INPUT-TO-STATE STABILITY FOR DISCRETE-TIME NONLINEAR SYSTEMS

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Abstract: In this paper the input-to-state stability (ISS) property is studied for discrete-time nonlinear systems. We show that many ISS results for continuous-time nonlinear systems in earlier papers (Sontag, 1989; Sontag, 1990; Sontag and Wang, 1996; Jiang *et al.*, 1994; Coron *et al.*, 1995) can be extended to the discrete-time case. More precisely, we provide a Lyapunov-like sufficient condition for ISS, and we show the equivalence between the ISS property and various other properties. Utilizing the notion of ISS, we present a small gain theorem for nonlinear discrete time systems. ISS stabilizability is discussed and connections with the continuous-time case are made. As in the continuous time case, where the notion ISS found wide applications, we expect that this notion will provide a useful tool in areas related to stability for nonlinear discrete time systems as well.

Keywords: discrete-time nonlinear systems, input-to-state stability, Lyapunov methods.

1. INTRODUCTION

The notion of input-to-state stability (for short, ISS) for nonlinear control systems was proposed by one of the authors (Sontag, 1989; Sontag, 1990) and has been used in the stability analysis and control synthesis of nonlinear systems by several authors – see, e.g., (Sontag, 1989; Sontag, 1990; Sontag and Wang, 1996; Tsinias, 1993; Kazakos and Tsinias, 1994; Praly and Jiang, 1993; Jiang *et al.*, 1994; Coron *et al.*, 1995; Krstić *et al.*, 1995). Basically, ISS gives a quantitative bound of the state trajectories in terms of the magnitude of the control input and their initial conditions.

The purpose of this paper is to study the ISS property for discrete-time nonlinear systems. We show that many ISS results for continuous-time nonlinear systems in earlier papers (Sontag, 1989; Sontag, 1990; Sontag and Wang, 1996; Jiang *et al.*, 1994; Coron *et al.*, 1995) can be extended to the discrete-time case. Among these extensions, we prove that various equivalent characterizations of the ISS condition proposed in (Sontag and Wang, 1996) also hold for discrete-time nonlinear systems. It is also shown that ISS small gain theorems in (Jiang *et al.*, 1994; Coron *et al.*, 1995) are extendable to the discrete-time setting. However, new phenomena arise in the extension from continuous-time to discrete-time. For continuous-

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time affine systems, continuous stabilization implies ISS stabilization by means of state-feedback change $u = K(x) + v$. For discrete-time affine or nonaffine systems, a more complex feedback transformation of the form $u = K_1(x)+K_2(x)v$ is in general required. The construction of the feedback terms K_1 and K_2 turns out to be nontrivial; nevertheless, we show that they can be explicitly obtained, for a class of feedback linearizable systems.

Section 2 starts with basic definitions and states some preliminary results for ISS systems. Section 3 gives some equivalent definitions of ISS, and closes with the statement of the main theorem. Section 4 proposes a nonlinear small gain theorem for discrete-time interconnected systems. Section 5 proves the important fact that continuous stabilization implies ISS stabilization. An illustration is given via a class of feedback linearizable systems.

2. BASIC DEFINITIONS AND RESULTS

We consider general nonlinear systems:

$$x(k+1) = f(x(k),\, u(k)), \qquad (1)$$

where states $x(k)$ are in $\mathbb{R}^n$, and control values $u(k)$ in $\mathbb{R}^m$, for some n and m, for each time instant $k \in \mathbb{J}_+$, the set of all nonnegative integers. We assume that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and satisfies $f(0,0) = 0$.

Controls or *inputs* are functions $u : \mathbb{J}_+ \to \mathbb{R}^m$. The set of all such functions with the supremum norm $\|u\| = \sup\{|u(k)| : k \in \mathbb{J}_+\} < \infty$ is denoted by l_∞^m, where $|\cdot|$ denotes the usual Euclidean norm. For a given system, we often consider the same system but with controls restricted to take values in some subset $\Omega \subset \mathbb{R}^m$; we use $\mathcal{M}_\Omega$ for the set of all such controls. We use Id to denote the identity function.

For each $\xi \in \mathbb{R}^n$ and each input u, we denote by $x(\cdot, \xi, u)$ the trajectory of system (1) with initial state $x(0) = \xi$ and the input u. Clearly such a trajectory is defined uniquely on $\mathbb{J}_+$, and for each u and each $k \in \mathbb{J}_+$, $x(k, \xi, u)$ depends on ξ continuously.

Recall that a function $\gamma : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is a $\mathcal{K}$-*function* if it is continuous, strictly increasing and $\gamma(0) = 0$; it is a $\mathcal{K}_\infty$-*function* if it is a $\mathcal{K}$-function and also $\gamma(s) \to \infty$ as $s \to \infty$; and it is a *positive definite* function if $\gamma(s) > 0$ for all $s > 0$, and $\gamma(0) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$-*function* if, for each fixed $t \geq 0$, the function $\beta(\cdot,\, t)$ is a $\mathcal{K}$-function, and for each fixed $s \geq 0$, the function $\beta(s, \cdot)$ is decreasing and $\beta(s,t) \to 0$ as $t \to \infty$.

Definition 2.1. System (1) is (*globally*) *input-to-state stable* (ISS) if there exist a $\mathcal{KL}$-function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ and a $\mathcal{K}$-function γ such that, for each input $u \in l_\infty^m$ and each $\xi \in \mathbb{R}^n$, it holds that

$$|x(k,\, \xi,\, u)| \;\leq\; \beta(|\xi|\,,\, k) + \gamma(\|u\|) \qquad (2)$$

for each $k \in \mathbb{J}_+$. □

Note that, by causality, the same definition would result if one would replace (2) by

$$|x(k,\, \xi,\, u)| \;\leq\; \beta(|\xi|\,,\, k) + \gamma(\|u_{[k-1]}\|) \quad (3)$$

where $k \geq 1$ and, for each $l \geq 0$, $u_{[l]}$ denotes the truncation of u at l; i.e., $u_{[l]}(j) = u(j)$ if $j \leq l$, and $u_{[l]}(j) = 0$ if $j > l$.

Definition 2.2. A continuous function V on $\mathbb{R}^n$ is called an ISS-*Lyapunov function* for system (1) if

$$\alpha_1(|\xi|) \;\leq\; V(\xi) \;\leq\; \alpha_2(|\xi|) \qquad (4)$$

holds for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and

$$V(f(\xi,\, \mu)) - V(\xi) \leq -\alpha_3(|\xi|) + \sigma(|\mu|), \quad (5)$$

for some $\alpha_3 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$.

A smooth ISS-Lyapunov function is one which is smooth. □

Observe that if V is an ISS-Lyapunov function for (1), then V is a Lyapunov function for the 0-input system $x(k+1) = f(x(k),\, 0)$.

Proposition 2.3. If system (1) admits an ISS-Lyapunov function, then it is ISS.

The converse of this result, as well as a converse Lyapunov theorem regarding uniform global asymptotic stability for systems with disturbances taking values in compacts, are also true, and will be proved in a forthcoming paper.

Sketch of the proof of Proposition p-liss-iss. Assume that system (1) admits an ISS-Lyapnov function V. Let α_i $(i = 1,2,3)$ and σ be as in (4) and (5). First observe that (5) can be rewritten as

$$V(f(\xi,\, \mu)) - V(\xi) \leq -\alpha_4(V(\xi)) + \sigma(|\mu|),$$

for all ξ and μ, where $\alpha_4 = \alpha_3 \circ \alpha_2^{-1}$. To prove the lemma, one needs the following:

Lemma 2.4. For any $\mathcal{K}_\infty$-function α, there is a $\mathcal{K}_\infty$-function $\widehat{\alpha}$ satisfying the following:

- $\widehat{\alpha}(s) \leq \alpha(s)$ for all $s \geq 0$; and

- $\mathrm{Id} - \widehat{\alpha} \in \mathcal{K}$. □

We now return to the proof of Lemma 2.3. For the $\mathcal{K}_\infty$-function α_4, let $\widehat{\alpha}_4$ be a function picked as in Lemma 2.4. We then have

$$V(f(\xi,\,\mu)) - V(\xi) \le -\widehat{\alpha}_4(V(\xi)) + \sigma(|\mu|). \quad (6)$$

Fix a point $\xi \in \mathbb{R}^n$ and pick an input u. Let $x(k)$ denote the corresponding trajectory $x(k,\xi,u)$ of (1). Let c be any number in $(0,1)$, and consider the set defined by $D = \{\xi :\ V(\xi) \le b\}$, where $b = \widehat{\alpha}_4^{-1}\left(\frac{\sigma(\|u\|)}{c}\right)$.

Claim: If there is some $k_0 \in \mathbb{J}_+$ such that $x(k_0) \in D$, then $x(k) \in D$ for all $k \ge k_0$.

Proof. Assume that $x(k_0) \in D$. Then $V(x(k_0)) \le b$, that is, $c\,\widehat{\alpha}_4(V(x(k_0))) \le \sigma(\|u\|)$. By (6),

$$\begin{aligned} V(x(k_0+1)) \le & -(1-c)\widehat{\alpha}_4(V(x(k_0))) \\ & + \widetilde{\alpha}_4(V(x(k_0))) + \sigma(\|u\|) \end{aligned}$$

where $\widetilde{\alpha}_4 = \mathrm{Id} - c\,\widehat{\alpha}_4$. Observe that $\widetilde{\alpha}_4 \in \mathcal{K}$ because $\mathrm{Id} - \widehat{\alpha}_4 \in \mathcal{K}$. Combining this fact with the assumption that $V(x(k_0)) \le b$, one can show that $V(x(k_0+1)) \le -(1-c)\widehat{\alpha}_4(V(x(k_0))) + b \le b$. Using induction, one can show that $V(x(k_0+j)) \le b$ for all $j \in \mathbb{J}_+$, that is, $V(x(k)) \in D$ for all $k \ge k_0$. □

We now let $j_0 = \min\{k \in \mathbb{J}_+ :\ x(k) \in D\} \le \infty$. Then it follows from the above conclusion that $V(x(k)) \le \widehat{\gamma}(\|u\|)$ for all $k \ge j_0$, where $\widehat{\gamma}(r) = \widehat{\alpha}_4^{-1}(\sigma(r)/c)$. For $k < j_0$, it holds that $c\,\widehat{\alpha}_4(V(x_k)) > \sigma(\|u\|)$, and hence, $V(x(k+1)) - V(x(k)) \le -(1-c)\widehat{\alpha}_4(V(x(k)))$. We let $\widehat{\beta}(s,k)$ be the solution of the scalar difference equation

$$z(k+1) = z(k) - (1-c)\widehat{\alpha}_4(z(k))$$

with initial condition $z(0) = s$. Observe that, for any $s > 0$, the sequence $z(t)$ decreases to zero (it never crosses zero, because $\mathrm{Id} - \widehat{\alpha}_4 \ge 0$); thus, $\widehat{\beta}$ is a $\mathcal{KL}$-function. It follows, by induction on k, that $V(x(k)) \le \widehat{\beta}(V(x(0)),k)$ for all $0 \le k \le j_0 + 1$. Thus, $V(x(k)) \le \max\{\widehat{\beta}(V(\xi),k),\ \widehat{\gamma}(\|u\|)\}$. From this one gets (2) with $\beta(s,r) = \alpha_1^{-1}(\widehat{\beta}(\alpha_2(|\xi|),r))$ and $\gamma(s) = \alpha_1^{-1}\widehat{\gamma}(s)$. □

Remark 2.5. From the proof of Lemma 2.3, one sees that if the ISS-Lyapunov function satisfies the inequality $V(x(k+1)) - V(x(k)) \le -\alpha(V(x(k))) + \sigma(\|u\|)$, with the property that $\mathrm{Id} - \alpha \in \mathcal{K}$, then for any $0 < c < 1$, the function $(\alpha_1^{-1} \circ \alpha^{-1})(\sigma(s)/c)$ can be taken as a gain function of the system. That is, for any $0 < c < 1$, there is a $\mathcal{KL}$-function β such that $|x(k,\xi,u)| \le \beta(|\xi|,k) + (\alpha_1^{-1} \circ \alpha^{-1})(\sigma(\|u\|)/c)K$ for all ξ and all u. □

3. EQUIVALENT NOTIONS OF ISS

In this section we show that, as in the continuous time case, there are various notions that are equivalent to ISS.

3.1 *Asymptotic Gains*

Consider system (1). We say that the system has *$\mathcal{K}$-asymptotic gain* if there exists some $\gamma \in \mathcal{K}$ such that

$$\overline{\lim_{k\to\infty}}\, |x(k,\xi,u)| \le \overline{\lim_{k\to\infty}}\, \gamma(|u(k)|) \quad (7)$$

for all $\xi \in \mathbb{R}^n$.

We say that system (1) is *uniformly bounded-input bounded-state* (UBIBS) stable if bounded initial states and controls produce uniformly bounded trajectories, i.e., there exist two $\mathcal{K}$-functions σ_1, σ_2 such that

$$\sup_k |x(k,\xi,u)| \le \max\{\sigma_1(|\xi|),\ \sigma_2(\|u\|)\} \quad (8)$$

Again, by the causality property of the system, the above is equivalent to $\sigma_1(s) \ge s$ and

$$|x(k,\xi,u)| \le \max_{0 \le j \le k-1}\{\sigma_1(|\xi|),\ \sigma_2(|u(j)|)\} \quad (9)$$

It is not hard to see that if a system is ISS, then it is UBIBS and it admits a $\mathcal{K}$-asymptotic gain. The converse is also true. It will be proved after we introduce another related notion in the next section.

3.2 *Robust Stability Margins*

As in the case of continuous time systems, there turns out to be also an interesting connection between the ISS and the robust stability.

By a (possibly time-varying) *feedback law* for a system (1) we will mean any function $w : \mathbb{J}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that for each fixed k, $w(k,\cdot)$ is a continuous function.

Let ρ be any $\mathcal{K}_\infty$ function. For any $d \in \mathcal{M}_\Omega$ with Ω as the closed unit ball in $\mathbb{R}^m$, we view the system

$$x(k+1) = f(x(k), d(k)\rho(|x(k)|)) \quad (10)$$

as a system $x(k+1) := g(x(k), d(k))$ with time-varying parameters $d \in \mathcal{M}_\Omega$. We use $x_\rho(k,\xi,d)$ to denote the solution to (10) corresponding to the initial state ξ and the disturbance signal d.

We will say that the system (1) is *robustly stable* if there exists a $\mathcal{K}_\infty$ function ρ (called a *stability margin*) such that system (10) is globally asymptotically stable uniformly in d. To be more precise, this means that there exists $\beta \in \mathcal{KL}$ such that the following holds for all trajectories of (10):

$$|x_\rho(k,\xi,d)| \le \beta(|\xi|,k)$$

for all $k \ge 0$, all $d \in \mathcal{M}_\Omega$.

Note that for a nonlinear GAS system, in general only small perturbations can be tolerated while preserving stability. The requirement $\rho \in \mathcal{K}_\infty$ is thus nontrivial.

Following the same steps as in Section V of (Sontag and Wang, 1996), one may show the next result:

Lemma 3.1. System (1) is ISS if and only if it is robustly stable. □

(As a matter of fact, the proof turns out to be far simpler, because a key technical point, that asymptotic stability of systems with disturbances is equivalent to uniform asymptotic stability, is much easier in discrete-time.)

The following lemma provides a key link in our main results. Its proof is analogous to the proof of the corresponding continuous time result in (Sontag and Wang, 1996, Section V)).

Lemma 3.2. If system (1) is UBIBS and if it admits $\mathcal{K}$-asymptotic gain, then it is robustly stable. □

To close this section, we summarize our results in this section:

Theorem 1. Consider system (1). The following are equivalent:

(1) It is ISS.
(2) It is UBIBS and it admits $\mathcal{K}$-asymptotic gain.
(3) It is robustly stable.

Proof. We have: [1 $\Rightarrow$ 2] (clear by Definitions), [2 $\Rightarrow$ 3] (see Lemma 3.2), [3 $\Rightarrow$ 1] (see Lemma 3.1). □

4. NONLINEAR SMALL GAIN THEOREMS

In this section, we will discuss the ISS property for interconnected systems of the following type:

$$\begin{aligned} x_1(k+1) &= f_1(x_1(k), x_2(k), u_1(k)), \\ x_2(k+1) &= f_2(x_1(k), x_2(k), u_2(k)), \end{aligned} \qquad (11)$$

where for $i = 1, 2$ and for each $k \in \mathbb{J}_+$, $x_i(k) \in \mathbb{R}^{n_i}$, $u_i(k) \in \mathbb{R}^{m_i}$, and $f_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$ is continuous.

It turns out that, in the continuous time case, ISS small gain theorems are very powerful in treating stability problems for such interconnected systems. The first such results were obtained in (Jiang *et al.*, 1994). Later the proofs of the main results were considerably simplified, using asymptotic gains, in (Coron *et al.*, 1995); in conjunction with the results in (Sontag and Wang, 1996) this gives ISS results. In (Jiang *et al.*, 1996), a small gain theorem was presented in terms of Lyapunov functions. By following the continuous-time approach used in (Coron *et al.*, 1995, Sec. 4), one can get the following small theorem. For the interconnected system (11), we assume that both of the subsystems are ISS. To be more precise, we assume that, for the following systems,

$$x_1(k+1) = f_1(x_1(k), v_1(k), u_1(k)), \qquad (12)$$
$$x_2(k+1) = f_2(x_2(k), v_2(k), u_2(k)), \qquad (13)$$

the trajectories satisfy

$$|x_1(k)| \le \max\Big\{\beta_1(|\xi_1|,k),\ \gamma_1^x(\|v_1\|),\ \gamma_1^u(\|u_1\|)\Big\}$$

$$|x_2(k)| \le \max\Big\{\beta_2(|\xi_2|,k),\ \gamma_2^x(\|v_2\|),\ \gamma_1^u(\|u_2\|)\Big\}$$

Theorem 2. Assume that systems (12) and (13) are ISS, with $\gamma_1^x(\gamma_2^x(s)) < s$ (or, $\gamma_2^x(\gamma_1^x(s)) < s$) for all $s > 0$. Then, the interconnected system (11) is ISS with (u_1, u_2) as input. □

The above small gain theorem can also be stated in terms of ISS-Lyapunov functions. Assume that both subsystems in (11) admit ISS-Lyapunov functions. Let V_1 (respectively, V_2) be an ISS-Lyapunov function of (12) (respectively, (13)). That is, there exist class $\mathcal{K}_\infty$-functions α_{ij}, σ_i, ρ_i^x and ρ_i^u ($1 \le i, j \le 2$) such that $\alpha_{i1}(|\xi|) \le V_i(\xi) \le \alpha_{i2}(|\xi|)$, and

$$\begin{aligned} V_1(f_1(\xi_1,\xi_2,\mu_1)) - V_1(\xi_1) &\le -\sigma_1(V_1(\xi_1)) \\ &\quad + \rho_1^x(V_2(\xi_2)) + \rho_1^u(|\mu_1|), \qquad (14) \\ V_2(f_2(\xi_2,\xi_1,\mu_2)) - V_2(\xi_2) &\le -\sigma_2(V_2(\xi_2)) \\ &\quad + \rho_2^x(V_1(\xi_1)) + \rho_2^u(|\mu_2|). \qquad (15) \end{aligned}$$

In view of Lemma 2.4, we may assume that $\mathrm{Id} - \sigma_i \in \mathcal{K}$ for $i = 1, 2$.

Theorem 3. Assume that systems (12) and (13) admit ISS-Lyapunov functions V_1 and V_2 respectively that satisfy (14)-(15), with $\mathrm{Id} - \sigma_i \in \mathcal{K}$ for $i = 1, 2$. If there exists a $\mathcal{K}_\infty$-function ρ such that $\sigma_1^{-1} \circ (\mathrm{Id}+\rho) \circ \rho_1^x \circ \sigma_2^{-1} \circ (\mathrm{Id}+\rho) \circ \rho_2^x < \mathrm{Id}$, then the interconnected system (11) is ISS with (u_1, u_2) as input.

5. ISS-STABILIZABILITY

Consider system (1). We say that the system is *continuously stabilizable* if there is a continuous function $w : \mathbb{R}^n \to \mathbb{R}^m$ with $w(0) = 0$ such that under the feedback $u = w(x)$, the closed-loop system

$$x(k+1) = f(x(k), w(x(k)))$$

is globally asymptotically stable (GAS), i.e., there exists some $\mathcal{KL}$-function $\beta(\cdot,\cdot)$ such that the following holds for all trajectories of the system:

$$|x(k,\xi)| \le \beta(|\xi|, k)$$

for all $k \ge 0$.

We say that system (1) is *continuously* ISS *stabilizable* if there exist a continuous map $w : \mathbb{R}^n \to \mathbb{R}^m$ with $w(0) = 0$ and an $n\times n$ matrix Γ of continuous functions, invertible for each x, such that under the control law $u = w(x) + \Gamma(x)v$ the closed-loop system

$$x(k+1) = f(x(k), w(x(k)) + \Gamma(x(k))v(k)) \qquad (16)$$

is ISS (with v as new input).

Clearly, if a system is ISS-stabilizable, then it is stabilizable. By following the approach used in the continuous case (see (Sontag, 1990)), one can prove the following:

Theorem 4. System (1) is continuously stabilizable if and only if it is ISS-stabilizable. □

Remark 5.1. In contrast to the continuous case, the construction of an ISSstabilizing control law is far from explicit even for affine systems. Nevertheless, it is shown in the next section that an alternative and simple design of ISS controllers is possible for a class of feedback linearizable systems. □

5.1 *Feedback Linearizable Systems*

The main purpose of this section is to illustrate our main results in the preceding section via feedback linearizable systems. As done in (Sontag, 1989) for continuous-time, we derive explicit formulas for the special case of such systems. Both continuous- and discrete-time linearizable systems have been fairly studied in the recent control literature, see, e.g., (Isidori, 1995; Nijmeijer and van der Schaft, 1990; Sontag, 1998) and references therein. It is worthing noting that a large body of research papers on nonlinear adaptive control have been based on globally feedback linearizable systems, see the recent texts (Krstić *et al.*, 1995; Marino and Tomei, 1995) and the references cited therein.

We restrict ourselves to systems (1) which are *affine* in u, that is,

$$x(k+1) \;=\; a(x(k)) \;+\; b(x(k))u \qquad (17)$$

A system (17) is said to be globally feedback linearizable if there exist a state-coordinate change, that is, an isomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ which vanishes at zero,

$$z \;=\; \phi(x) \qquad (18)$$

and a regular state-feedback transformation

$$u \;=\; \vartheta_1(x) \;+\; \vartheta_2(x)v \qquad (19)$$

with $\vartheta_2(x)$ a globally invertible matrix, such that system (17) is brought to a linear controllable system

$$z(k+1) \;=\; Az(k) \;+\; Bv \qquad (20)$$

Take any constant matrix K_0 so that $A + BK_0$ is asymptotically stable. The linear feedback $v(k) = K_0 z(k)$ globally asymptotically stabilizes the linear system (20) and therefore its original nonlinear system (17). In addition, the feedback $v(k) = K_0 z(k) + w(k)$ makes the closed-loop system

$$z(k+1) = (A + BK_0)z(k) + Bw(k)$$

ISS (with respect to w). This can be seen through the following:

$$|z(k)| \le \lambda_{\max}(A + BK_0)^k |z(0)| + \frac{|B|}{1-\lambda_{\max}(A+BK_0)}\|w\| \qquad (21)$$

where $\lambda_{\max}(A+BK_0)$ denotes the spectral radius of $A + BK_0$.

Since ϕ is a global isomorphism and vanishes at zero, there exist two class $\mathcal{K}_\infty$-functions $\widehat{\phi}_1$ and $\widehat{\phi}_2$ such that

$$\widehat{\phi}_1(|x|) \;\le\; |\phi(x)| \;\le\; \widehat{\phi}_2(|x|) \qquad (22)$$

This together with (21) and (18) implies that

$$|x(k)| \le \widehat{\phi}_1^{-1}\left(2\lambda_{\max}(A+BK_0)^k \widehat{\phi}_2(|x(0)|)\right) + \widehat{\phi}_1^{-1}\left(\frac{2|B|}{1-\lambda_{\max}(A+BK_0)}\|w\|\right) \qquad (23)$$

Therefore, system (17) is made ISS with a feedback law $u = \vartheta_1(x(k)) + \vartheta_2(x(k))K_0\phi(x(k)) + \vartheta_2(x(k))w(k)$.

By analogy with the case of continuous-time systems (Sontag, 1989), we would ask whether or not it is possible to render system (17) ISS using a nonlinear feedback of the form

$$u = K(x) + w \tag{24}$$

where $K : \mathbb{R}^n \to \mathbb{R}^m$ is continuous with $K(0) = 0$.

First of all, we point out that, similar as in the continuous time case, a linear feedback does not necessarily produce a system which is ISS with respect to additive disturbances occuring at the level of real control input u. More precisely, when taking K as the linearizing feedback, i.e. $K(x(k)) = \vartheta_1(x(k)) + \vartheta_2(x(k))K_0\phi(x(k))$, the corresponding closed-loop system $x(k+1) = a(x(k)) + b(x(k))(K(x(k)) + w)$ may fail to be ISS with respect to the signal w.

As an illustration of this important point, we consider a discrete-time scalar system

$$x(k+1) = x(k)^3 + (|x(k)| + 1)u \tag{25}$$

which is globally rendered linear by a nonlinear feedback of the form

$$K(x) = -\frac{x^3}{|x|+1} + \frac{1}{|x|+1}v. \tag{26}$$

A further choice of $v(k) = \frac{1}{2}x(k)$ gives a feedback law that globally asymptotically stabilizes system (25). However, this linear feedback controller does not make the closed-loop system ISS with respect to additive disturbances w at the true control input u:

$$x(k+1) = \frac{1}{2}x(k) + (|x(k)| + 1)w. \tag{27}$$

Indeed, letting $w = 1$ in (27) yields that the solutions $x(k)$ with $x(0) = 0$ of (27) diverge to $+\infty$ as $k \to +\infty$.

To compare with the continuous time case, a natural question to ask is if there is *any* feedback that make the closed-loop system ISS? Unfortunately, in contrast to the continuous time situation, there does generally not exist a feedback of the form (24) that produces an ISS system. As a counterexample, we still consider system (25).

The system (25) in closed-loop with (24) is as the following

$$x(k+1) = G(x(k)) + (|x(k)| + 1)w(k)$$

with $G(x(k)) = x(k)^3 + (|x(k)| + 1)K(x(k))$. Now consider the signal $w(k)$ given by:

$$w(k) = \begin{cases} 1 \text{ if } G(x(k)) \geq 0, \\ -1 \text{ if } G(x(k)) < 0. \end{cases}$$

It can be seen that with such a choice of w, $|x(k+1)| \geq |x(k)| + 1$, and hence, $x(k+1) \to \infty$. The above argument shows that the system fails to be ISS no matter what G is. Observe that this is still true even if G is dead-beat stable, i.e., G is such a function that every trajectory of $x(k+1) = G(x(k))$ can reach the origin in finite time (e.g., $G(x) \equiv 0$).

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