

# Further Results on Controllability Properties of Discrete-Time Nonlinear Systems\*

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**Abstract.** Controllability questions for discrete-time nonlinear systems are addressed in this paper. In particular, we continue the search for conditions under which the group-like notion of transitivity implies the stronger and semigroup-like property of forward accessibility. We show that this implication holds, pointwise, for states which have a weak Poisson stability property, and globally, if there exists a global “attractor” for the system.

## 1. Introduction

This paper continues the study, initiated in [4] and then developed in [1], of some controllability properties of discrete-time nonlinear systems, of the type:

$$x(t+1) = f(x(t), u(t)), \quad t = 0, 1, 2, \dots, \quad (1)$$

where  $x(t) \in \mathcal{X}$  and  $u(t) \in U$ . We will deal, as in the above-mentioned papers, with the class of *invertible* systems, that is with those systems for which the function  $f(\cdot, u)$  is a diffeomorphism. Such systems arise, for instance, when dealing with continuous-time models under digital controls via *sampling*. For further motivations of the study of this class we refer to [4].

If  $\Sigma$  is a system of type (1), and  $x_0$  is any state, then we can define the reachable set from  $x_0$ ,  $R(x_0)$ , and the orbit from  $x_0$ ,  $O(x_0)$ . We will see later (section 2.1) the precise definition of these objects; intuitively,  $R(x_0)$  is the set of all those states that can be reached from  $x_0$  using arbitrary controls, and  $O(x_0)$  consists of all those states to which we can steer  $x_0$  using both the motions of  $\Sigma$  and negative-time motions. The concept of reachable sets is certainly more natural than the concept of orbits, since negative-time motions are not allowed. However, orbits are usually easier to study—they arise from group actions—and they have nicer properties. For instance, it is known that each orbit has a natural structure of submanifold of  $\mathcal{X}$ .

This paper studies some relations between these two concepts. In particular, we will focus our attention on the relation between the notion of forward accessibility (i.e.  $\text{int } R(x_0) \neq \emptyset$ )

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and the weaker notion of transitivity (i.e.  $\text{int } O(x_0) \neq \emptyset$ ). We would like to see when transitivity implies forward accessibility. It is a classical result, in the continuous-time framework, that this implications holds always for analytic systems, and under some appropriate Lie-rank conditions in the  $C^\infty$  case (this fact is often called the “positive form of Chow’s Lemma”). For discrete-time systems it is known that, in general, this implication fails. However, for analytic discrete-time systems, there are some cases in which it has been already established that transitivity implies forward accessibility. For instance, this is known when  $f$  is a rational map (see [5]), or when  $x_0$  is a positively Poisson stable point for a some fixed diffeomorphism  $f(\cdot, u_0)$  (see [1]). In this paper we will strengthen considerably this last result, by proving the implication when  $x_0$  has a weak type of Poisson stability. Moreover it is shown that transitivity implies forward accessibility when there exists a transitive state  $x_0$  which is also a global “attractor” for  $\Sigma$ .

The paper is organized as follows. In section 2 we introduce some basic definitions and notations. In section 3 we associate to a discrete-time system  $\Sigma$  some families of vector fields whose orbits will be correlated to the geometry of reachable sets and orbits. These families of vector fields are an extension of those considered in the previous work [1], [2], [3], [4]; we consider their introduction one of the main contributions of this paper. Section 4 presents some of the connections between the sets of vector fields so introduced and reachability. In section 5 we prove partial results for smooth systems. Finally, in section 6, which deals with the analytic case, we give our main results.

## 2. Basic definitions

In this paper we study discrete-time nonlinear systems  $\Sigma$  of the type (1), where the state space  $\mathcal{X}$  and the control space  $U$  satisfy the following properties:

- $\mathcal{X}$  is a connected, second countable, Hausdorff, differentiable manifold of dimension  $n$ ,
- $U$  is a subset of  $\mathbb{R}^m$  such that  $U \subseteq \text{clos int } U$ , and any two points in the same connected component of  $U$  can be joined by a smooth curve lying entirely in  $\text{int } U$  (except possibly for endpoints).

Notice that when  $U \subseteq \mathbb{R}$  then the second assumption on  $U$  is automatically satisfied.

The system is of class  $C^k$ , with  $k = \infty$  or  $\omega$ , if the manifold  $\mathcal{X}$  is of class  $C^k$  and the function  $f : \mathcal{X} \times U \rightarrow \mathcal{X}$  is of class  $C^k$  (i.e., there exists a  $C^k$  extension of  $f$  to an open neighborhood of  $\mathcal{X} \times U$  in  $\mathcal{X} \times \mathbb{R}^m$ ). We call systems of class  $C^\infty$  *smooth systems* and those of class  $C^\omega$  *analytic systems*.

*Definition 1.* A system  $\Sigma$  is said to be *invertible* if for all  $u \in U$ , the function  $f_u : \mathcal{X} \rightarrow \mathcal{X}$  with  $f_u(x) = f(x, u)$  is a diffeomorphism.

We will be dealing with the class of invertible systems. For each  $u \in U$ , we will denote by  $f_u^{-1}$  the inverse function of  $f_u$ .

From now on, and unless otherwise stated, we assume that a fixed smooth invertible system  $\Sigma$  is given.

### 2.1. Some notations

For any fixed state  $x$  and any nonnegative integer  $k$  define:

$$\psi_{k,x}(\mathbf{u}) := f_{u_k, \dots, u_1}(x) \quad (2)$$

where  $\mathbf{u} = (u_k, \dots, u_1) \in U^k$ , and where  $f_{u_k, \dots, u_1}$  denotes  $f_{u_k} \circ \dots \circ f_{u_1}$ . If there exists an integer  $k \geq 0$  and a  $\mathbf{u} = (u_k, \dots, u_1) \in U^k$  such that  $\psi_{k,x}(\mathbf{u}) = z$ , we will write:

$$x \overset{\sim}{\underset{k}{\rightarrow}} z.$$

For each  $\mathbf{u}$ , let  $\rho_{k,x}(\mathbf{u})$  be the rank of  $\frac{\partial}{\partial \mathbf{u}} \psi_{k,x}[\mathbf{u}]$ . For each  $x \in \mathcal{X}$ , let:

$$\bar{\rho}_x := \max_{k \geq 0} \max_{\mathbf{u} \in U^k} \rho_{k,x}(\mathbf{u}). \quad (3)$$

Let:

$$\begin{cases} R^0(x) & := \{x\} \\ R^k(x) & := \{z \mid x \overset{\sim}{\underset{k}{\rightarrow}} z\}, \quad k = 1, 2, \dots, \end{cases}$$

$R^k(x)$  is the set of states *reachable from  $x$  in (exactly)  $k$  steps*. The following sets will also be very helpful:

$$\tilde{R}^k(x) := \{\psi_{k,x}(\mathbf{u}) \mid \mathbf{u} \in U^k, \rho_{k,x}(\mathbf{u}) = \bar{\rho}_x\},$$

which represents the set of states that are *maximal-rank reachable from  $x$  in (exactly)  $k$  steps*, and

$$\bar{R}^k(x) := \{\psi_{k,x}(\mathbf{u}) \mid \mathbf{u} \in U^k, \rho_{k,x}(\mathbf{u}) = n\}$$

which represents the set of states that are *nonsingularly reachable from  $x$  in  $k$  steps*. We let:

$$R(x) := \bigcup_{k \geq 0} R^k(x)$$

and analogously for  $\tilde{R}(x)$  and  $\bar{R}(x)$ .  $R(x)$  is the set of states reachable from  $x$ . The set  $\tilde{R}(x)$  is always nonempty, but the set  $\bar{R}(x)$  may be empty.

Recall that  $\Sigma$  is said to be *forward accessible from  $x$*  if and only if  $\text{int } R(x) \neq \emptyset$ . It can be proved that, for analytic systems  $\Sigma$ ,

$$\boxed{U \text{ connected} \Rightarrow \forall x \in \mathcal{X}, \tilde{R}(x) \text{ is dense in } R(x).} \quad (4)$$

For the proof of (4) we refer to Proposition 3.3 of [1]; the basic idea is to combine the analyticity assumption, which guarantees that the set of control sequences giving maximal rank is open and dense, with the assumption that  $U \subseteq \text{clos int } U$ , which gives that for each  $u \in U$  there exists a sequence  $u_n \in \text{int } U$  converging to  $u$ .

We also define the *controllable set to  $x$* , and the *orbit from  $x$* , as follows. Let:

$$\begin{cases} C^0(x) & := \{x\} \\ C^k(x) & := \{z \mid z \overset{\sim}{\underset{k}{\rightsquigarrow}} x\}, \quad k = 1, 2, \dots, \end{cases}$$

then the controllable set to  $x$  is:

$$C(x) := \bigcup_{k \geq 0} C^k(x).$$

Let:

$$\begin{cases} O^0(x) & := \{x\} \\ O^k(x) & := \{z \mid \exists z_1 \in O^{k-1} \text{ and } (z_1 \overset{\sim}{\underset{1}{\rightsquigarrow}} z \text{ or } z \overset{\sim}{\underset{1}{\rightsquigarrow}} z_1)\}, \quad k = 1, 2, \dots, \end{cases}$$

then the orbit from  $x$  is:

$$O(x) := \bigcup_{k \geq 0} O^k(x).$$

The system  $\Sigma$  is said to be *backward accessible from  $x$*  if and only if  $\text{int } C(x) \neq \emptyset$ ;  $\Sigma$  is said to be *transitive from  $x$*  if and only if  $\text{int } O(x) \neq \emptyset$ .

The following Lemma states a well known criterion for forward accessibility (for the proof see Proposition 3.2 in [1]).

**LEMMA 1** *Let  $\Sigma$  be a smooth invertible system. For each  $x \in \mathcal{X}$ , the following are equivalent:*

1.  $\text{int } R(x) \neq \emptyset$ ,
2.  $\text{int } \bar{R}(x) \neq \emptyset$ .

There is an analogous result for the transitivity property.

**LEMMA 2** *Let  $\Sigma$  be a smooth invertible system. Then, for each  $x \in \mathcal{X}$ , and for any positive integer  $k$  the following properties hold:*

1.  $\text{int } O^k(x) \neq \emptyset$  if and only if there exist a sequence of control values  $\mathbf{u} = (u_1, \dots, u_k)$ , and a sequence  $\epsilon = (\epsilon_1, \dots, \epsilon_k)$ , with  $\epsilon_i = \pm 1$ , such that, the following map:

$$\Psi_\epsilon : U^k \rightarrow \mathcal{X} : (v_1, \dots, v_k) \mapsto f_{v_k}^{\epsilon_k} \circ \dots \circ f_{v_1}^{\epsilon_1}(x)$$

has full rank at  $\mathbf{u}$ .

2.  $\text{int } O(x) \neq \emptyset$  if and only if there exists a positive integer  $k$  such that the previous conditions are satisfied for this  $k$ .

**Proof:** We prove 1. The sufficient part follows easily from the Implicit Function Theorem, so we need to see the converse. Notice that  $O^k(x)$  is given by the union, over all the different sequences of length  $k$  of  $\pm 1$ 's, of the images of maps of the type  $\Psi_\epsilon$ . Thus, arguing by contradiction, since this union is countable, the necessary part follows by Sard's Theorem and the fact that a countable union of set of measure zero has again measure zero. The second claim follows immediately from the first, by a similar argument. ■

### 3. A new class of vector fields associated to systems

Some Lie algebras of vector fields  $L, L^-, L^+, \Gamma, \Gamma^-, \Gamma^+$  were introduced in [4] (see also [2] and [3] for previous work) to study the controllability properties of invertible systems. Here, using the same vector fields, we will define slightly different Lie algebras, which allow us to derive stronger results.

Let  $\Sigma$  be a given smooth invertible system. First, for each  $u \in U$ , and each  $i = 1, \dots, m$ , we let  $X_{u,i}^+$ , and  $X_{u,i}^-$  be the following vector fields:

$$X_{u,i}^+(x) = \left. \frac{\partial}{\partial v_i} \right|_{v=0} f_u^{-1} \circ f_{u+v}(x), \tag{5}$$

$$X_{u,i}^-(x) = \left. \frac{\partial}{\partial v_i} \right|_{v=0} f_u \circ f_{u+v}^{-1}(x). \tag{6}$$

Given a vector field  $Y$  and a control value  $u \in U$ , we can define another vector field from  $Y$  by applying the change of coordinates given by the diffeomorphism  $f_u$ ,

$$(\text{Ad}_u Y)(x) = (df_u(x))^{-1} Y(f_u(x)). \tag{7}$$

Here  $df_u$  stands for the differential of  $f_u$  with respect to  $x$ . This is sometimes called the "pull-back of  $Y$  under the diffeomorphism  $f_u$ ." In the same way, but now using the diffeomorphism  $f_u^{-1}$ , we also define  $\text{Ad}_u^{-1}$ . We let:

$$\text{Ad}_{u_k \dots u_1}^{\epsilon_k \dots \epsilon_1} Y = \text{Ad}_{u_1}^{\epsilon_1} \dots \text{Ad}_{u_k}^{\epsilon_k} Y. \tag{8}$$

We define now:

$$\begin{aligned} \Gamma^+ &= \{ \text{Ad}_{u_k \dots u_1} X_{u_0,i}^+ \mid k \geq 0, 1 \leq i \leq m, u_0, \dots, u_k \in U \}, \\ \Gamma^- &= \{ \text{Ad}_{u_k \dots u_1}^{-1 \dots -1} X_{u_0,i}^- \mid k \geq 0, 1 \leq i \leq m, u_0, \dots, u_k \in U \}, \\ \Gamma &= \{ \text{Ad}_{u_k \dots u_1}^{\epsilon_k \dots \epsilon_1} X_{u_0,i}^{\epsilon_0} \mid k \geq 0, 1 \leq i \leq m, u_0, \dots, u_k \in U, \epsilon_0, \dots, \epsilon_k = \pm 1 \}. \end{aligned} \tag{9}$$

These previous definitions are the same as those given in [4].

For any finite sequence of controls  $\mu = (v_1, \dots, v_k)$ , we will denote by  $|\mu|$  the length  $k$  of this sequence. Now, for any such  $\mu$  we define the following Lie algebras:

$$\begin{aligned} L_\mu^+ &= \text{Lie} \{X_{u,i}^+, \text{Ad}_{v_1} \cdots \text{Ad}_{v_l} X_{u,i}^+ \mid 1 \leq l \leq k, 1 \leq i \leq m, u \in U\}, \\ L_\mu^- &= \text{Lie} \{X_{u,i}^+, \text{Ad}_{v_1}^{-1} \cdots \text{Ad}_{v_l}^{-1} X_{u,i}^+ \mid 1 \leq l \leq k, 1 \leq i \leq m, u \in U\}, \\ L_\mu &= \text{Lie} \{X_{u,i}^+, \text{Ad}_{v_1}^\alpha \cdots \text{Ad}_{v_l}^\alpha X_{u,i}^+ \mid 1 \leq l \leq k, 1 \leq i \leq m, \alpha = \pm 1, u \in U\}. \end{aligned} \quad (10)$$

Notice that, if  $\mu = (0, \dots, 0)$ , then the previous Lie algebras coincide respectively with the Lie algebras  $L_k^+$ ,  $L_k^-$ , and  $L_k$  defined in [4].

The way in which these algebras of vector fields will be used is as follows. We will show that the integral manifolds that they give rise to help in describing the geometry of the sets  $R(x)$ ,  $C(x)$ , and  $O(x)$ , in the sense that the corresponding motions (forward and/or backwards) of our system lie in these orbits (see Propositions 3 and 4 given below). Moreover we will see that the notions of forward and backward accessibility, as well as transitivity, are related to the full dimensionality of the tangent subspaces at each point, corresponding to these families of vector fields (see Proposition 7).

Along these lines, the following two results were proved in [4], Corollary 4.4 and Theorem 2 part (a) respectively. We repeat their statements here for the convenience of the reader.

**PROPOSITION 1** *Let  $\Sigma$  be a smooth invertible system. If  $y \in \tilde{R}(x)$ , and  $\dim \text{Lie } \Gamma^+(y) = n$ , then  $\Sigma$  is forward accessible from  $x$ .*

**PROPOSITION 2** *Let  $\Sigma$  be a smooth invertible system, then the following conditions are equivalent:*

1.  $\Sigma$  is forward accessible from all  $x \in \mathcal{X}$ ,
2.  $\dim \Gamma^+(x) = n$  for all  $x \in \mathcal{X}$ ,
3.  $\dim \text{Lie } \Gamma^+(x) = n$  for all  $x \in \mathcal{X}$ .

#### 4. Some general properties

**Definition 2.** A map  $h : \mathcal{X} \rightarrow \mathcal{X}$  is *balanced* (with respect to the given system  $\Sigma$ ) if it can be written as  $f_{u_1}^{\epsilon_1} \circ \cdots \circ f_{u_k}^{\epsilon_k}$  for some  $u_i \in U$ ,  $\epsilon_i = \pm 1$  and with  $\sum_{i=1}^k \epsilon_i = 0$ .

**LEMMA 3** *Assume that  $y = f_{u_1}^{\epsilon_1} \circ \cdots \circ f_{u_k}^{\epsilon_k}(x)$ , with  $u_i \in U$ ,  $\epsilon_i = \pm 1$  for each  $i = 1, \dots, k$ . Consider the partial sums:*

$$l_i = \sum_{j=1}^i \epsilon_j \quad \text{for } i = 1, \dots, k.$$

Assume that  $l_k = 0$ , and let  $l = \max \{ |l_i| \mid i = 1, \dots, k \}$ . Pick any sequence  $v_1, \dots, v_l$  of elements in  $U$ . Then one can write:

$$y = g_k \circ \dots \circ g_1(x), \quad (11)$$

where each  $g_i$  is a balanced map of the form:

$$g_i = h^i \circ f_{u_i}^{\epsilon_i} \circ k^i,$$

and, for each  $i = 1, \dots, k$ , the maps  $h^i$  and  $k^i$  have the form:

$$\begin{cases} h^i &= f_{v_1}^{\alpha(i)} \circ \dots \circ f_{v_{\mu(i)}}^{\alpha(i)} \\ k^i &= f_{v_{\nu(i)}}^{-\alpha(i)} \circ \dots \circ f_{v_1}^{-\alpha(i)} \end{cases}$$

and  $\alpha(i) = \pm 1$ ,  $|\mu(i) - \nu(i)| = 1$ .

**Proof:** Let  $\alpha(i) = \text{sgn}(l_i) \in \{0, 1, -1\}$  and write also  $l_0 = 0$ . For  $i = 1, \dots, k$ , define:

$$g_i = h^i \circ f_{u_i}^{\epsilon_i} \circ k^i$$

as follows:

$$\begin{cases} k^i &= f_{v_{|l_{i-1}|}}^{\alpha_{i-1}} \circ \dots \circ f_{v_1}^{\alpha_{i-1}} \\ h^i &= f_{v_1}^{-\alpha_{i-1}} \circ \dots \circ f_{v_{|l_i|}}^{-\alpha_{i-1}}. \end{cases}$$

The  $g_i$  are balanced maps, since  $l_i = \epsilon_i + l_{i-1}$ . Moreover, by definition, it holds that:

$$k^i = (h^{i-1})^{-1} \quad i = 2, \dots, k.$$

We now prove, by induction on  $i$ , that:

$$g_i \circ \dots \circ g_1 = h^i \circ f_{u_i}^{\epsilon_i} \circ \dots \circ f_{u_1}^{\epsilon_1}. \quad (12)$$

If  $i = 1$ , then  $l_1 = \epsilon_1$  and  $l_0 = 0$ , so (12) is obvious. The induction step for  $i > 1$  is provided by:

$$g_i \circ \dots \circ g_1 = (h^i \circ f_{u_i}^{\epsilon_i} \circ k^i)(h^{i-1} \circ f_{u_{i-1}}^{\epsilon_{i-1}} \circ \dots \circ f_{u_1}^{\epsilon_1}) = h^i \circ f_{u_i}^{\epsilon_i} \circ \dots \circ f_{u_1}^{\epsilon_1}.$$

Now, since  $l_k = 0$ ,  $h^k = \text{identity}$ , so (12) gives the desired conclusion. ■

If  $\Delta$  is a set of smooth vector fields and  $x \in \mathcal{X}$ , we denote by  $\text{Orb}_\Delta(x)$  the orbit of  $\Delta$  passing through  $x$ . Recall that, by definition,  $y \in \text{Orb}_\Delta(x)$  if and only if there exists an absolute continuous curve  $\gamma : [a, b] \rightarrow \mathcal{X}$  such that  $\gamma(a) = x$ ,  $\gamma(b) = y$ , and there exist  $t_i$  with  $a = t_0 < t_1 < \dots < t_r = b$ , and vector fields  $X_i \in \Delta$  such that  $\gamma$  restricted to  $[t_i, t_{i+1}]$  is an integral curve of  $X_i$  or  $-X_i$ .

The following fact about continuous-time systems is well known, we repeat it here since it is needed in the proof of the next Proposition. Let  $\Sigma$  be a continuous-time system described by the following set of controlled differential equations:

$$\dot{x}(t) = f(x(t), u(t)). \quad (13)$$

(For a precise definition of continuous-time systems see for instance [7], [8], or [6].) Given two states  $x_1, x_2$ , we say that  $x_1$  can be controlled to  $x_2$  if there exist some interval  $[0, T]$ ,  $T \geq 0$ , and an essentially bounded measurable map  $u(\cdot)$  defined on  $[0, T]$ , such that the solution of the differential equation (13) with this  $u(\cdot)$ , and with  $x(0) = x_1$ , is defined on all the interval  $[0, T]$ , and  $x(T) = x_2$ . We say that  $x_2$  is weakly reachable from  $x_1$  if there exists a finite sequence of states  $z_1 = x_1, z_2, \dots, z_k = x_2$  such that for each  $l = 2, \dots, k$  either  $z_{l-1}$  can be controlled to  $z_l$  or  $z_l$  can be controlled to  $z_{l-1}$ . Let:

$$\Delta = \{f(\cdot, u)\}_{u \in U}.$$

Given these notations, the following fact holds (for the proof see Proposition 2.16 in [7]):

LEMMA 4  $x_1$  is weakly accessible from  $x_2 \Leftrightarrow x_1 \in \text{Orb}_\Delta(x_2)$ .

The following result generalizes [4] Proposition 5.2, which deals with the very special case in which  $\mu = \underbrace{0, \dots, 0}_k$ .

PROPOSITION 3 Let  $k$  be any nonnegative integer. Assume that  $\Sigma$  is a smooth system with connected  $U$ . Then:

1.  $R^k(x) \subseteq \text{Orb}_{L_\mu^-}(y)$  for all  $y \in R^k(x)$  and for all  $\mu$  with  $|\mu| \geq k$ ,
2.  $C^k(x) \subseteq \text{Orb}_{L_\mu^+}(y)$  for all  $y \in C^k(x)$  and for all  $\mu$  with  $|\mu| \geq k - 1$ .

**Proof:** Note that in the above statements, it is always sufficient to prove the inclusion for any particular  $y$  in  $R^k(x)$  or, for the second part, in  $C^k(x)$ , since  $\text{Orb}_\Delta(z) = \text{Orb}_\Delta(y)$  if  $z \in \text{Orb}_\Delta(y)$ , for any set of vector fields  $\Delta$ .

We prove now the first part. Let  $z \in R^k(x)$ . Then:

$$z = f_{u_1} \circ \dots \circ f_{u_k}(x),$$

for some  $(u_1, \dots, u_k) \in U^k$ . Now take any  $\mu$  with  $|\mu| \geq k$ ,  $\mu = (v_1, \dots, v_k, \dots, v_{|\mu|})$ , and consider the state:

$$y = f_{v_1} \circ \dots \circ f_{v_k}(x) \in R^k(x).$$

We will prove that  $z$  is in  $\text{Orb}_{L_\mu^-}(y)$ . We have:

$$z = f_{u_1} \circ \dots \circ f_{u_k} \circ f_{v_k}^{-1} \circ \dots \circ f_{v_1}^{-1}(y).$$



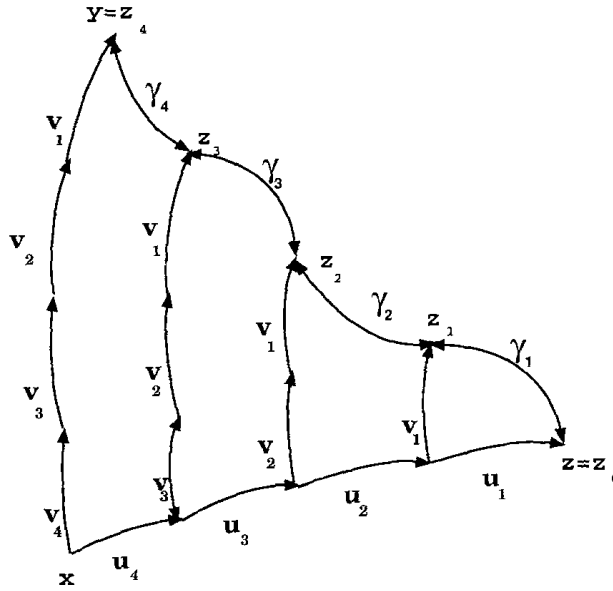


Figure 1. Case with  $k = 4$ .

We can write the previous equation as:

$$z = g_1 \circ \dots \circ g_k(y),$$

where:

$$\begin{cases} g_1 &= f_{u_1} \circ f_{v_1}^{-1} \\ g_i &= f_{v_1} \circ \dots \circ f_{v_{i-1}} \circ f_{u_i} \circ f_{v_i}^{-1} \circ \dots \circ f_{v_1}^{-1} \quad \text{for } i = 2, \dots, k. \end{cases}$$

Letting  $z_0 = z$ , and, for  $i = 1, \dots, k$ ,  $z_i = g_i^{-1}(z_{i-1})$ , we have  $z_k = y$  (see Fig. 1).

We prove that  $z_i \in \text{Orb}_{L_\mu^-}(z_{i-1})$ , from which the desired conclusion follows by transitivity.

Given the assumptions on the set  $U$ , for  $i = 1, \dots, k$ , there exist smooth curves  $\alpha_i : [0, T] \rightarrow U$ , such that  $\alpha_i(0) = v_i$ ,  $\alpha_i(T) = u_i$ , and  $\alpha_i(t) \in \text{int } U$ . Let

$$\gamma_i(t) = f_{v_1} \circ \dots \circ f_{v_i} \circ f_{\alpha_i(t)}^{-1} \circ f_{v_{i-1}}^{-1} \circ \dots \circ f_{v_1}^{-1}(z_{i-1}), \tag{14}$$

for  $i = 1, \dots, k$ , and  $t \in [0, T]$ .

Note that  $\gamma_i(0) = z_{i-1}$ ,  $\gamma_i(T) = z_i$ , and, for any fixed  $t$ ,

$$\gamma_i(t + \epsilon) = f_{v_1} \circ \dots \circ f_{v_i} \circ f_{\alpha_i(t+\epsilon)}^{-1}(q_i),$$

where

$$q_i = f_{v_{i-1}}^{-1} \circ \dots \circ f_{v_1}^{-1}(z_{i-1}) = f_{\alpha_i(t)} \circ f_{v_i}^{-1} \circ \dots \circ f_{v_1}^{-1}(\gamma_i(t)).$$

Therefore:

$$\begin{aligned} \frac{\partial}{\partial t} \gamma_i(t) &= \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} f_{v_1} \circ \cdots \circ f_{v_i} \circ f_{\alpha_i(t+\epsilon)}^{-1} \circ f_{\alpha_i(t)} \circ f_{v_i}^{-1} \circ \cdots \circ f_{v_1}^{-1}(\gamma_i(t)) \\ &= - \sum_{j=1}^m \alpha'_{ij}(t) \text{Ad}_{v_1}^{-1} \cdots \text{Ad}_{v_i}^{-1} X_{\alpha_i(t),j}^+(\gamma_i(t)), \end{aligned} \quad (15)$$

where  $\alpha_{ij}$  is the  $j$ -th component of the curve  $\alpha_i$ . For each  $i = 1, \dots, k$ , we may interpret equation (15) as the equation of a continuous-time system with state trajectory  $\gamma_i(t)$  and control  $(\alpha_i(t), \alpha'_{i1}(t), \dots, \alpha'_{im}(t))$ , so by Lemma 4 it follows that  $z_i \in \text{Orb}_{L_\mu^-}(z_{i-1})$ , as desired.

Now we prove the second part. The proof follows the same lines as the proof of the first part. Let  $z \in C^k(x)$ . Then:

$$z = f_{u_1}^{-1} \circ \cdots \circ f_{u_k}^{-1}(x),$$

for some  $(u_1, \dots, u_k) \in U^k$ . Now take any  $\mu$  with  $|\mu| \geq k - 1$ ,  $\mu = (v_1, \dots, v_{k-1}, v_k, \dots, v_l)$ . (If  $|\mu| = k - 1$ , we choose  $v_k$  arbitrarily; it will be clear later that we need  $v_k$  only for technical reasons.) Let  $y \in C^k(x)$  be the following state:

$$y = f_{v_1}^{-1} \circ \cdots \circ f_{v_k}^{-1}(x).$$

We will prove that  $z$  is in  $\text{Orb}_{L_\mu^+}(y)$ . We have:

$$z = f_{u_1}^{-1} \circ \cdots \circ f_{u_k}^{-1} \circ f_{v_k} \circ \cdots \circ f_{v_1}(y).$$

We can write the previous equation as:

$$z = g_1 \circ \cdots \circ g_k(y),$$

where:

$$\begin{cases} g_1 &= f_{u_1}^{-1} \circ f_{v_1} \\ g_i &= f_{v_1}^{-1} \circ \cdots \circ f_{v_{i-1}}^{-1} \circ f_{u_i}^{-1} \circ f_{v_i} \circ \cdots \circ f_{v_1} \quad \text{for } i = 2, \dots, k. \end{cases}$$

Letting  $z_k = y$ , and, for  $i = 1, \dots, k$ ,  $z_{k-i} = g_{k-i+1}(z_{k-i+1})$ , we have (see Fig. 2):

$$\begin{cases} z_0 &= z \\ z_{i-1} &= f_{v_1}^{-1} \circ \cdots \circ f_{v_{i-1}}^{-1} \circ f_{u_i}^{-1} \circ f_{v_i} \circ \cdots \circ f_{v_1}(z_i). \end{cases}$$

To prove our statement, it is now sufficient to show that, for  $i = 1, \dots, k$ ,  $z_{i-1} \in \text{Orb}_{L_\mu^+}(z_i)$ .

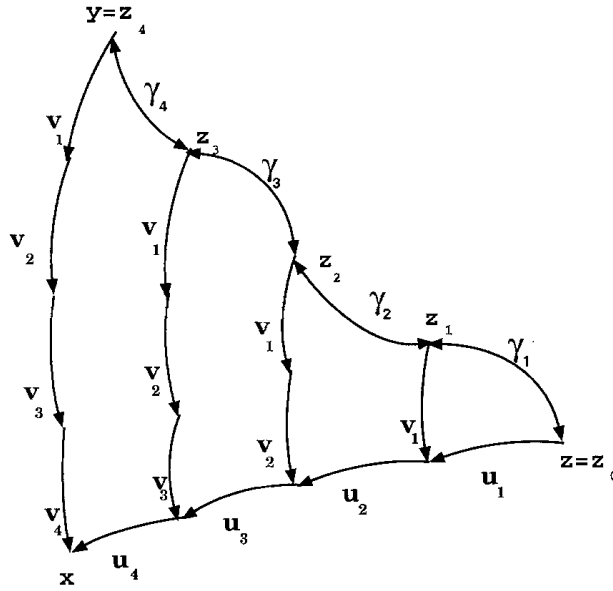


Figure 2. Case with  $k = 4$ .

As before, for each  $i = 1, \dots, k$ , there exists a smooth curve  $\alpha_i : [0, T] \rightarrow U$  such that  $\alpha_i(0) = v_i$ ,  $\alpha_i(T) = u_i$ , and  $\alpha_i(t) \in \text{int } U$ . Let

$$\gamma_i(t) = f_{v_1}^{-1} \circ \dots \circ f_{v_{i-1}}^{-1} \circ f_{\alpha_i(t)}^{-1} \circ f_{v_i} \circ \dots \circ f_{v_1}(z_i), \tag{16}$$

for  $i = 1, \dots, k$ , and  $t \in [0, T]$ .

Arguing as in the first case, we conclude:

$$\begin{aligned} \frac{\partial}{\partial t} \gamma_i(t) &= \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} f_{v_1}^{-1} \circ \dots \circ f_{v_{i-1}}^{-1} \circ f_{\alpha_i(t+\epsilon)}^{-1} \circ f_{\alpha_i(t)} \circ f_{v_{i-1}} \circ \dots \circ f_{v_1}(\gamma_i(t)) \\ &= - \sum_{j=1}^m \alpha'_{ij}(t) \text{Ad}_{v_1} \dots \text{Ad}_{v_{i-1}} X_{\alpha_i(t),j}^+(\gamma_i(t)). \end{aligned} \tag{17}$$

■

We now give a similar result for the *zero-time* orbit of a point  $x$ . We introduce the following notation:

$$\begin{aligned} O_k^0(x) &= \{y = f_{u_s}^{\epsilon_s} \circ \dots \circ f_{u_1}^{\epsilon_1}(x) \mid s \geq 0, \sum_{i=1}^s \epsilon_i = 0, u_j \in U, \\ &\text{and } |\sum_{i=1}^j \epsilon_i| \leq k \ \forall j = 1, \dots, s\}, \end{aligned} \tag{18}$$

with  $k \in \mathbb{Z}$ ,  $k \geq 0$ .

PROPOSITION 4 *Let  $k$  be any nonnegative integer, and assume that  $\Sigma$  is a smooth invertible system with connected  $U$ . Then, for all  $x \in \mathcal{X}$ , all  $y \in O_k^0(x)$ , and all sequences of control values  $\mu$  with  $|\mu| \geq k$ , we have:*

$$O_k^0(x) \subseteq \text{Orb}_{L_\mu}(y).$$

**Proof.** Notice first that  $x \in O_k^0(x)$ , by using the empty sequence. We will prove that for each  $x$ ,

$$O_k^0(x) \subseteq \text{Orb}_{L_\mu}(x). \quad (19)$$

The general statement follows from this, since  $y \in O_k^0(x) \subseteq \text{Orb}_{L_\mu}(x)$  implies

$$\text{Orb}_{L_\mu}(x) = \text{Orb}_{L_\mu}(y).$$

Let  $y \in O_k^0(x)$ . Then:

$$y = f_{u_s}^{\epsilon_s} \circ \cdots \circ f_{u_1}^{\epsilon_1}(x)$$

with  $\sum_{i=1}^s \epsilon_i = 0$  and  $k \geq |\sum_{i=1}^j \epsilon_i|$  for any  $j = 1, \dots, s$ . Now take any  $\mu$  with  $|\mu| \geq k$ ,  $\mu = (v_1, \dots, v_k, \dots, v_{|\mu|})$ . We can now apply Lemma 3, with  $l = k$ , and we obtain:

$$y = g_s \circ \cdots \circ g_1(x).$$

Here, each  $g_i$  is a balanced map of the form:

$$g_i = f_{v_1}^{\alpha(i)} \circ \cdots \circ f_{v_{\mu(i)}}^{\alpha(i)} \circ f_{u_i}^{\epsilon_i} \circ f_{v_{\nu(i)}}^{-\alpha(i)} \circ \cdots \circ f_{v_1}^{-\alpha(i)}$$

with  $\alpha(i) = \pm 1$ , and  $|\mu(i) - \nu(i)| = 1$ . Now let:

$$\begin{cases} z_0 & = x \\ z_{i+1} & = g_{i+1}(z_i) \quad \text{for } i = 0, \dots, s-1; \end{cases}$$

thus  $z_s = y$  (see Fig. 3).

Let  $\rho(i) = \max\{\mu(i), \nu(i)\}$ . Given the assumptions on the set  $U$ , for each  $i = 1, \dots, s$ , there exists a smooth curve  $\beta_i : [0, T] \rightarrow U$ , such that  $\beta_i(0) = v_{\rho(i)}$ ,  $\beta_i(T) = u_i$ , and  $\beta_i(t) \in \text{int } U$ . Let:

$$\gamma_i(t) = f_{v_1}^{\alpha(i)} \circ \cdots \circ f_{v_{\mu(i)}}^{\alpha(i)} \circ f_{\beta_i(t)}^{\epsilon_i} \circ f_{v_{\nu(i)}}^{-\alpha(i)} \circ \cdots \circ f_{v_1}^{-\alpha(i)}(z_{i-1}) \quad (20)$$

for  $i = 1, \dots, s$ , and  $t \in [0, T]$ . Thus it holds that  $\gamma_i(0) = z_{i-1}$  and  $\gamma_i(T) = z_i$ . To prove (19), it is enough to establish that  $z_{i-1} \in \text{Orb}_{L_\mu}(z_i)$ .

If  $\alpha(i) = 1$  and  $\rho(i) = \mu(i)$ , then necessarily  $\epsilon_i = -1$ . Thus equation (20) is of the same type as equation (14), and, as in (15), we may conclude that:

$$\frac{\partial}{\partial t} \gamma_i(t) = - \sum_{j=1}^m \beta_{ij}^t(t) \text{Ad}_{v_1}^{-1} \cdots \text{Ad}_{v_{\mu(i)}}^{-1} X_{\beta_i(t), j}^+(\gamma_i(t)),$$

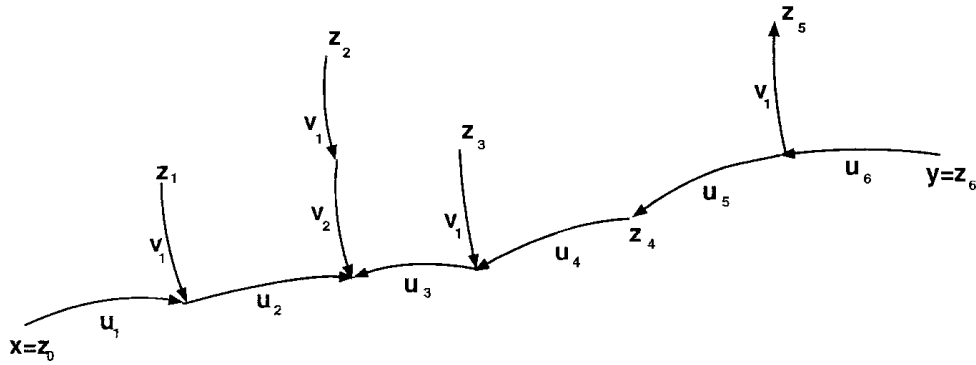


Figure 3. Case with  $s = 6, k = 3$ .

where  $\beta_{ij}$  denotes the  $j$ -th component of the curve  $\beta_i$ . If  $\alpha(i) = 1$  and  $\rho(i) = \nu(i)$ , then necessarily  $\epsilon_i = 1$ . In this case, instead of considering the curve  $\gamma_i(t)$ , we consider the following curve:

$$\gamma_i^{-1}(t) = f_{v_1}^{\alpha(i)} \circ \dots \circ f_{v_{\nu(i)}}^{\alpha(i)} \circ f_{\beta_i(t)}^{-\epsilon_i} \circ f_{v_{\mu(i)}}^{-\alpha(i)} \circ \dots \circ f_{v_1}^{-\alpha(i)}(z_i), \quad (21)$$

which joins  $z_i$  to  $z_{i-1}$ . Since equation (21) is again of the same type as equation (14), now we can conclude as before.

If  $\alpha(i) = -1$  and  $\rho(i) = \nu(i)$ , then necessarily  $\epsilon_i = -1$ . Thus equation (20) is of the same type as equation (16), and, as in (17), we may conclude that:

$$\frac{\partial}{\partial t} \gamma_i(t) = - \sum_{j=1}^m \beta'_{ij}(t) \text{Ad}_{v_1} \cdots \text{Ad}_{v_{\mu(i)}} X_{\beta_i(t),j}^+(\gamma_i(t)).$$

Finally, if  $\alpha(i) = -1$  and  $\rho(i) = \mu(i)$ , then necessarily  $\epsilon_i = 1$ . So we can argue as before by considering the curve  $\gamma_i^{-1}(t)$  instead of  $\gamma_i$ . ■

### 5. New results on accessibility

In this section we present some new results for systems with connected control space  $U$ .

PROPOSITION 5 *Let  $\Sigma$  be any smooth invertible system. If  $y \in R^k(x)$ , and  $\mu = (v_1, \dots, v_k)$  are such that:*

$$y = f_{v_1} \circ \dots \circ f_{v_k}(x),$$

then

$$\dim \text{Lie } \Gamma^+(x) \geq \dim L_\mu(y). \quad (22)$$

**Proof:** Assume that  $\dim L_\mu(y) = r$ , and let  $Y_1, \dots, Y_r$  be vector fields in  $L_\mu$  so that  $\{Y_1(y), \dots, Y_r(y)\}$  is a basis of  $L_\mu(y)$ . Without loss of generality, we may assume that each  $Y_i$  is a vector field involving Lie brackets of a finite numbers of vector fields of the type:

$$X_{u,j_1}^+, \text{ or } \text{Ad}_{v_1} \cdots \text{Ad}_{v_l} X_{u,j_2}^+, \text{ or } \text{Ad}_{v_1}^{-1} \cdots \text{Ad}_{v_l}^{-1} X_{u,j_3}^+,$$

with  $l \leq k$ , and  $j_i \in \{1, \dots, m\}$ .

Consider, for  $i = 1, \dots, r$ , the following vector fields:

$$Z_i = \text{Ad}_{v_k} \cdots \text{Ad}_{v_1} Y_i.$$

These are linearly independent at  $x$ ; in fact:

$$\begin{aligned} \text{Ad}_{v_k} \cdots \text{Ad}_{v_1} Y_i(x) &= d(f_{v_1} \circ \cdots \circ f_{v_k})^{-1}(x) Y_i(f_{v_1} \circ \cdots \circ f_{v_k}(x)) \\ &= d(f_{v_1} \circ \cdots \circ f_{v_k})^{-1}(x) Y_i(y). \end{aligned}$$

Moreover, it follows recursively from the fact that, for any two vector fields  $X_1$  and  $X_2$ ,

$$\text{Ad}_v[X_1, X_2] = [\text{Ad}_v X_1, \text{Ad}_v X_2], \quad \text{for any } v \in U,$$

that  $Z_i \in \text{Lie } \Gamma^+$  for each  $i = 1, \dots, r$ . Thus (22) holds. ■

*Definition 3.* For any nonnegative integer  $k$ , and for  $x \in \mathcal{X}$  we say that:

1.  $x$  is  $k$ -forward accessible if  $\text{int } R^k(x) \neq \emptyset$ ,
2.  $x$  is  $k$ -backward accessible if  $\text{int } C^k(x) \neq \emptyset$ ,
3.  $x$  is  $k$ -transitive if  $O_k^0(x)$  is open.

Recall that if  $\mu = (v_1, \dots, v_k)$  is any finite sequence of controls, then we will denote by  $|\mu|$  the length  $k$  of this sequence.

**PROPOSITION 6** *Let  $\Sigma$  be a smooth invertible system, with connected  $U$ . Then:*

1. *if  $x$  is  $k$ -forward accessible, then for all  $\mu$  with  $|\mu| \geq k - 1$ ,  $\text{Orb}_{L_\mu^+}(x)$  is open;*
2. *if  $x$  is  $k$ -backward accessible, then for all  $\mu$  with  $|\mu| \geq k$ ,  $\text{Orb}_{L_\mu^-}(x)$  is open;*
3. *if  $x$  is  $k$ -transitive, then for all  $\mu$  with  $|\mu| \geq k$ ,  $\text{Orb}_{L_\mu}(x)$  is open.*

**Proof:** (1) Since  $x$  is  $k$ -forward accessible, there exists an open set  $V$  contained in  $R^k(x)$ . Let  $\mu = (v_1, \dots, v_{k-1}, v_k, \dots, v_l)$ , where if  $|\mu| = k - 1$ , we choose  $v_k$  arbitrarily, since it will be clear later that  $v_k$  is needed only for technical reasons. Let

$$W = f_{v_1}^{-1} \circ \cdots \circ f_{v_k}^{-1}(V).$$

We will prove that  $W \subset \text{Orb}_{L_\mu^+}(x)$ , from which (1) follows.

Pick any  $y \in W$ ; then there exists  $z \in V$  such that  $y \in C^k(z)$ . Moreover, since  $z \in R^k(x)$ , we also have  $x \in C^k(z)$ ; thus, applying the second result in Proposition 3, we can conclude:

$$y \in \text{Orb}_{L_\mu^+}(x)$$

as desired.

(2) We proceed as in (1). Thus if  $\mu = (v_1, \dots, v_k, \dots, v_{|\mu|})$ , we let  $V, W$  be two open sets chosen so that:

$$V \subset C^k(x),$$

$$W = f_{v_1} \circ \dots \circ f_{v_k}(V),$$

and we will prove that  $W \subset \text{Orb}_{L_\mu^-}(x)$ .

Pick any  $y \in W$ , then  $y \in R^k(z)$  for some  $z \in V$ . Moreover, we also have  $x \in R^k(z)$ ; thus, applying the first result in Proposition 3, we can conclude:

$$y \in \text{Orb}_{L_\mu^-}(x)$$

as desired.

(3) This part is an obvious consequence of Proposition 4, since  $x \in O_k^0(x)$  implies:

$$O_k^0(x) \subset \text{Orb}_{L_\mu}(x).$$

Thus  $\text{Orb}_{L_\mu}(x)$  is open. ■

## 6. Analytic case

Throughout all this section we assume that an analytic invertible system  $\Sigma$ , with connected control space  $U$ , is given. All the results presented here hold under these assumptions.

**PROPOSITION 7** *Denote by  $\mu$  a sequence of control values.*

1. *If  $x$  is  $k$ -forward accessible then for all  $\mu$  with  $|\mu| \geq k - 1$ ,  $\dim L_\mu^+(x) = n$ .*
2. *If  $x$  is  $k$ -backward accessible then for all  $\mu$  with  $|\mu| \geq k$ ,  $\dim L_\mu^-(x) = n$ .*
3. *If  $x$  is  $k$ -transitive then for all  $\mu$  with  $|\mu| \geq k$ ,  $\dim L_\mu(x) = n$ .*

**Proof:** We will prove only the first statement; the second and the third follow using the same arguments. Since  $\Sigma$  is analytic, applying a theorem of Nagano (see [9], section 9, or [7], Theorem 5), we know that the distribution associated to  $L_\mu^+$  is integrable. Moreover, if  $x$  is  $k$ -forward accessible then, by the first result in Proposition 6, we have that  $\text{Orb}_{L_\mu^+}(x)$  is open. Thus we can conclude that  $\dim L_\mu^+(x) = n$ , as desired. ■

PROPOSITION 8 *If  $y \in R^l(x)$ , and  $y$  is  $k$ -transitive for some  $k \leq l$ , then:*

$$\dim \text{Lie } \Gamma^+(x) = n.$$

**Proof:** The statement is an immediate consequence of Proposition 5, and Proposition 7 part 3. ■

We say that a system  $\Sigma$  is *bounded transitive* if there exists a nonnegative integer  $k$  such that  $O^k(x)$  has non-empty interior for all  $x \in \mathcal{X}$ . Notice that if  $O^k(x)$  has non-empty interior then, clearly,  $O_{2k}^0(x)$  is open. Thus the following Corollary is a consequence of the previous Proposition and of Proposition 2:

COROLLARY 1 *If  $\Sigma$  is bounded transitive, then  $\Sigma$  is also forward accessible.* ■

*Remark 1.* Notice that if the state space  $\mathcal{X}$  is compact,  $\Sigma$  transitive implies  $\Sigma$  bounded transitive. To see this fact we argue as follows. For each  $x \in \mathcal{X}$ , denote by  $k_x$  any positive integer such that  $x$  is  $k_x$ -transitive. Then, by continuity, there exists an open neighborhood  $O_x$  of  $x$  such that if  $y \in O_x$  then  $y$  is  $k_x$ -transitive. These open sets  $\{O_x\}_{x \in \mathcal{X}}$  cover  $\mathcal{X}$ . Let  $\{O_{x_i}\}_{i=1, \dots, l}$  be a finite subcover. Then, letting  $k = \max\{k_{x_i}\}_{i=1, \dots, l}$ , we have that  $O^k(x)$  has nonempty interior for all  $x \in \mathcal{X}$ ; thus  $\Sigma$  is bounded transitive.

So, in particular, the previous result implies that, if the state space  $\mathcal{X}$  is compact, then a system  $\Sigma$  is transitive if and only if it is forward accessible. This result was proved in a very different way before, see [1] Theorem 2.

*Definition 4.* Given  $x \in \mathcal{X}$ , we let:

$$\omega(x) = \{y \in \mathcal{X} \mid \exists x_{n_k} \in R^{n_k}(x), x_{n_k} \rightarrow y, n_k \rightarrow \infty\}.$$

Note that, for control space consisting of only one element,  $U = \{u\}$ , the previous Definition is the standard  $\omega$ -limit set.

PROPOSITION 9 *If  $y \in \omega(x)$  and  $y$  is transitive, then  $\dim \text{Lie } \Gamma^+(x) = n$ .*

**Proof:** Since  $y$  is transitive, there exists a nonnegative integer  $k$  such that  $O_k^0(y)$  is open. Thus there exists a neighborhood  $W$  of  $y$ , such that  $O_k^0(z)$  is open for all  $z \in W$ . Since  $y \in \omega(x)$ , there exists  $\tilde{y} \in R^l(x) \cap W$  with  $l \geq k$ ; thus, by Proposition 8, we conclude the desired result. ■

Recall that if  $Y$  is a vector field on a manifold  $\mathcal{X}$ , one says that  $x \in \mathcal{X}$  is a *positively Poisson stable point* for  $Y$  if and only if for each neighbourhood  $V$  of  $x$  and each  $T \geq 0$  there exists some  $t > T$  such that  $e^{tY}(x) \in V$ , where  $e^{tY}(\cdot)$  represents the flow of  $Y$ .

Analogously, one can define positive Poisson stability in discrete time, for a given diffeomorphism  $f : \mathcal{X} \rightarrow \mathcal{X}$ , as done in [1]. Next we define a generalization of this concept to systems.



*Definition 5.* Given a system  $\Sigma$ , the point  $x \in \mathcal{X}$  is said to be *positively Poisson stable* for  $\Sigma$  if and only if for each neighbourhood  $V$  of  $x$  and each integer  $N \geq 0$  there exist some integer  $k > N$ , and  $(v_1, \dots, v_k) \in U^k$  such that  $f_{v_k} \circ \dots \circ f_{v_1}(x) \in V$ .

The following result generalizes [1] Theorem 1, which deals with the very special case of states which are positively Poisson stable for the diffeomorphism  $f_0$ . The proof in this case requires the full machinery just introduced.

**THEOREM 1** *Let  $x \in \mathcal{X}$  be a positively Poisson stable point for  $\Sigma$ . Then  $x$  is transitive if and only if  $x$  is forward accessible.*

**Proof:** Notice that  $x$  positively Poisson stable for  $\Sigma$  means that  $x \in \omega(x)$ . Thus from Proposition 9, we know that if  $x$  is transitive then  $\dim \text{Lie } \Gamma^+(x) = n$ . So there exists a neighbourhood  $W$  of  $x$  such that  $\dim \text{Lie } \Gamma^+(y) = n$  for all  $y \in W$ . Since  $x \in \omega(x)$ , and, since by the analyticity assumption  $\tilde{R}(x)$  is dense in  $R(x)$  (see (4)), there exists some  $y \in W \cap \tilde{R}(x)$ . Thus we can conclude that  $x$  is forward accessible, using Proposition 1.

The other implication being obvious, the statement is proved. ■

*Definition 6.* Given a system  $\Sigma$ , we say that  $\Sigma$  is *weakly asymptotically controllable* to a state  $\bar{x}$  if  $\bar{x} \in \text{clos } R(x)$  for all  $x \in \mathcal{X}$ .

*Remark 2.* Saying that  $\bar{x} \in \text{clos } R(x)$  is *not* equivalent to saying that there exists a fixed infinite sequence  $u_i, i > 0$ , such that the sequence  $x_i = f_{u_i, \dots, u_1}(x)$  converges to  $\bar{x}$ , notion which is sometimes is called *asymptotic controllability*. Our notion is weaker, in fact, as it is proved in the next Lemma, it is equivalent to  $\bar{x} \in \omega(x)$ , which, in general, implies only that a subsequence of the  $x_i$ 's converges to  $\bar{x}$ .

**LEMMA 5** *Assume that  $(\bar{x}, \bar{u}) \in \mathcal{X} \times U$  is not an equilibrium pair, that is,  $f(\bar{x}, \bar{u}) \neq \bar{x}$ . Then the following properties are equivalent:*

1.  $\Sigma$  is weakly asymptotically controllable to  $\bar{x}$ ,
2.  $\bar{x} \in \omega(x)$  for all  $x \in \mathcal{X}$ .

**Proof:** Obviously (2) implies (1); thus we need only to establish the converse. Let  $W_n$  be a sequence of neighborhoods of  $\bar{x}$  such that if  $x_n \in W_n$  then  $x_n \rightarrow \bar{x}$ . We may assume that  $W_0 = \mathcal{X}$ . We will prove, by induction, that for any  $x \in \mathcal{X}$ , there exists a sequence  $\{x_n\}_{n \geq 0}$  such that:

$$x_n \in R^{k_n}(x) \cap W_n \quad \text{with } k_n \geq n. \quad (23)$$

Clearly, from (23), Property 2 follows.

Pick any  $x \in \mathcal{X}$ . Define  $x_0 = x$ , and  $k_0 = 0$ ; then  $x_0 \in R^{k_0}(x) \cap W_0$ . Assume that we have already defined  $x_0, \dots, x_{n-1}$ . We let:

$$\tilde{x}_n = \begin{cases} f(x_{n-1}, \bar{u}) & \text{if } f(x_{n-1}, \bar{u}) \neq \bar{x} \\ f(f(x_{n-1}, \bar{u}), \bar{u}) & \text{if } f(x_{n-1}, \bar{u}) = \bar{x} \end{cases}$$

then  $\tilde{x}_n \neq \bar{x}$ . By assumption there exists some  $x_n \in R^{\tilde{k}_n}(\tilde{x}_n) \cap W_n$  for some  $\tilde{k}_n > 0$ . Thus  $x_n \in R^{k_n}(x) \cap W_n$ , with  $k_n = k_{n-1} + 1 + \tilde{k}_n$ , if  $\tilde{x}_n$  was defined in the first way, and  $k_n = k_{n-1} + 2 + \tilde{k}_n$  otherwise. In any case, we have  $k_n \geq n$  since  $k_{n-1} \geq n - 1$ . Thus (23) holds. ■

We now obtain one of our main results:

**THEOREM 2** *Let  $\Sigma$  be an analytic invertible system. If  $\Sigma$  is weakly asymptotically controllable to  $\bar{x}$ , and  $\bar{x}$  is transitive then  $\Sigma$  is forward accessible from all  $x \in \mathcal{X}$ .*

**Proof:** Unless we are in the trivial case  $\mathcal{X} = \{\bar{x}\}$ ,  $\bar{x}$  transitive implies that there exists  $\bar{u}$  such that  $f(\bar{x}, \bar{u}) \neq \bar{x}$ . Thus, by Lemma 5,  $\bar{x} \in \omega(x)$  for all  $x \in \mathcal{X}$ , which, using Proposition 9, gives  $\dim \text{Lie } \Gamma^+(x) = n$  for all  $x \in \mathcal{X}$ . So the statement follows from Proposition 2. ■

*Example.* Let's consider the following class of systems:

$$\Sigma : x(t+1) = Ax(t) + Bu_1(t) + g(x(t), u_2(t)),$$

with  $\mathcal{X} = \mathbb{R}^n$ ,  $U = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ ,  $u_i \in \mathbb{R}^{m_i}$ , and where  $A$ ,  $B$  are matrices with  $A \in \mathbb{R}^{n \times n}$ , and  $B \in \mathbb{R}^{n \times m_1}$ . Assume that  $g$  is an analytic function such that:

$$g(x, 0) = 0 \quad \forall x \in \mathcal{X}.$$

It is easy to see that the assumption that  $(A, B)$  is a stabilizable pair implies that  $\Sigma$  is weakly asymptotically controllable to 0. Recall that  $(A, B)$  stabilizable is equivalent to say that there exists a matrix  $F \in \mathbb{R}^{m_1 \times n}$  such that the matrix  $A + BF$  is Hurwitz, i.e. all its eigenvalues have negative real part. So the previous Theorem applies in this case, in particular, we have the following conclusion:

If  $A, B$  is a stabilizable pair and  $\Sigma$  is transitive from 0 then  $\Sigma$  is forward accessible from all  $x \in \mathcal{X}$ .

## References

1. Albertini, F., and Sontag, E. D., "Discrete-time transitivity and accessibility: analytic systems," *SIAM J. Control and Opt.*, vol. 31, pp. 1599–1622, 1993.
2. Fliess, M. and Normand-Cyrot, D., "A group-theoretic approach to discrete-time nonlinear controllability," *Proc. IEEE Conf. Dec. Control*, San Diego, Dec. 1981.
3. Jakubczyk, B., and Normand-Cyrot, D., "Orbites de pseudo groupes de difféomorphismes et commandabilité des systèmes non linéaires en temps discret," *C.R. Acad. Sciences de Paris*, vol. 298-I, pp. 257–260, 1984.
4. Jakubczyk, B., and Sontag, E. D., "Controllability of nonlinear discrete-time systems: A Lie-algebraic approach," *SIAM J. Control and Opt.*, vol. 28, pp. 1–33, 1990.
5. Mokkadem, A., "Orbites de semi-groupes de morphismes réguliers et systèmes non linéaires en temps discret," *Forum Math.*, vol. 1, pp. 359–376, 1989.

6. Nijmeijer, H., and Van der Schaft, A. V., *Nonlinear Dynamical Control Systems*, Springer-Verlag: New York, 1990.
7. Sontag, E.D., "Integrability of certain distributions associated with actions on manifolds and applications to control problems," in *Nonlinear Controllability and Optimal Control*, edited by H. J. Sussmann, pp. 81–131. Marcel Dekker: New York, 1990.
8. Sontag, E.D., *Mathematical Control Theory, Deterministic Finite Dimensional Systems*. Springer-Verlag: New York, 1990.
9. Sussmann, H.J., "Orbits of families of vector fields and integrability of distributions," *Trans. American Mathematical Society*, vol. 180, pp. 172–188, 1973.