

The main result in this section is the following characterization of D^kISS:

Theorem 3
Let k be a positive integer. The following facts are equivalent:

1. System (1) is D^kISS.
2. There exists a smoothly invertible ISS filter

$$\dot{\eta} = g(\eta, \mu) \tag{36}$$

   such that the system

$$\dot{x} = f(x, \eta)$$
$$\dot{\eta} = g(\eta, \mu) \tag{37}$$

   is D^{k-1}ISS.
3. For each smoothly invertible ISS filter as in (36) the cascade system (37) is D^{k-1}ISS.

Proof
The implication (iii) $\Rightarrow$ (ii) is obvious. Let us consider (ii) $\Rightarrow$ (i). Since system (37) is D^{k-1}ISS, there exist some $\beta \in \mathcal{KL}$ and some $\gamma \in \mathcal{K}$ such that along any trajectory $(x(t), \eta(t))$ of (37), it

holds that

$$\begin{aligned}|x(t)| &\leqslant \beta(|x(0)| + |\eta(0)|, t) + \gamma(\|d\|^{[k-1]}) \\ &\leqslant \beta(2|x(0)|, t) + \beta_0(2\|\eta\|) + \gamma(\|d\|^{[k-1]}) \end{aligned} \tag{38}$$

where $\beta_0(s) = \beta(s, 0)$. Let G be the smooth function as in Definition 7.1 for the function g in system (37).

Observe that any trajectory $x(t)$ of (1) with an input $\eta \in W^{k,\infty}$, $(x(t), \eta(t))$ is a trajectory of (37) with the input $d \in W^{k,\infty}$ defined by $d(t) = G(\eta(t), \dot{\eta}(t))$.

Notice that $g(0,0) = 0$ implies $G(0,0) = 0$. Thus, by continuity of G, there exists $\gamma_0 \in \mathcal{K}_\infty$ such that $G(a,b) \leqslant \gamma_0(|a|) + \gamma_0(|b|)$. Take any trajectory $x(t)$ of (1) with an input $\eta \in W^{k,\infty}$, and let $d(t) = G(\eta(t), \dot{\eta}(t))$. Then

$$\|d\| \leqslant \gamma_0(\|\eta\|) + \gamma_0(\|\dot{\eta}\|) \tag{39}$$

Hence, in the case when $k = 1$, that is, when system (37) is ISS, (38) combined with (39) implies that

$$|x(t)| \leqslant \beta(2|x(0)|, t) + \beta_0(2\|\eta\|) + \gamma_0(\|\eta\|) + \gamma_0(\|\dot{\eta}\|) \tag{40}$$

This shows that system (1) is DISS. To consider the more general case for $k \geqslant 2$, we consider inductively the following functions defined by

$$G_i(a_0, a_1, \ldots, a_i, a_{i+1}) = \sum_{j=0}^{i} \frac{\partial G_{i-1}}{\partial a_j}(a_0, a_1, \ldots, a_i) a_{j+1}$$

with $G_0(a_0, a_1) := G(a_0, a_1)$. Observe that for each i, $G_i(0, \ldots, 0) = 0$, and hence, there exists some $\gamma_i \in \mathcal{K}$ such that

$$|G_i(a_0, a_1, \ldots, a_{i+1})| \leqslant \gamma_i(|a^{[i+1]}|)$$

where $a^{[i]} = (a_0, a_1, \ldots, a_i)$.

By induction, one can show that for any trajectory $\eta(t)$ of the η-subsystem of (37) with the input $d \in W^{k-1,\infty}$, it holds that, for any $0 \leqslant i \leqslant k-1$,

$$d^{(i)}(t) = G_i(\eta(t), \dot{\eta}(t), \ldots, \eta^{(i+1)}(t))$$

Consequently, one has

$$\|d\|^{[k-1]} \leqslant \gamma_k(\|\eta\|^{[k]})$$

Thus, for any trajectory $x(t, \xi, \eta)$ of (1), it holds that, with $d = G(\eta, \dot{\eta})$,

$$\begin{aligned}|x(t,\xi,\eta)| &\leqslant \beta(2|\xi|, t) + \beta_0(2\|\eta\|) + \gamma(\|d\|^{[k-1]}) \\ &\leqslant \beta(2|\xi|, t) + \beta_0(2\|\eta\|) + \tilde{\gamma}_k(\|\eta\|^{[k]})\end{aligned}$$

where $\tilde{\gamma}_k = \gamma \circ \gamma_k$. Hence, system (1) is D^kISS.

To complete the proof of Theorem 3, it only remains to show the implication (i) ⇒ (iii). But this implication is an immediate consequence of Lemma 6.1. □

8. ARE DISS SYSTEMS ALWAYS ISS?

In this section we will discuss an example of a DISS system which is not ISS. This shows that DISS is indeed *strictly weaker* than ISS. First, however, we show that DISS and ISS are equivalent for scalar systems.

Proposition 8.1
A one-dimensional system in the form of (1) is ISS if and only if it is DISS.

Proof
Clearly, we only need to show one direction of the implication. Let a one-dimensional system (1) be DISS. Then, there exists a smooth function $V:\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}_{\geqslant 0}$ such that

$$\underline{\alpha}(|x|+|\mu_0|)\leqslant V(x,\mu_0)\leqslant\bar{\alpha}(|x|+|\mu_0|)\quad\forall(x,\mu_0)\in\mathbb{R}\times\mathbb{R}^m \tag{41}$$

for some $\underline{\alpha},\bar{\alpha}\in\mathscr{K}_\infty$, and

$$\frac{\partial V}{\partial x}(x,\mu_0)f(x,\mu_0)+\frac{\partial V}{\partial\mu_0}\mu_1\leqslant-\alpha(|x|)+\gamma(|\mu_0|+|\mu_1|),\quad\forall x,\ \forall(\mu_0,\mu_1) \tag{42}$$

for some α,γ of class $\mathscr{K}_\infty$. In particular, with $\mu_1=0$, (42) yields

$$\frac{\partial V}{\partial x}(x,\mu_0)f(x,\mu_0)\leqslant-\alpha(|x|)+\gamma(|\mu_0|) \tag{43}$$

This implies that there exists a $\mathscr{K}_\infty$ gain margin χ (for instance $\chi=\alpha^{-1}\circ 2\gamma$) such that

$$|x|\geqslant\chi(|\mu_0|)\Rightarrow\frac{\partial V}{\partial x}(x,\mu_0)f(x,\mu_0)\leqslant-\tilde{\alpha}(|x|) \tag{44}$$

where $\tilde{\alpha}$ is of class $\mathscr{K}_\infty$. If V is independent of μ_0, this would already provide an ISS-Lyapunov function for the system, and the ISS property would follow. For the general case, let $V_0(x)=V(x,0)$. From (41), one sees that

$$\underline{\alpha}(|x|)\leqslant V_0(x)\leqslant\bar{\alpha}(|x|)\quad\forall x$$

and from (44), one sees that

$$DV_0(x)f(x,0)<-\tilde{\alpha}(|x|)\quad\forall x \tag{45}$$

Since both DV_0 and $f(x,0)$ are scalar functions, it follows that $DV_0(x)\neq 0$ for all $x\neq 0$. Since the 0-input system $\dot{x}=f(x,0)$ is GAS, it follows that $xf(x,0)<0$ for all $x\neq 0$. This together with (45) implies that $xDV_0(x)<0$ for all $x\neq 0$. We will complete the proof by showing the following:

$$|x|\geqslant\chi(|\mu_0|)\Rightarrow DV_0(x)f(x,\mu_0)<0 \tag{46}$$

for all $x\neq 0$, from which it follows that V_0 is an ISS-Lyapunov function for the system.

Suppose (46) fails for some $x_0\neq 0$. Applying the intermediate value theorem to the continuous function $DV_0(x_0)f(x_0,\mu)$ with the property that $DV_0(x_0)f(x_0,0)<0$, one sees that there exists some $\bar{\mu}_0$ for which $\chi(|\bar{\mu}_0|)\leqslant|x_0|$ such that $DV_0(x_0)f(x_0,\bar{\mu}_0)=0$. It then follows from the fact that $DV_0(x_0)\neq 0$ that $f(x_0,\bar{\mu}_0)=0$. This is impossible since it contradicts (44). Hence, (46) holds for all x. □

Remark 8.2

Applying Theorem 3 and Proposition 8.1, one sees that for a scalar system as in (1), the following are equivalent:

1. The system is ISS.
2. For some smoothly invertible ISS filter as in (36), the corresponding cascade system (37) is ISS.
3. For each smoothly invertible ISS filter as in (36) the corresponding cascade system (37) is ISS.

In what follows we show by example that Proposition 8.1 in general fails in higher dimensions.

Example 8.3

Take any 2×2 matrix Φ with the property that Φ is Hurwitz but $\Phi^{\mathrm{T}} + \Phi$ has at least one positive eigenvalue (where A^{T} denotes the transpose of A). An example of such a matrix is

$$\Phi = \begin{bmatrix} -1 & 4 \\ -1 & -1 \end{bmatrix}$$

Let $\bar{\lambda}$ be such an eigenvalue of $(\Phi^{\mathrm{T}} + \Phi)$, and let v_1 be a unit eigenvector of Φ corresponding to $\bar{\lambda}$. For $\theta \in \mathbb{R}$, let $U(\theta)$ be defined by

$$U(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \tag{47}$$

Notice that $U(\theta)^{\mathrm{T}} U(\theta) = \mathrm{I}$. Consider now the system:

$$\dot{x} = (x^{\mathrm{T}} x) U(\theta)^{\mathrm{T}} \Phi U(\theta) x \tag{48}$$

where $\theta(\cdot)$ is taken to be the input to the system.

To see that this system is not ISS, we will show that there is some input which is bounded and for which the solution of (48) with $x(0) = (0, 1)'$ is not defined for all $t > 0$.

To define this input, we proceed as follows. We start by writing the eigenvector v_1 in polar form: $v_1 = (\cos\phi_0, \sin\phi_0)$, with $0 \leqslant \phi_0 < 2\pi$. Viewing the system away from zero as a system on $\mathbb{R}^2 \setminus \{0\}$, we consider the feedback law $\theta(x) := \phi - \phi_0$, where, using polar coordinates, $x_1 = r\cos\phi$ and $x_2 = r\sin\phi$. In defining the feedback, we may assume that arguments are taken in the following range: $0 \leqslant \phi < 2\pi$. However, the choice is irrelevant, since only trigonometric functions of θ appear in the system description.

In principle, there is no reason for a solution to exist for (48), under this feedback law, since the feedback law is discontinuous. However, again from periodicity of the equations, substitution into the right-hand side of (48) results in a smooth differential equation. Thus there is a unique solution, defined on some maximal interval $[0, T_{\max})$, starting from the initial state $x(0) = (0, 1)'$. We consider the input u which coincides with $\theta(x(t))$ on the maximal interval $[0, T_{\max})$, and equals some arbitrary value, let us say zero, for $t > T_{\max}$. This input is bounded (by 4π). We now show that $r = |x(t)| \nearrow \infty$ as $t \nearrow T_{\max}$.

Transforming to polar coordinates, we have that along trajectories of (48):

$$2r\dot{r} = 2(x^{\mathrm{T}}x)x^{\mathrm{T}}U(\theta)^{\mathrm{T}}\Phi U(\theta)x = 2r^4(\cos\phi \quad \sin\phi)U(\theta)^{\mathrm{T}}\Phi U(\theta)\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}$$

$$= r^4(\cos(\phi-\theta) \quad \sin(\phi-\theta))\Psi\begin{pmatrix}\cos(\phi-\theta)\\ \sin(\phi-\theta)\end{pmatrix}$$

where $\Psi = \Phi + \Phi^{\mathrm{T}}$, and

$$\dot{\phi} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{x_1^2 + x_2^2} = r^2(-\sin\phi \quad \cos\phi)U(\theta)\Phi U(\theta)\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}$$

$$= r^2(\sin(\theta-\phi) \quad \cos(\theta-\phi))\Phi\begin{pmatrix}\cos(\phi-\theta)\\ \sin(\phi-\theta)\end{pmatrix}$$

Thus, away from the equilibrium $x = 0$, we have that the system (48) on $\mathbb{R}^2\setminus\{0\}$ is, up to a co-ordinate change, the same as the following system which evolves on $\mathbb{R}_{>0} \times S^1$:

$$\dot{r} = \frac{1}{2}r^3(\cos(\phi-\theta) \quad \sin(\phi-\theta))\Psi\begin{pmatrix}\cos(\phi-\theta)\\ \sin(\phi-\theta)\end{pmatrix} \tag{49}$$

$$\dot{\phi} = r^2(\sin(\theta-\phi) \quad \cos(\theta-\phi))\Phi\begin{pmatrix}\cos(\phi-\theta)\\ \sin(\phi-\theta)\end{pmatrix} \tag{50}$$

With the feedback law $\theta = \phi - \phi_0$, Equation (49) becomes

$$\dot{r} = \frac{1}{2}r^3 v_1^{\mathrm{T}}\Psi v_1 = \frac{\bar{\lambda}}{2}r^3 \tag{51}$$

Thus r diverges monotonically to infinity in finite time, as claimed.

Nevertheless we claim that (48) is DISS. For this purpose, consider the system

$$\dot{x} = (x'x)U(\theta)'\Phi U(\theta)x$$
$$\dot{\theta} = -\theta + d \tag{52}$$

Here d takes value in $\mathbb{R}$ and plays the role of an exogenous input, whereas θ is a component of the extended state $[x', \theta]'$. By virtue of the main result in Section 7, the DISS property for (48) is equivalent to the ISS property for (52). Pick as a candidate Lyapunov function:

$$W(x,\theta) = x^{\mathrm{T}}U(\theta)^{\mathrm{T}}PU(\theta)x + k\theta^2 \tag{53}$$

where $P = P' > 0$ is the solution of the Lyapunov equation

$$\Phi^{\mathrm{T}}P + P\Phi = -\mathrm{I}_2 \tag{54}$$

Notice that

$$\lambda_{\min}(P)|x|^2 + k\theta^2 \leqslant W(x,\theta) \leqslant \lambda_{\max}(P)|x|^2 + k\theta^2 \tag{55}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ denote the largest and smallest eigenvalues of P, respectively. Thus W is proper. Taking derivatives of W along any trajectory $(x(t), \theta(t))$ of (52) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}W(x(t),\theta(t)) = |x(t)|^2[(U(\theta(t))^{\mathrm{T}}\Phi U(\theta(t))x(t))^{\mathrm{T}}U(\theta(t))^{\mathrm{T}}PU(\theta(t))x(t)$$
$$+ x(t)^{\mathrm{T}}U(\theta(t))^{\mathrm{T}}PU(\theta(t))(U(\theta(t))^{\mathrm{T}}\Phi U(\theta(t))x(t))]$$
$$+ \left[2x(t)^{\mathrm{T}}U(\theta(t))^{\mathrm{T}}P\frac{\partial}{\partial\theta}U(\theta(t))x(t) + 2k\theta(t)\right](-\theta(t)+d(t)) \tag{56}$$

Since $U(\omega)$ is orthonormal for all $\omega \in \mathbb{R}$, it follows from (54) that

$$U(\omega)^{\mathrm{T}}PU(\omega)U(\omega)^{\mathrm{T}}\Phi U(\omega) + U(\omega)^{\mathrm{T}}\Phi^{\mathrm{T}}U(\omega)U(\omega)^{\mathrm{T}}PU(\omega) = -\mathrm{I}_2 \tag{57}$$

for all ω. Let $c > 0$ be such that $|U(\omega)\,P(\partial/\partial\omega)U(\omega)| \leqslant c$ for all ω, where $|A|$ denotes the operator norm of $A \in \mathbb{R}^{2\times 2}$. Then

$$\left[2x(t)^{\mathrm{T}}U(\theta(t))^{\mathrm{T}}P\frac{\partial}{\partial\theta}U(\theta(t))x(t) + 2k\theta(t)\right](-\theta(t)+d(t))$$
$$\leqslant 2c|x(t)|^2(|\theta(t)| + |d(t)|) - 2k\theta(t)^2 + 2k\theta(t)d(t)$$
$$\leqslant \frac{1}{4}|x(t)|^4 + 4c^2(|\theta(t)| + |d(t)|)^2 - k\theta(t)^2 + k\,d(t)^2$$
$$\leqslant \frac{1}{4}|x(t)|^4 + 8c^2\theta(t)^2 + 8c^2d(t)^2 - k\theta(t)^2 + k\,d(t)^2 \tag{58}$$

Combining (56)–(58), one sees that

$$\frac{\mathrm{d}}{\mathrm{d}t}W(x(t),\theta(t)) \leqslant -\frac{3}{4}|x(t)|^4 - (k - 8c^2)\theta(t)^2 + (8c^2 + k)d(t)$$

along any trajectory of (52). It then can be seen that if $k > 8c$, W is an ISS-Lyapunov function for system (52). Consequently, system (52) is ISS as we wanted to show.

9. MORE EXAMPLES

It is clear that one has the following implications for each $k \geqslant 1$:

$$\text{ISS} \Rightarrow D^k\text{ISS} \Rightarrow \text{ISS in constant } u$$

Below we show by examples how the converse implications may fail. For this purpose, we first show the following.

Lemma 9.1

Consider a locally Lipschitz map $f: \mathbb{R}^n \to \mathbb{R}^n$.

1. If the system $\dot{x} = f(x)$ is GAS, then the system $\dot{x} = f(x+u)$ is ISS in constant u.
2. If the system $\dot{x} = f(x) + u$ is ISS, then the system $\dot{x} = f(x+u)$ is DISS.

Proof

(i) Suppose that the system $\dot{z} = f(z)$ is GAS. Then, there is some $\beta_0 \in \mathcal{KL}$ such that

$$|z(t)| \leqslant \beta_0(|z(0)|, t) \quad \forall t \geqslant 0 \tag{59}$$

holds for every trajectory $z(\cdot)$ of the system. Let $u(t) \equiv \mu$ be a constant input, and consider a trajectory $x(t)$ of the system $\dot{x} = f(x+u)$. Let $z(t) = x(t) + u(t)$ Then $z(t)$ is a trajectory of $\dot{z} = f(z)$. Hence, $z(\cdot)$ satisfies (59). Combining this with the fact that $|x(t)| \leqslant |z(t)| + \|u\|$, we get

$$|x(t)| \leqslant \beta_0(|z(0)|, t) + \|u\| \leqslant \beta_0(2|x(0)|, t) + \beta_0(2|\mu|, t) + \|u\|$$
$$\leqslant \beta_0(2|x(0)|, t) + \gamma(\|u\|)$$

where $\gamma(s) = \beta_0(2s, 0) + s$. This shows that the system is ISS in constant inputs.

(ii) Suppose that system $\dot{z} = f(z) + u$ is ISS. Then, for some $\beta_0 \in \mathcal{KL}$ and some $\gamma_0 \in \mathcal{K}$, it holds that

$$|z(t, \xi, u)| \leqslant \beta_0(|\xi|, t) + \gamma_0(\|u\|)$$

for the trajectory $z(t, \xi, u)$ of the system with initial state $z(0) = \xi$ and input u. Take a trajectory $x(t, \xi, u)$ of the system $\dot{x} = f(x+u)$ for some input $u \in W^{1,\infty}$. Let $z(t) = x(t, \xi, u) + u(t)$. Obviously, $z(\cdot)$ is a solution of $\dot{z} = f(z) + \dot{u}$. Hence,

$$|z(t, \xi, u)| \leqslant \beta_0(|z(0)|, t) + \gamma_0(\|\dot{u}\|)$$

Arguing as in the proof of (i), it can be seen that

$$|x(t, \xi, u)| \leqslant \beta_0(2|\xi|, t) + \gamma_0(\|\dot{u}\|) + \gamma(\|u\|)$$

where $\gamma(s) = \beta_0(2s, 0) + s$. Hence, the system is DISS. □

To show that (ISS in constant u) $\not\Rightarrow D^k$ISS, we first show the following.

Lemma 9.2
There exists a smooth system $\dot{x} = f(x)$ in $\mathbb{R}^2$ with the following properties:

1. The origin is globally asymptotically stable for $\dot{x} = f(x)$.
2. For each $a > 2$ there exists an input u^a such that:
   - u^a is smooth, periodic, and u^a as well as all its derivatives are bounded in norm by 1;
   - the solution of
   $$\dot{x} = f(x + u^a(t)), \quad x(0) = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{60}$$
   is $x(t) = (a\cos t, a\sin t)'$.

Proof
We fix a smooth non-increasing function $\gamma : [0, \infty) \to [-1, 0]$ such that $\gamma(r) = -r$ on $[0, 1/2]$, and $\gamma(r) = -1$ for all $r \geqslant 1$. In terms of this γ, we define the following system:

$$\dot{x}_1 = \frac{\gamma\left(\sqrt{x_1^2 + x_2^2}\right)}{\sqrt{x_1^2 + x_2^2}} x_1 - x_2$$

$$\dot{x}_2 = \frac{\gamma\left(\sqrt{x_1^2 + x_2^2}\right)}{\sqrt{x_1^2 + x_2^2}} x_2 + x_1$$

Note that this is a smooth system on $\mathbb{R}^2$, since $\gamma\left(\sqrt{x_1^2+x_2^2}\right)/\sqrt{x_1^2+x_2^2}\equiv -1$ for $x\approx 0$ (the dynamics are, in fact, linear near 0). In polar co-ordinates, we have

$$\dot{r}=\gamma(r),\quad \dot{\varphi}=-1$$

so the origin is indeed globally asymptotically stable. Observe that near the origin we have $\dot{r}=-r$, but for $|x|\geqslant 1$ we have $\dot{r}=-1$. This 'slowing down' will allow us to obtain the desired result.

For each $a>2$, we define the input u_a as follows:

$$u^a(t)=\begin{pmatrix}u_1^a(t)\\ u_2^a(t)\end{pmatrix}:=\begin{pmatrix}\frac{\sqrt{a^2-1}\,\sin t-\cos t}{a}\\ -\frac{\sin t+\sqrt{a^2-1}\,\cos t}{a}\end{pmatrix}$$

and observe that $\mathrm{d}u_1^a/\mathrm{d}t=u_2^a$, $\mathrm{d}u_2^a/\mathrm{d}t=-u_1^a$, and $(u_1^a)^2+(u_2^a)^2\equiv 1$. These facts imply that u^a and all its derivatives are bounded by 1.

The form $x^a(t)=(a\cos t,a\sin t)'$ for the solutions of (60) may be verified by substitution into the equation: one needs only to check that

$$\frac{\gamma(\sqrt{(a\cos t+u_1^a(t))^2+(a\sin t+u_2^a(t))^2})}{\sqrt{(a\cos t+u_1^a(t))^2+(a\sin t+u_2^a(t))^2}}(a\cos t+u_1^a(t))-(a\sin t+u_2^a(t))=-a\sin t$$

$$\frac{\gamma(\sqrt{(a\cos t+u_1^a(t))^2+(a\sin t+u_2^a(t))^2})}{\sqrt{(a\cos t+u_1^a(t))^2+(a\sin t+u_2^a(t))^2}}(a\sin t+u_2^a(t))+(a\cos t+u_1^a(t))=a\cos t$$

As $(a\cos t,a\sin t)'$ has constant norm $a>2$ and u has unit norm, the vector $x+u$ has norm always bigger than one, so the two multipliers of the form $\gamma(r)/r$ reduce to $-1/r$. In summary, it suffices to verify that

$$-a\sin t=-\frac{(a\cos t+u_1^a(t))}{\sqrt{(a\cos t+u_1^a(t))^2+(a\sin t+u_2^a(t))^2}}-(a\sin t+u_2^a(t))$$

$$a\cos t=-\frac{(a\sin t+u_2^a(t))}{\sqrt{(a\cos t+u_1^a(t))^2+(a\sin t+u_2^a(t))^2}}+(a\cos t+u_1^a(t))$$

with the above choice of u^a. It can be checked that this is indeed the case. □

Note that in the above example, $\|u^a\|^{[k]}=1$ for all $k\geqslant 0$. Hence, the system $\dot{x}=f(x+u)$ is not D^kISS for any $k\geqslant 0$. To see this, suppose that the system is D^kISS for some $k\geqslant 0$. Then there exist some $\beta\in\mathcal{KL}$ and $\sigma\in\mathcal{K}$ such that

$$|x^a(t)|\leqslant\beta(|x^a(0)|,t)+\sigma(\|u^a\|^{[k]})=\beta(a,t)+\sigma(1)$$

for any $a>2$. Consequently,

$$\limsup_{t\to\infty}|x^a(t)|\leqslant\sigma(1)$$

for any $a\geqslant 2$. This is a contradiction since $|x^a(t)|\equiv a$ for all $a>2$.

Also note that since the system $\dot{x} = f(x)$ as in the example is GAS, the system $\dot{x} = f(x+u)$ is ISS in constant inputs. Thus, the lemma provides an example where a system is ISS in constant inputs but fails to be D^kISS for any $k \geqslant 0$.

Below we modify the system to get a system that is DISS but not ISS. Thus we obtain an alternative to the counterexample in Section 9.

Let f be as defined in Lemma 9.2 and consider the system

$$\dot{z} = \varphi(|z|) f(z) + u \tag{61}$$

where $\varphi(r) = \sqrt{1+r^2} \geqslant |r|$. Let $V(z) = (z_1^2 + z_2^2)/2$. One has

$$DV(z)(\varphi(|z|)f(z)+u) = \varphi(|z|)\frac{\gamma(\sqrt{z_2^1 + z_2^2})}{|z|}|z|^2 + z\cdot u \leqslant \gamma(\sqrt{z_2^1+z_2^2})|z|^2 + z\cdot u$$

It follows that

$$|u| \leqslant \frac{\gamma(|z|)|z|}{2} \Rightarrow DV(z)(\varphi(|z|)f(z)+u) \leqslant \frac{\gamma(|z|)|z|^2}{2}$$

Consequently, system (61) is ISS. According to Lemma 9.2, the system

$$\dot{x} = \varphi(|x+u|) f(x+u) \tag{62}$$

is DISS. Below we show that system (62) is not ISS. To see this, consider, for each $a > 2$, the input $\tilde{u}^a$ defined by $\tilde{u}^a(t) = u(at)$. Let $\tilde{x}^a(t) = (a\cos at, a\sin at)$. One has:

- $|\tilde{u}_a(t)| \equiv 1$ and $|\tilde{x}^a(t)| \equiv a$.
- For any $a > 2$, $\tilde{x}^a(t)\cdot\tilde{u}^a(t) = -1$ and

$$|\tilde{x}^a(t) + \tilde{u}^a(t)|^2 = |\tilde{x}^a(t)|^2 + 2\tilde{x}^a(t)\cdot\tilde{u}^a(t) + |\tilde{u}^a(t)|^2 = -1$$

Hence, $\varphi(|\tilde{x}^a(t) + \tilde{u}^a(t)|) = a$.

Since $\tilde{x}^a(t) = x^a(at)$ and $\tilde{u}^a(t) = u_a(at)$, and since $x^a(t)$ is a solution of (60), it follows that $\tilde{x}^a(t)$ is a solution of the equation

$$\dot{\tilde{x}}^a(t) = a f(\tilde{x}^a(t), \tilde{u}^a(t))$$

Combining this with the fact that $\varphi(|\tilde{x}^a(t) + \tilde{u}^a(t)|) = a$, one sees that $\tilde{x}^a(t)$ is a solution of (62) with the input $\tilde{u}^a$. It follows from the fact that $|\tilde{u}^a(t)| \equiv 1$ and $|\tilde{x}^a(t)| \equiv a$ that it is impossible for system (62) to be ISS.

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APPENDIX A: SMOOTH APPROXIMATION TO MEASURABLE, ESSENTIALLY BOUNDED FUNCTIONS

We need in the text several routine smooth approximation results; for ease of reference, we provide proofs here.

Let φ be measurable, essentially bounded on $[a, b]$. Then there exists a sequence of measurable simple functions $\{\varphi_j\}$ that converges to φ almost everywhere on $[a, b]$ such that $\|\varphi_j\| \leqslant \|\varphi\|$ (c.f.

[17, Theorem 4.13]), (where $\|\cdot\|$ stands for the L_∞ norm on $[a,b]$). Furthermore, it is also easy to see that for every measurable simple function ρ, there exist a sequence of measurable piecewise constant functions $\{\rho_j\}$ that converges to ρ almost everywhere on $[a,b]$, and the $\{\rho_j\}$ can be chosen so that $\|\rho_j\| \leqslant \|\rho\|$ for all j (see, for instance, Remark C.1.2 in Reference [18]). In turn, for each piecewise constant function $\psi : [a,b] \to \mathbb{R}$, one can find a sequence of continuous functions $\{\psi_j\}$ that approaches ψ almost everywhere with the property that $\|\psi_j\| \leqslant \|\psi\|$. Finally, by the Weierstrass theorem, each continuous function $\sigma : [a,b] \to \mathbb{R}$ can be approximated by a sequence of polynomial functions $\{\sigma_j\}$ uniformly on $[a,b]$. Since the convergence is uniform, one sees that $\lim_{j\to\infty} \|\sigma_j\| \leqslant \|\sigma\|$. Combining the above arguments together, we have the following small variation of Remarks C.1.1 and C.1.2 in Reference [18]:

Lemma A.1
Let $\varphi : [a,b] \to \mathbb{R}$ be measurable, essentially bounded. Then there exists an equibounded sequence $\{\varphi_j\}$ of smooth functions such that

- $\varphi_j \to \varphi$ a.e. on $[a,b]$;
- $\limsup_{j\to\infty} \|\varphi_j\| \leqslant \|\varphi\|$; and
- by the Lebesgue dominated convergence theorem, $\lim_{j\to\infty} \|\varphi_j - \varphi\|_1 = 0$, where $\|\cdot\|_1$ is the L_1 norm on $[a,b]$.

Observe that the above approximation result also holds for functions from $[a,b]$ to $\mathbb{R}^m$.

Let $k \geqslant 1$. Suppose that $u \in W^{k,\infty}(a,b)$. Let $\{\varphi_j\}$ be a sequence of smooth functions that approaches $\varphi := u^{(k)}$ as in Lemma A.1. Define inductively, for $i = 1, 2, \ldots, k$

$$\varphi_j^i(t) = u^{(k-i)}(a) + \int_a^t \varphi_j^{i-1}(s)\,\mathrm{d}s$$

where $\varphi_j^0(t) = \varphi_j(t)$. Let $u_j(t) = \varphi_j^k(t)$. It can be seen that, for $i = 1, \ldots, k$, $u_j^{(i)} = \varphi_j^{k-i}$. Since

$$|u_j^{(k-1)}(t) - u^{(k-1)}(t)| \leqslant \int_a^t |\varphi_j(s) - u^{(k)}(s)|\,\mathrm{d}s \leqslant \|\varphi_j - \varphi\|_1$$

it follows that $u_j^{(k-1)} \to u^{(k-1)}$ uniformly on $[a,b]$. Processing inductively, one shows that, for $i = 0, 1, \ldots, k-1$, $\{u_j^{(i)}\}$ converges to $u^{(i)}$ uniformly on $[a,b]$. It then follows from the uniform convergence that $\lim_{j\to\infty} \|u_j^{(i)}\| = \lim_{j\to\infty} \|u^{(i)}\|$ for all $i = 0, 1, \ldots, k-1$. Hence, we have shown that, for $k \geqslant 1$, if $u \in W^{k,\infty}(a,b)$, then there exists a sequence of smooth functions $\{u_j\}$ that converges to u uniformly with the property that $\limsup_{j\to\infty} \|u_j\|^{[k]} \leqslant \|u\|^{[k]}$. Combining with the case of $k = 0$ as stated in Lemma A.1, we get the following:

Corollary A.2
Let $k \geqslant 0$. Suppose that $u \in W^{k,\infty}(a,b)$. Then there exists an equibounded sequence of smooth functions $\{u_j\}$ that converges to u pointwise on $[a,b]$ with the property that

$$\limsup_{j\to\infty} \|u_j\|^{[k]} \leqslant \|u\|^{[k]}$$

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