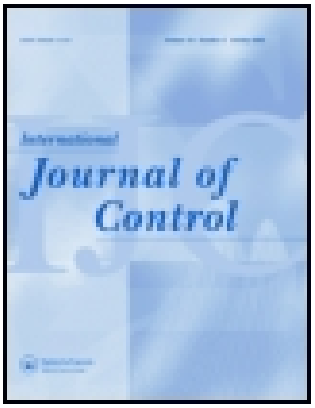

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Finite-dimensional open-loop control generators for non-linear systems

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Finite-dimensional open-loop control generators for non-linear systems

EDUARDO D. SONTAG†

This paper concerns itself with the existence of open-loop control generators for non-linear (continuous-time) systems. The main result is that, under relatively mild assumptions on the original system, and for each fixed compact subset of the state space, there always exists one such generator. This is a new system with the property that the controls it produces are sufficiently rich to preserve complete controllability along non-singular trajectories. General results are also given on the continuity and differentiability of the input-to-state mapping for various p-norms on controls, as well as a comparison of various non-linear controllability notions.

1. Introduction

In this paper we consider non-linear finite-dimensional systems of the type

$$\dot{x}(t) = f(x(t), u(t)) \tag{1.1}$$

where $x(t)$ is the state, and $u(t)$ is the control, at time t. (More precise definitions are given later.) An *open-loop control generator* for such a system is a new system, described by equations

$$\left.\begin{aligned}\dot{\omega}(t) &= P(\omega(t))\\ u(t) &= Q(\omega(t))\end{aligned}\right\} \tag{1.2}$$

This is a system with no controls but with an output map whose values are in the input space to (1.1). For any initial condition $\omega(0)$ of (1.2), there is (at least for small enough times t) a generated control $u(t) = Q(\omega(t))$, where $\omega(\cdot)$ is the solution of (1.2) with initial condition $\omega(0)$. For any initial state $x(0)$ for the original system, this control gives rise to a trajectory.

It is often the case in systems problems that such models are used for control generation; for instance, when dealing with tracking and the study of responses to ramps (polynomials of degree at most 1), one introduces the control generator with dynamics

$$\dot{\omega}_1 = 0, \quad \dot{\omega}_2 = \omega_1$$

and output $u(t) = \omega_2(t)$. Different initial conditions $\omega_1(0)$, $\omega_2(0)$ will give rise to all possible ramps.

A natural question to ask is: if the system (1.1) is known to be completely controllable, does there exist also a system as in (1.2) with the following property: for each state x_0 and x_1, there should exist some initial condition $\omega(0)$ and time T such that the control $\omega(\cdot)$ is well defined for t in $[0, T]$ and so that the trajectory induced by ω on the original system takes x_0 into x_1 at time T. Even more interestingly, one may demand that all these trajectories be non-singular in the sense of optimal control, or equivalently, that the time-varying linear systems obtained by linearizing along the

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obtained trajectories be themselves completely controllable. This last requirement is important if linear-feedback techniques are to be applied in order to regulate for small perturbations along the trajectories in question.

In this paper, we provide a positive answer to the above question. Some technical conditions are imposed, some of which are probably not essential (Assumption 2.1 below) and could be dropped if the proof is based on a different argument, as suggested later. Other assumptions, dealing with properties of the Lie algebra of vector fields generated by the control fields $f(\cdot, u)$, are unavoidable, as shown by counterexamples later.

A companion paper to the present one (Sontag 1987 b) *starts* with the assumption that such a control generator exists, and provides a 'universal' method for regulation along trajectories obtained as described above. The notion of a control generator is essential in the proofs given there, since all arguments depend on having a suitable parametrization of trajectories. Given a control generator, trajectories can be indeed parametrized by $\omega(0)$, $x(0)$ and T. The paper (Sontag 1987 b) deals with 'pseudolinearization' properties of non-linear systems, in the sense of work by Rugh (1983), Baumann and Rugh (1986) and by Reboulet and Champetier (1984), Champetier *et al.* (1985). These authors have dealt with the study of families of linearizations of non-linear systems around different operating points, and in particular the problem of obtaining compensators with the property that all closed-loop linearizations have the same dynamic behaviour. In contrast, Sontag (1987 b) studies linearizations along *trajectories* of non-linear systems. The basic result there, when coupled with the theorem proved here, establishes the following fact: provided that a system satisfies certain reasonable assumptions, it is possible to affect any desired state transfer using a suitable open loop signal generator, and to regulate for small deviations from the corresponding trajectories using linear control design techniques. The desired regulator has a form independent of the open-loop trajectory, which is fed on-line. An explicit form for the controller, as well as experimental results including the control of angular velocity of a rotating satellite, are given by Sontag (1987 a).

Central to both the results here and in Sontag (1987 b) is a study of those trajectories of a given system along which the linearization (as a time-varying linear system) is controllable. Such *non-singular trajectories* play a central role in the construction of precompensators. If a system is controllable, that is, if we may go from any state to any other state, one may expect that it should also be true that one can affect transfers in a non-singular manner. Unfortunately there is no 'Sard theorem' in infinite dimensions (controls belong to an infinite dimensional space) that would allow such a conclusion. We shall prove, however, that such non-singular controllability indeed holds, if (and only if) the given system satisfies a certain non-degeneracy property. (Roughly, there must be no periodic autonomous subsystems.) Sufficient conditions for this to happen are that there be no finite escape times and that the state space be simply connected, *or* that there be some equilibrium state for the system.

After setting up definitions and the statement of the main result, we provide various results dealing with the continuity and differentiability of the input-to-state mapping for various p-norms on controls. These results are needed later, but we have not been able to find them in the literature in the generality needed. Later, we provide a comparison of various non-linear controllability notions; these results should also be of interest in themselves. Finally, we give in the last section the proof of the main theorem.

2. Definitions and statement of the main theorem

A *system* Ξ is described by a set of controlled ordinary differential equations

$$\dot{x}(t) = f(x(t), u(t)) \tag{2.1}$$

where for each t, $x(t)$ is in the state space S_Ξ, which we take to be an arbitrary open subset of $\mathbb{R}^n$, and $u(t)$ is in the control-value space U_Ξ, which for simplicity we take to be an euclidean space $\mathbb{R}^m$, m an integer. We assume that the dynamics map $f: S_\Xi \times U_\Xi \to \mathbb{R}^n$ is real-analytic, and that the following property holds.

Assumption 2.1

There is a continuous function $\beta: S_\Xi \to \mathbb{R}$ such that

$$\|f_u(\xi, \mu)\| \leqslant \beta(\xi)$$

for all $\xi \in S_\Xi$ and all $\mu \in U_\Xi$. (By $\|f_u\|$ we denote the norm of the jacobian of f with respect to u, for any fixed operator norm.)

We often omit the argument t. The system is *polynomial* if each component of f is a polynomial and *rational* if each component of f is a rational function having no poles on $S_\Xi \times U_\Xi$. It is *autonomous* if f is independent of u; autonomous systems will be used in order to model control generators.

Other definitions of a system could be used. Generalizing the results to systems on manifolds would be straightforward but notationally somewhat cumbersome; on the other hand, the generalization of the results given here to smooth but non-analytic systems would be an interesting topic for further research.

We need to define carefully the notion of control. A $u: [0, T] \to U_\Xi$ for which there is a compact subset $K = K_u$ of U_Ξ such that $u(t) \in K$ for almost all t is an *admissible control*; the defining property says that u is essentially bounded, and $T = T_u$ is the *length* of u. Given any such u and any $\xi \in \Xi$, the unique absolutely continuous solution $x(\cdot)$ of (2.1) with $x(0) = \xi$ at time $t \leqslant T$, if defined, is denoted by $x(t) = \psi(t, \xi, u)$. A pair (x, u) of functions on an interval $[0, T]$, with u an admissible control and x satisfying (2.1), i.e.,

$$x(t) = \psi(t, x(0), u)$$

for all $t \in [0, T]$, is an *admissible trajectory* on $[0, T]$. If u has length T and ξ is such that there exists an admissible trajectory (x, u) on $[0, T]$ with $x(0) = \xi$, we say that u *can be applied to* ξ. If there is an admissible trajectory on $[0, T]$ with initial $x(0) = \xi_1$ and final $x(T) = \xi_2$, we say that ξ_1 can be *controlled to* ξ_2 *in time* T, or that ξ_2 *can be reached from* ξ_1, and that u *steers* ξ_1 to ξ_2. If there is some $T > 0$ such that ξ_1 can be controlled to ξ_2 in time T, we just say ξ_1 can be controlled to ξ_2.

The *variational system* of Ξ along the admissible trajectory (x, u) is the linearization of Ξ along this trajectory, that is, the linear time-varying system $D_{x,u}\Xi$ defined as follows (strictly speaking, time-varying systems are not 'systems' with our definition):

$$\dot{\lambda}(t) = f_x(x(t), u(t))\lambda(t) + f_u(x(t), u(t))v(t), \quad t \in [0, T] \tag{2.2}$$

where f_x, f_u denote jacobians of f with respect to the first n variables and the last m variables respectively, and where $\lambda(t) \in \mathbb{R}^n$ and $v(t) \in \mathbb{R}^m$ for all t. The original system Ξ is *linearly controllable* along (x, u) if (2.2) is (completely) controllable in $[0, T]$, i.e. for each λ_1 and λ_2 in $\mathbb{R}^n$ there is an essentially bounded v such that, solving (2.2) with this v and with initial condition $\lambda(0) = \lambda_1$ results in $\lambda(T) = \lambda_2$. Linear controllability along a given (x, u) is equivalent to the map $\alpha(u) := \psi(T, \xi, u)$ having full rank at u, seen as a map on an appropriate space of controls (see below).

Assume that we are now given both a system Ξ as in (2.1) and an autonomous system Ω, with state space $S_\Omega \subseteq \mathbb{R}^r$ and dynamics denoted by P, as well as an analytic map $Q: S_\Omega \to U_\Xi$. We shall use $\Omega\downarrow\Xi$ to denote the system obtained by feeding the output of Ω as a control to Ξ, and think of the corresponding combination as an autonomous system

$$\left.\begin{aligned}\dot{\omega} &= P(\omega)\\ \dot{\xi} &= f(\xi, Q(\omega))\end{aligned}\right\} \tag{2.3}$$

with state space $S_\Omega \times S_\Xi$. Consider a pair $(w_0, x_0) \in S_\Omega \times S_\Xi$. For small enough times $T > 0$, the solution $(\omega(t), \xi(t))$ of (2.3) with $\omega(0) = w_0$ and $\xi(0) = x_0$ is defined on the interval $[0, T]$. We shall say that (w_0, x_0) is *non-degenerate* iff for some such T, Ξ is linearly controllable along the ensuing admissible trajectory $(\xi, Q(\omega))$ of Ξ, and denote the set of such pairs by $ND(\Omega\downarrow\Xi)$. We also call the trajectory (ω, ξ) non-degenerate if (w_0, x_0) is. It is easy to see that $ND(\Omega\downarrow\Xi)$ is an open set, and that $(\omega(t), \xi(t))$ is again in $ND(\Omega\downarrow\Xi)$, for each $t \leqslant T$ (see Sontag 1987 b).

Finally, we say that the system Ξ is *complete* if for every $\xi \in S_\Xi$, every $T > 0$, and every admissible control u, the solution $\psi(t, \xi, u)$ is well-defined for all $t \leqslant T$, i.e. every control can be applied to every state; it is *controllable* iff for each ξ_1 and ξ_2 in S_Ξ, ξ_1 can be controlled to ξ_2. An *equilibrium point* for Ξ is a pair (ξ, μ), $\xi \in S_\Xi$, $\mu \in U_\Xi$, such that $f(\xi, \mu) = 0$.

Theorem 2.1: *Main theorem*

Assume that the system Ξ is controllable and that, *either* it has some equilibrium point, *or* it is complete and S_Ξ is simply connected. Let C be any compact subset of S_Ξ. There exists then an autonomous polynomial system Ω, a polynomial map Q, and a compact subset $\mathcal{O}$ of $ND(\Omega\downarrow\Xi)$ such that the following property holds. For each ξ_1 and ξ_2 in C there is a $T > 0$ and an admissible trajectory (ω, ξ) of the system (2.3) with $\xi(0) = \xi_1$ and $\xi(T) = \xi_2$ such that $(\omega(t), \xi(t)) \in \mathcal{O}$ for all $t \in [0, T]$.

Controllability (and Assumption 2.1) alone are *not* sufficient to ensure the desired conclusions. A counter-example will be provided later. Note that systems with a non-simply connected state space appear naturally in robotics, when there are workspace obstacles.

Assumption 1 is made mainly for simplicity of exposition, and it can be relaxed considerably. In any case, most types of systems can be modelled in this way. Certainly, the usual case of systems linear in controls is included. More general non-linearities can also be included if there are control bounds. For instance, a system with $f(x, u) = u + u^2$ and $|u| \leqslant 1$ can be modelled by

$$f(x, v) = \sin(v) + \sin^2(v)$$

where $U_\Xi = \mathbb{R}$ and Assumption 2.1 is satisfied. Similarly, open-constraint sets can be included: for instance if u above is restricted to the interval $|u| < 1$ then we may reparametrize controls as

$$u = \left(\frac{2}{\pi}\right) \arctan(v)$$

In any case, the need for Assumption 2.1 is most probably due only to our method of proof; one could rewrite all our treatment using instead 'almost uniform' convergence. This would be less elegant than using p-convergence as below, but would afford greater generality.

We make the following notational convention in order to save space: to display column vectors, we use also the alternative notation

$$(a_1 : \dots : a_r) \tag{2.4}$$

(note the :) instead of

$$\begin{vmatrix} a_1 \\ \vdots \\ a_r \end{vmatrix}$$

If the a_i are scalars, this is the same as the transpose of the row vector $(a_1, \dots, a_r)$, but we will mostly deal with cases in which the a_i are themselves column vectors, in which case (2.4) would correspond to, in more usual but cumbersome notation, $(a_1', \dots, a_r')'$.

3. Approximation results

We need next a number of approximation results. Section 3.1 deals with continuity properties and differentiability of the input/state map, and § 3.2 develops a construction related to standard proofs of the Stone–Weirstrass theorem.

3.1. *Continuous dependence theorems*

The results in this section are probably known at the 'folk' level, but we have been unable to find a suitable reference in the form needed for this paper, so we give a self-contained presentation.

A system Ξ will be fixed for the rest of this section. A real number $T > 0$ will be also fixed. We let L_∞^m be the Banach space of all essentially bounded measurable functions $[0, T] \to U_\Xi = \mathbb{R}^m$, endowed with the sup norm

$$\|u\|_\infty := \text{ess sup}\,\{|u(t)|, t \in [0, T]\}$$

where $|\mu|$ is the euclidean norm in $\mathbb{R}^m$—any other norm could be used instead. (We also use the notation $|\xi|$ for the euclidean norm in the state space $S_\Xi \in \mathbb{R}^n$.) Since the interval of definition is finite, the spaces L_p^m (p-integrable functions), $p \geqslant 1$, all contain L_∞^m. We shall be interested in the latter space viewed as a subspace of each L_p^m; to avoid confusion, we use a different notation. Thus, B_p^m will denote L_∞^m with the norm

$$\|u\|_p := \left\{ \int_0^T |u(t)|^{1/p}\, dt \right\}^p$$

for any $p \geqslant 1$. For simplicity of statement, we also let B_∞^m be the same as L_∞^m (with the sup norm). It is a standard fact that whenever $1 \leqslant p < q \leqslant \infty$, then

$$\|u\|_p \leqslant c_1 \|u\|_q$$

for all $u \in L_\infty^m$, for some constant c_1. In fact,

$$c_1 = T^{(1/p)\cdot(1/q)}$$

will do. Conversely, if $1 \leqslant p < q < \infty$ and if k is a given constant, then there is another constant c_2 such that

$$\|u\|_q^q \leqslant c_2 \|u\|_p^p$$

whenever $\|u\|_\infty \leqslant k$.

(*Proof.* Write $|u|^q = |u|^{q-p}|u|^p \leqslant k^{q-p}|u|^p$ and integrate.)

Thus, as long as one remains in a bounded subset of L_∞^m all the p-topologies ($p < \infty$) are equivalent.

Recall that the continuous mapping $f: D \to N_2$, where D is an open subset of the normed space N_1, and N_2 is another normed space, is (Fréchet) differentiable at a point $\xi \in D$ iff there is a linear mapping $Df(\xi): N_1 \to N_2$ such that

$$\|f(x) - f(\xi) - Df(\xi)(x - \xi)\| = o(\|x - \xi\|)$$

The *derivative* of f is the mapping $Df: D \to L(N_1, N_2)$ sending ξ into $Df(\xi)$. One defines second derivatives of f via derivatives of Df, and so on inductively. A *smooth* f is one that has derivatives of all orders. We shall say that f *has full rank at* ξ iff f is a submersion there, i.e. $Df(\xi)$ is onto. The normed spaces B_p^m, $p < \infty$, are not Banach—they are dense in the respective L_p^m—but we shall apply the implicit function theorem to differentiable mappings $B_p^m \to \mathbb{R}^n$. This will present no difficulty because such a map already has full rank when restricted to an appropriate finite-dimensional subspace, and the implicit function theorem can be applied to the restricted map.

The following situation will arise below. Assume that

$$f: S_\Xi \times U_\Xi \to \mathbb{R}^n$$

is a C^2 map, and that $K_1 \subseteq S_\Xi$ is a given compact convex set such that, for some constant k,

$$\|f_u(\xi, \mu)\| \leqslant k \tag{3.1}$$

whenever $\xi \in K_1$ and $\mu \in U_\Xi$. When f has linear growth in u, as assumed for the map defining the dynamics of Ξ, this property will hold for all compact sets K_1. We have the following observation.

Lemma 3.1

Assume that f, K_1, k are as above. Pick a compact subset $K_2 \subseteq U_\Xi$. Then, there exist constants M and N such that, if $\xi, \eta \in K_1$, $\mu \in K_2$, $v \in U_\Xi$, and $1 \leqslant p \leqslant 2$, then

(*a*) $|f(\xi, \mu) - f(\eta, v)| \leqslant M\{|\xi - \eta| + |\mu - v|\}$, and

(*b*) if $g(x, u) := f(x, u) - f(\xi, \mu) - f_x(\xi, \mu)(x - \xi) - f_u(\xi, \mu)(u - \mu)$, then

$|g(\eta, v)| \leqslant N\{|\xi - \eta|^2 + |\mu - v|^p\}$

Proof

Property (*a*) is obtained by separately bounding $|f(\xi, \mu) - f(\eta, \mu)|$ and $|f(\eta, \mu) - f(\eta, v)|$, and by using property (3.1) and the mean-value theorem. We now prove property (*b*). Assume first that $|\mu - v| < 1$. Then v is in the compact set

$$K_3 := \{v \mid |v - u| \leqslant 1 \quad \text{for some} \quad u \in K_2\}$$

By Taylor's formula with remainder (recall that f is twice differentiable), we known that, for ξ, η in K_1 and $\mu, v \in K_3$,

$$|g(\eta, v)| \leqslant a|\xi - \eta|^2 + b|\mu - v|^2$$

for some constants a, b (which depend on K_1 and K_3, and hence on K_2). Since $|\mu - v| \leqslant 1$ and $p \leqslant 2$, this means also that

$$|g(\eta, v)| \leqslant a|\xi - \eta|^2 + b|\mu - v|^p \tag{3.2}$$

Next note that $g_u(x, u) = f_u(x, u) - f_u(\xi, \mu)$, so that g also satisfies (3.1) (with $2k$ instead of k). We can then apply part (a) to g to conclude that, for η, μ, v as in the statement,

$$|g(\eta, v) - g(\eta, \mu)| \leqslant M'|\mu - v|$$

for a constant M' that depends only on K_1. If $|\mu - v| \geqslant 1$, then also $|\mu - v|^{1-p} \leqslant 1$, so $|\mu - v| \leqslant |\mu - v|^p$. Thus

$$|g(\eta, v)| \leqslant |g(\eta, \mu)| + |g(\eta, v) - g(\eta, \mu)| \leqslant a|\xi - \eta|^2 + M'|\mu - v|^2$$

Choosing $N := \max\{a, b, M'\}$, the result follows. □

For the given system Ξ, let $\mathscr{D}$ be the set of triples

$$(t, \xi, u) \in [0, T] \times S_\Xi \times L_\infty^m$$

for which the solution $\psi(\tau, \xi, u)$ is defined for all $0 \leqslant \tau \leqslant t$. It is a standard fact that $\mathscr{D}$ is open, and that $\psi(t, \cdot, \cdot)$ is smooth on

$$\mathscr{D}_t := \{(x, u) \mid (\tau, x, u) \in \mathscr{D} \quad \text{for all } 0 \leqslant \tau \leqslant t\}$$

for each t. (See for instance Grasse 1981, Theorem 2.9 and Proposition 2.11.) We shall need differentiability and continuity with respect to p-norms, $p < \infty$, as well.

Lemma 3.2

Pick any $1 \leqslant p \leqslant \infty$, and consider $\mathscr{D}_T$ as a subset of $S_\Xi \times B_p^m$. Then $\mathscr{D}_T$ is open and the mapping

$$\alpha : \mathscr{D}_T \to S_\Xi, \quad \alpha(\xi, u) := \psi(T, \xi, u)$$

is continuous. If $p > 1$, then α is also differentiable, and in that case

$$D\alpha(\xi, u)[\lambda_0, v]$$

is the solution $\lambda(T)$ of the variational equation (2.2), where $x(t) = \psi(t, \xi, u)$ and $\lambda(0) = \lambda_0$. In particular, $\alpha(\xi, \cdot)$ has full rank at u if and only if Ξ is linearly controllable along (x, u). (Thus the full rank property is independent of the particular $p > 1$.)

Proof

We first assume that $S_\Xi = \mathbb{R}^n$ and that the map f defining the evolution is defined on all of $\mathbb{R}^n \times \mathbb{R}^m$ and is globally Lipschitz, meaning that there is a constant M such that

$$|f(\xi, \mu) - f(\eta, v)| \leqslant M\{|\xi - \eta| + |\mu - v|\}$$

for all ξ, η in $\mathbb{R}^n$ and all μ, v in $\mathbb{R}^m$. In this case, solutions are always defined, so that $\mathscr{D}_T = \mathbb{R}^n \times B_p^m$. Assume that (x, u) and (y, v) are both admissible trajectories. We have that, for each $0 \leqslant t \leqslant T$,

$$x(t) - y(t) = \int_0^t \{f(x(\tau), u(\tau)) - f(y(\tau), v(\tau))\}\, d\tau + x(0) - y(0)$$

By the Lipschitz condition,

$$|x(t) - y(t)| \leqslant M \int_0^T |x(\tau) - y(\tau)|\, d\tau + |x(0) - y(0)| + M\|u - v\|_1$$

By the Bellman–Gronwall lemma, we conclude that

$$|x(t) - y(t)| \leqslant \exp Mt\, \{|x(0) - y(0)| + M\|u - v\|_1\}$$

for all $0 \leqslant t \leqslant T$. Thus, for each $1 \leqslant p \leqslant \infty$ there are constants a, b such that

$$|\psi(t, \xi, u) - \psi(t, \eta, v)| \leqslant a|\xi - \eta| + b\|u - v\|_p \tag{3.3}$$

for all ξ, η in $\mathbb{R}^n$, all u, v in B_p^m, and all $0 \leqslant t \leqslant T$.

Assume now that S_Ξ and f are arbitrary. Pick any element (ξ, u) in $\mathscr{D}_T$. Choose open neighbourhoods $\mathscr{V}_\xi$ and $\mathscr{W}_\xi$ of ξ such that

$$\mathscr{V}_\xi \subseteq \operatorname{clos} \mathscr{V}_\xi \subseteq \mathscr{W}_\xi \subseteq S_\Xi$$

Let $\theta: \mathbb{R}^n \to \mathbb{R}$ be any smooth function which is identically 1 on clos $(\mathscr{V}_\xi)$ and vanishes outside $\mathscr{W}_\xi$. Consider the system obtained with $S'_\Xi = \mathbb{R}^n$, same U_Ξ, and f replaced by

$$h(\xi, \mu) := \theta(\xi) f(\xi, \mu)$$

Since f has linear growth in u, h does also, and hence since θ has compact support we are in the situation of (a) in Lemma 3.1, thus h is globally Lipschitz. So the arguments in the previous paragraph apply to the system with dynamics $h(\xi, \mu)$. We let ϕ be the transition map ψ for this system. By (3.3), there is then a neighbourhood $\mathscr{W}_\xi$ of ξ and an $\varepsilon > 0$ such that $\phi(t, \eta, v)$ is in $\mathscr{V}_\xi$ for all $0 \leqslant t \leqslant T$ whenever $\eta \in \mathscr{U}_\xi$ and $\|u - v\|_p < \varepsilon$. Since $h(\cdot, \mu)$ and $f(\cdot, \mu)$ coincide on $\mathscr{V}_\xi$, it follows that $\psi(t, \eta, v)$ solves the original differential equation, i.e. it equals $\phi(t, \eta, v)$ for these t, η, v. In particular, $\mathscr{D}_T$ contains a neighbourhood of (ξ, u) and is therefore open. Continuity of α follows from (3.3).

We now prove differentiability when $p > 1$. Let (x, u) be an admissible trajectory, and

$$A(t) = f_x(x(t), u(t)), \quad B(t) = f_u(x(t), u(t))$$

Pick a convex compact neighbourhood $\mathscr{V}_\xi$ of $\xi = x(0)$ and any η in $\mathscr{V}_\xi$. For any other control v (of length T) sufficiently near to u in B_p^m, let (z, v) be the trajectory that results when applying v to η. By continuity, we may choose a suitable neighbourhood of u so that this trajectory always stays in a given compact convex neighbourhood of ξ in S_Ξ, say K_1. Let K_2 be any compact set such that the (essentially bounded) control u satisfies $u(t) \in K_2$ for almost all t. Let $\delta(t) := z(t) - x(t)$ and $\upsilon(t) := v(t) - u(t)$. From part ($b$) of Lemma 3.1 it follows that, if $1 \leqslant p \leqslant 2$ then

$$\dot{\delta}(t) = A(t)\delta(t) + B(t)\upsilon(t) + \phi(t)$$

where

$$|\phi(t)| \leqslant N\{|\delta(t)|^2 + |v(t)|^p\}$$

for a suitable constant N. Thus, if λ solves (2.2) with $\lambda(0) = \delta(0) = \lambda_0 = \xi - \eta$, it follows that

$$\delta(T) = \lambda(T) + \int_0^T \Phi(T, \tau)\phi(\tau)\, d\tau$$

So

$$|\delta(T) - \lambda(T)| \leqslant M\left\{\int_0^T |\delta(t)|^2\, dt + \|v\|_p^p\right\}$$

for some constant M. Applying (3.3) to the first term there results that there is a constant M' such that

$$|\delta(T) - \lambda(T)| \leqslant M'\{|\lambda_0|^2 + \|v\|_p^2 + \|v\|_p^p\} \tag{3.4}$$

Since $p > 1$, it follows that $|\delta(T) - \lambda(T)|$ is indeed $o(|\lambda_0| + \|v\|_p)$, as required to establish differentiability. If instead $2 < p$, we consider (3.4) for the case $p = 2$. Since

$$\|v\|_2 \leqslant c\|v\|_p$$

for some constant, it follows that $|\delta(T) - \lambda(T)|$ is majorized by an expression

$$M''\{|\lambda_0|^2 + 2\|v\|_p^2\}$$

again as desired. □

Remark 3.1

The differentiability result is false if $p = 1$. For instance, consider the system

$$\dot{x} = \sin^2 u$$

with $S_\Xi = U_\Xi = \mathbb{R}$, and the controls u_ε on [0, 1] with

$$u_\varepsilon(t) = 1 \quad \text{on} \quad [0, \varepsilon] \quad \text{and} \quad u_\varepsilon(t) = 0 \quad \text{for} \quad t > \varepsilon$$

Let also $u \equiv 0$, $x \equiv 0$, and $x_\varepsilon :=$ solution when applying u_ε to 0. Note that $u_\varepsilon \to u$ when $\varepsilon \to 0$. The differential of $u(\cdot) \to \psi(T, 0, u)$ as a map on B_1^1, if it exists, would have to be the mapping which is identically zero. Thus differentiability would mean that $|x_\varepsilon(T)|$ is $o(\|u_\varepsilon\|_1)$ as $\varepsilon \to 0$. Since $|x_\varepsilon(T)| = \varepsilon = \|u_\varepsilon\|_1$, this is false. (Note that, on the other hand, for $p > 1$ one has for this example that

$$\|u_\varepsilon\|_p = \varepsilon^{1/p}$$

and there is no contradiction.)

Remark 3.2

For $p < \infty$ the linear growth condition is essential. Otherwise, not even continuity holds. Indeed, take any finite p and consider the equation $\dot{x} = u^q$, where $q > p$ is arbitrary. Pick any r with $q > (1/r) > p$. The control u_ε, defined now by

$$u_\varepsilon(t) := \varepsilon^{-r} \quad \text{for} \quad t \leqslant \varepsilon$$

and zero otherwise, has $\|u_\varepsilon\|_p \to 0$ as $\varepsilon \to 0$, but the corresponding solution x_ε has

$$|x_\varepsilon(1)| = \varepsilon^{1-rq} \to \infty$$

3.2. *Approximation of PC controls by polynomial controls*

We fix a $T > 0$ and an integer σ. For any elements $v_0, \ldots, v_\sigma$ in U_Ξ, and for any real numbers $0 = s_0 < s_1 < \ldots < s_\sigma < s_{\sigma+1} = T$, consider the piecewise constant admissible control on $[0, T]$ defined as follows:

$$u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; t) := v_1 \quad \text{if} \quad s_i \leqslant t \leqslant s_{i+1}, \quad i = 0, \ldots, \sigma$$

Let $\{Q_n(t)\}$ be any fixed sequence of polynomial kernels of degree $2n$, that is, each Q_n is a polynomial of degree $2n$ and the following properties hold.

$$\text{For each } \delta > 0,\ Q_n \to 0 \text{ as } n \to \infty, \text{ uniformly on } |t| > \delta,\ t \in [-T, T] \tag{3.5}$$

$$\int_{-T}^{T} Q_n(\tau)\, d\tau = 1, \quad \text{for all } n \tag{3.6}$$

$$Q_n(t) \geqslant 0, \quad \text{for all } t \in \mathbb{R} \text{ and all } n \tag{3.7}$$

For instance, we may take

$$Q_n(t) := k_n(T^2 - t^2)^n$$

where the k_n are appropriately chosen constants. Consider now the convolution:

$$p_n(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; t) := \int_0^T Q_n(t - \tau) u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; \tau)\, d\tau \tag{3.8}$$

as a function on $t \in [0, T]$ with values in $\mathbb{R}^m$. (Integration of vector functions is understood componentwise.) If we expand

$$Q_n(t - s) = \sum_{j=0}^{2n} q_j(t) s^j$$

then $p_n(t)$ equals

$$\sum_{i=0}^{\sigma} \left[\sum_{j=0}^{2n} \frac{q_j(t)}{j+1} (s_{i+1}^{j+1} - s_i^{j+1}) \right] v_i \tag{3.9}$$

Thus p_n is a (vector) polynomial in $(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; t)$, of degree $2n$ in t. The expression in the right-hand side of (3.8) can be written as

$$\int_{-T}^{T} Q_n(\tau) u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; t - \tau)\, d\tau$$

if we identify u with its extension to $(-\infty, \infty)$ obtained by setting $u \equiv 0$ outside $[0, T]$. Let $|u|$ denote the euclidean norm of $u \in \mathbb{R}^m$, and fix any $p \geqslant 1$. Note that

$$\|p_n(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; \cdot) - u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; \cdot)\|_p^p$$
$$= \int_0^T \|p_n(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; t) - u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; t)\|^p\, dt \tag{3.10}$$

By property (3.6), we may write

$$u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; t) = \int_{-T}^{T} Q_n(s) u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; t)\, ds$$

so the expression in (3.10) is bounded above by

$$\int_0^T \left| \int_{-T}^{T} Q_n(\tau)\{u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; t-\tau) - u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; t)\}\, d\tau \right|^p dt \tag{3.11}$$

Assume that we are now given a sequence $(s_1, \ldots, s_\sigma)$ as above, a real number δ with $T > \delta > 0$, and a $t \in [0, T]$ which does not belong to any of the intervals $[s_i - \delta, s_i + \delta]$, for any $i = 0, 1, \ldots, \sigma + 1$. (Where $s_0 := 0$ and $s_{\sigma+1} := T$.) For any τ such that $|\tau| < \delta$, it follows that $u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; t-\tau) = u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; t)$, and hence the inside integral of (3.11) can be replaced by the integral over $|\tau| > \delta$ (as long as t is of this type). Since the differences

$$|u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; t-s) - u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; t)|$$

are always bounded by $c_1 := 2 \cdot \max\{|v_i'|\}$, it follows that for such t the term inside the integral is bounded by

$$\{c_1 \textstyle\int_{J_\delta} Q_n(\tau)\, d\tau\}^p$$

where J_δ is

$$[-T, -\delta] \cup [\delta, T]$$

When t is not in any of the above intervals, the inside term is in any case bounded by $\{c_1\}^p$, and the set of such exceptional t has a measure of at most $(\sigma + 2)\delta$. We conclude that the expression in (3.11) is majorized by

$$(\sigma + 2)\delta(c_1)^p + \{c_1 \textstyle\int_{J_\delta} Q_n(\tau)\, d\tau\}^p$$

By property (3.5), the following result holds.

Lemma 3.3

For any $1 \leqslant p < \infty$, $p_n(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; \cdot)$ converges in B_p^m to $u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; \cdot)$. This convergence is uniform on the real numbers $0 < s_1 < \ldots < s_\sigma < T$, and is also uniform on the vectors $v_0, \ldots, v_\sigma$ on compact subsets of $U_\Xi^{\sigma+1}$.

The result is of course false for $p = \infty$, since a limit of polynomials in L_∞^m is necessarily continuous. For this reason we have introduced the spaces B_p^m, $p \neq \infty$. An alternative approach would be based on the notion of 'almost uniform' convergence, for which a similar result can be proved.

4. Several controllability notions

We next introduce various very natural strong notions of controllability, and eventually prove that they are all in fact equivalent.

4.1. *Non-singular controllability*

Let a system Ξ and a $T > 0$ be fixed. We shall say that a control u on $[0, T]$ *non-singularly* (or, *ns-*) *steers* ξ into ζ iff u can be applied to ξ and the resulting trajectory (x, u) is such that $x(T) = \zeta$ and Ξ is linearly controllable along (x, u). We also say that u can be 'non-singularly applied' to ξ. If such a u exists, ξ can be *ns-controlled* into ζ. If every ξ can be non-singularly controlled to every other ζ, the system Ξ is *ns-controllable.* The notion of ns-controllability is transitive in the following strong sense. Assume that ξ can be ns-controlled to η and that η can be controlled to ζ (not necessarily non-singularly). Then it is also true that ξ can be ns-controlled to ζ. This is because if w is the concatenation of a control u (of length T) which ns-steers ξ into η with a control v (of length S) which steers η into ζ, then w is a control (of length $T+S$) which ns-steers ξ into ζ—the differential of $\psi(T+S, \xi, \cdot)$ is full rank already at those variations v which are zero on $[T, T+S]$.

A particular type of non-singularity is as follows. Assume that the control u is piecewise constant, $u = u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; \cdot)$, of length T and it steers the state ξ into ζ. We shall say that u *pcns-* (piecewise constant non-singularly) steers ξ to ζ if the mapping

$$(v_0, \ldots, v_\sigma) \mapsto \psi(T, \xi, u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; \cdot)) \tag{4.1}$$

defined on a neighbourhood of $(v_0', \ldots, v_\sigma')$, has differential of full rank at $(v_0', \ldots, v_\sigma')$. In that case, u also ns-steers ξ and ζ. This is because the mapping in (4.1) is the composition of the linear bounded map

$$U_\Xi^{\sigma+1} \to L_\infty^m, (v_0, \ldots, v_\sigma) \mapsto u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; \cdot) \tag{4.2}$$

with $\psi(T, \xi, \cdot)$, hence the latter must have full rank at u. As above, we define ξ to be pcns-controllable to ζ if such a u exists, and the system Ξ is pcns-controllable if this happens for any pair of states. This notion is also transitive in the sense discussed above.

4.2. *Strong normal controllability*

We shall say that the control u of length T *strongly normally* (or, *sn-*) *steers* the state ξ to the state ζ in time T iff there exist an integer σ, elements $v_0', \ldots, v_\sigma'$ in U_Ξ, a neighbourhood $\mathscr{V}_\zeta$ of ζ, and a smooth mapping

$$\beta : \mathscr{V}_\zeta \to \mathbb{R}^\sigma$$

such that, for each $z \in \mathscr{V}_\zeta$, if $\beta(z) = (s_1, \ldots, s_\sigma)$ then $0 < s_1 < \ldots < s_\sigma < T$, the control $u(\beta(z); v_0, \ldots, v_\sigma; \cdot)$ of length T, steers ξ into z in time T, and $u(\beta(\zeta); v_0, \ldots, v_\sigma; \cdot) = u$. The state ξ can be *sn-controlled* to ζ (in time T) if such a u exists. The system Ξ is *sn-controllable* iff for each ξ, ζ in S_Ξ there is a $T > 0$ such that ξ can be sn-controlled to ζ in time T. This notion is closely related to that of normal controllability given by Sussmann (1976); the qualifier 'strong' refers to the fact that the controls u are required to (locally) all have a uniform length T. As with the other definitions, if ξ is sn-controllable into η and η is controllable into ζ, then ξ is sn-controllable into ζ.

Remark 4.1

For particular states and controls, sn- and ns-controllability are (for systems not linear in controls) different. For example, consider the system (with $U_\Xi = S_\Xi = \mathbb{R}$)

$$\dot{x} = x + 1 - \sin(u)$$

and $\xi = 0$, $\zeta = 2(\exp(\frac{1}{2}) - 1)$, $T = 1$, $v_0' = \pi/2$, $v_1' = -\pi/2$, $s = 1/2$, and

$$\beta(z) := 1 + \ln 2 - \ln(2 + z)$$

for z near ζ. Then

$$u = \begin{cases} \pi/2 & \text{if } 0 \leqslant s < 1/2 \\ -\pi/2 & \text{if } 1/2 \leqslant s \leqslant 1 \end{cases}$$

sn-controls ξ to ζ. But it does not do so nonsingularly, since along the corresponding trajectory the linearization is the autonomous system $\dot{x} = x$. The corresponding notions for *systems* (rather than individual controls) do coincide (proved below). The situation is closed related to that in the context of orbit theorems for non-linear continuous time systems (see Sontag 1986), where different topologies are induced for the same state space depending on whether one takes the finest topology that makes all motions continuous with respect to switching times or instead with respect to control values.

Remark 4.2

Controllability is by itself *not* equivalent to sn-controllability. For instance consider the system with $U_\Xi = \mathbb{R}$, $S_\Xi = \mathbb{R}^2 - \{(0,0)\}$, and with equations in polar coordinates:

$$\dot{\theta} = 1, \quad \dot{r} = ru$$

The system is controllable (any state can be steered to every other state in time at most 2π). But the set of states reached in precisely time T is a half-line, and hence has no interior. Note that the state space is not simply-connected, and that there are no equilibrium points. The 'clock' coordinate θ is responsible for the pathological behaviour of this example.

Let $\mathscr{L}$ be the Lie algebra associated to Ξ. This is the smallest Lie algebra of vector fields on S_Ξ which contains the vector fields

$$\{f(\cdot, \mu), \mu \in U_\Xi\}$$

For any $\xi \in S_\Xi$, we associate the following subset of the tangent space at ξ:

$$\mathscr{L}(\xi) := \{X(\xi), X \in \mathscr{L}\}$$

If Ξ is controllable, then it is a well-known fact that $\mathscr{L}$ has full rank (i.e. $\dim \mathscr{L}(\xi) = n$) at all ξ. (See Isidori 1985 for many basic results on the Lie algebraic aspects of control systems.) The ideal of $\mathscr{L}$ generated by all the differences

$$\{f(\cdot, \mu) - f(\cdot, \nu), \quad \mu, \nu \in U_\Xi\}$$

is the *zero-time* algebra $\mathscr{L}^0$. Similarly, we introduce the spaces $\mathscr{L}^0(\xi)$ as above. It follows from the definitions that, for any fixed $\mu \in U_\Xi$, $\mathscr{L}(\xi)$ is the span

$$\mathscr{L}(\xi) = \mathscr{L}^0(\xi) + \text{span}\,\{f(\xi, \mu)\} \tag{4.3}$$

for all $\xi \in S_\Xi$. It is also known that (because of controllability), $\mathscr{L}^0$ has constant rank, so that there are only two possibilities: either $\dim \mathscr{L}^0(\xi)$ is always n or it is always $n-1$. In the latter case, a local change of coordinates can always be found which results in dynamics with an autonomous coordinate

$$\dot{x}_1(t) = f_1(x_1)$$

Thus there is (locally) a 'clock' as in Remark 4.2. When there is an equilibrium point

$f(\xi, \mu) = 0$, (4.3) together with the constancy of rank, gives that $\mathscr{L}^0$ has full rank, meaning that $\mathscr{L}^0(\xi)$ has rank n at every point. When instead S_Ξ is simply connected and the system is complete, this is also true by a result of Elliot (see Elliot 1971, Sussmann and Jurdjevic 1972). We summarize then with a proposition.

Proposition 4.1

If the system Ξ is as in the statement of Theorem 2.1, then $\mathscr{L}^0$ has full rank.

We shall prove that the full rank of $\mathscr{L}^0$ is sufficient for the conclusions of the theorem to hold. This rank condition is also necessary, because the conclusions imply that the system is ns-controllable, and this will be shown below to be equivalent to the rank condition.

If $T > 0$, $\xi \in S_\Xi$, and $0 < s_1 < \ldots < s_\sigma < T$, we let

$$A^T_{s_1, \ldots, s_\sigma}(\xi) = \{\psi(T, \xi, u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; \cdot)) | v_0, \ldots, v_\sigma \in U_\Xi\} \tag{4.4}$$

Note that, by the implicit function theorem, if ξ can be pcns-controlled to some other state ζ, then some set as in (4.4) has a non-empty interior. We remark below that the converse is also true. Let $A_{PC}(\xi)$ denote the union of all sets as in (4.4), for all possible $T > 0$, i.e. the set of states reachable from ξ using piecewise constant controls.

Controllability using arbitrary controls is equivalent to controllability using just piecewise constant controls. The next lemma is well-known (see for instance, Sussmann 1979, Theorem 1).

Lemma 4.1

If Ξ is controllable, then $A_{PC}(\xi) = S_\Xi$ for all ξ.

4.3. *A fixed-point argument*

We shall need an argument based in the Brouwer fixed point theorem that has been used repeatedly in control theory ('Bronovsky–Lobry lemma'), and which is also used in the main step of the proof of the preservation of controllability under sampling (see for example, Sontag 1983). An abstract version is provided by Grasse (1981), Lemma 3.2, (see also Lee and Markus (1968), pp. 251–252), which we reproduce here in somewhat weaker form. Choose any norm in $\mathbb{R}^n$.

Lemma 4.2

Let $\mathscr{W}$ be a topological space and let $H : \mathscr{W} \to \mathbb{R}^n$ be continuous. Assume that for some ζ in $\mathbb{R}^n$ there is a neighbourhood $\mathscr{V}$ of ζ and a continuous map $\beta : \mathscr{V} \to \mathscr{W}$ such that $H(\beta(z)) = z$ for all z in $\mathscr{V}$. Then, there is an $\varepsilon > 0$ and an open neighbourhood $\mathscr{V}'$ of ζ such that, for any mapping

$$h : \mathscr{W} \to \mathbb{R}^n$$

with $\|h(x) - H(x)\| < \varepsilon$ for all x, necessarily $\mathscr{V}' \subseteq h(\mathscr{W})$.

Let $\mathscr{P}_{k,T}(\xi)$ denote the set of states that can be reached from ξ using polynomial controls of length T and degree at most k in t.

Lemma 4.3

If ξ can be sn-controlled to ζ in time T, then there exists a k such that ζ is in the interior of $\mathscr{P}_{k,T}(\xi)$.

Proof

Assume that ξ, ζ, u, T, $v_0, \ldots, v_\sigma$, β, $\mathscr{V}_\zeta$ are as in the definition of sn-steering. Restricting $\mathscr{V}_\zeta$ if necessary, we will assume that it is compact. Let $\mathscr{W} := \beta(\mathscr{V}_\zeta)$, a compact subset of $\mathbb{R}^\sigma$ which contains $\beta(\zeta) = s^0 = (s_1^0, \ldots, s_\sigma^0)$. Let

$$H : \mathscr{W} \to \mathbb{R}^n, (s_1, \ldots, s_\sigma) \mapsto \psi(T, \xi, u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; \cdot))$$

Thus we are in the situation of Lemma 4.2, with $\mathscr{V} = \mathscr{V}_\zeta$; let ε, $\mathscr{V}'$ be as in the conclusions there. We now construct a sequence of mappings $\{H_N\}$ from $\mathscr{W}$ into $\mathbb{R}^n$ which converges uniformly to H, and with the property that the image of each H_N is included in the set of all states reachable from ξ using polynomial controls of degree at most $2N$. So for n large enough one of these images contains $\mathscr{V}'$, and the lemma follows.

Consider, for the above $v_0, \ldots, v_\sigma$, the mapping

$$(s_1, \ldots, s_\sigma) \mapsto u(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; \cdot)$$

see as a map from $\mathscr{W}$ into (for instance) B_2^m. This is continuous (but is not differentiable). Let K be the image of $\mathscr{W}$ under this mapping. Thus K is a compact subset of the open subset of B_2^m consisting of all the controls (of length T) that can be applied to ξ. Thus there exists a $\delta > 0$ such that every admissible control p which is at a distance less than δ from K can also be applied to ξ. For each positive integer $N \geqslant 1$, let

$$H_N(s_1, \ldots, s_\sigma) := \psi(T, \xi, p_N(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; \cdot))$$

where the polynomials p_N are as in § 3.2. Since the p_N converge to u uniformly on the s_i, for N large enough they can be applied to ξ. We take the subsequence of the H_N that consists of such large N. Since ψ is continuous on B_2^m, we conclude that the H_N indeed converge uniformly to H. □

Lemma 4.4

If for some k, T the interior of $\mathscr{P}_{k,T}(\xi)$ is nonempty, then there exist $s_1, \ldots, s_\sigma$ such that $A^T_{s_1, \ldots, s_\sigma}(\xi)$ has a non-empty interior.

Proof

This is again proved as a corollary of Lemma 4.2, as follows. Let $H(a_0, \ldots, a_k)$ be the state reached when applying to ξ the control (of length T)

$$\sum_{i=0}^{k} a_i t^i \tag{4.5}$$

This is defined and continuous on an open subset of $\mathbb{R}^{mk}$ (the control must be applicable to ξ). It is also smooth, since we may see H as the composition of the (linear, bounded) mapping that sends $(a_0, \ldots, a_\sigma)$ into the element (4.5) of L_∞^m followed by the mapping $\psi(T, \xi, \cdot)$. The image of the smooth map H contains by assumption an open set. Since the domain of H is a finite-dimensional separable manifold, we may apply Sard's theorem to conclude that its differential is full rank at some point. Applying the implicit function theorem, and restricting the domain of H appropriately, we are again in the situation of Lemma 4.2.

Now we approximate H by mappings H_N constructed as follows. For each N and each $(a_0, \ldots, a_\sigma)$, consider the piecewise constant (sampled) control in L_∞^m which has in the interval $[kT/N, (k+1)T/N]$ the value of (4.5) at (say) kT/N. These converge uniformly to (4.5), and hence are admissible and can be applied to ξ for N sufficiently large. Then $H_N(a_0, \ldots, a_\sigma)$ is by definition the application of this control to ξ. Thus H_N contains an open neighbourhood of ζ for large enough N, and this proves the lemma (with $\sigma = N - 1$). □

4.4. *All notions are equivalent*

The following shows that the controllability notions introduced in this section are very natural. The set of all states to which ξ can be steered in time exactly T is $A^T(\xi)$.

Proposition 4.2

Assume that the system Ξ is controllable. Then the following properties are all equivalent.

(*a*) $\mathscr{L}^0$ has full rank.

(*b*) There exist ξ, ζ in S_Ξ such that ξ can be sn-controlled to ζ.

(*c*) Ξ is sn-controllable.

(*d*) There exist ξ, ζ in S_Ξ such that ξ can be ns-controlled to ζ.

(*e*) Ξ is ns-controllable.

(*f*) There exist ξ, ζ in S_Ξ such that ξ can be pcns-controlled to ζ.

(*g*) Ξ is pcns-controllable.

(*h*) Some set as in (4.4) has non-empty interior.

(*i*) Some set $A^T(\xi)$ has non-empty interior.

(*j*) Some set $\mathscr{P}_{k,T}$ has non-empty interior.

Proof

We pointed out above that (*g*) implies (*e*). That (*b*) implies (*j*) is proved in Lemma (4.3), while Lemma (4.4) shows that (*j*) implies (*h*).

The 'transitivity' properties discussed earlier establish that (*b*) is equivalent to (*c*), (*d*) is equivalent to (*e*), and (*f*) is equivalent to (*g*). Note also that (*h*) trivially implies (*i*).

The equivalence between (*i*) and (*a*) was proved by Sussmann and Jurdjevic (1972); see for instance Sontag (1986) for a somewhat more general result.

If (*a*) holds, then Lemma 4.1 together with (Sontag, 1983, Lemma 2.2), implies that (*b*) holds.

The implicit function theorem, applied to the map $\psi(T, \xi, \cdot)$ on L_∞^m, gives that (*d*) implies (*i*).

Finally, statements (*f*) and (*h*) are equivalent. Indeed, we have already remarked that pcns-controllability implies that some set as in (4.4) has a non-empty interior. Conversely, assume that such a given set has a non-empty interior. The map (4.1) is smooth (composition of the linear bounded map in (4.2) with $\psi(T, \xi, \cdot)$), and its image is $A^T_{s_1,\dots,s_\sigma}(\xi)$. Since the domain is a separable finite-dimensional manifold (open subset of $U_\Xi^{\sigma+1}$), we may apply Sard's theorem to conclude that (4.1) must be full rank at some point, i.e. ξ can be pcns-controlled to some ζ. □

5. Proof of Theorem 1.1

In this section we prove that the conclusions of the theorem follow from ns-controllability of the system. From the remarks in the previous section, this will be all that is needed in order to prove the theorem. The following is the main technical step needed. It establishes basically that if the system is ns- (or pcns-, or sn-) controllable, then one can transfer states *non-singularly* using polynomial controls. We know from Proposition 4.2, part (j), that we can do so using polynomial controls. An application of Sard's theorem will then ensure that from each state we can go nonsingularly and polynomially into *some* other state. But in order to transfer to a predetermined state it is necessary to concatenate this resulting control with another control, and the concatenation will in general not be polynomial but at best a spline. So a different argument, again based on the above fixed-point theorem, is needed.

Proposition 5.1

Assume that ξ and ζ are in S_Ξ and that ξ is pcns-controllable into ζ in time T. There exist then an integer ρ, a compact subset K of $U_\Xi^{\sigma+1}$, and neighbourhoods $\mathcal{W}_\xi$ and $\mathcal{W}_\zeta$ of ξ and ζ respectively, with the following properties.

(a) Let u be the control $v_0 + \dots + v_\rho t^\rho$ of length T, where $(v_0, \dots, v_\rho)$ is in K. Pick any $x \in \mathcal{W}_\xi$. Then u can be applied to x, and Ξ is linearly controllable along the ensuing trajectory.

(b) If $x \in \mathcal{W}_\xi$ and $z \in \mathcal{W}_\zeta$ then there exists some u as above, such that u steers x into z.

Proof

Let $\xi, \zeta, u, T, s_1, \dots, s_\sigma, \upsilon_0, \dots, \upsilon_\sigma$ be as in the definition of pcns-steering. Let

$$H(v_0, \dots, v_\sigma) := \psi(T, \xi, u(s_1, \dots, s_\sigma; v_0, \dots, v_\sigma; \cdot))$$

defined on a neighbourhood of $\upsilon_0, \dots, \upsilon_\sigma$. Note that u ns-steers ξ into ζ, as discussed earlier. There is by the implicit function theorem applied to H a neighbourhood $\mathcal{V}_\zeta$ of ζ and a smooth

$$\beta : \mathcal{V}_\zeta \to \mathbb{R}^{(\sigma+1)m}$$

with $H(\beta(z)) = z$ on $\mathcal{V}_\zeta$. By continuous differentiability of $\psi(T, \cdot, \cdot)$ there is a neighbourhood $\mathcal{A}$ of u (say, in B_2^m), and a neighbourhood $\mathcal{V}_\xi$ of ξ, such that each $\upsilon \in \mathcal{A}$ can be applied nonsingularly to each x in $\mathcal{V}_\xi$. Let $\mathcal{W}$ be a compact neighbourhood of $(\upsilon_0, \dots, \upsilon_\sigma)$ such that $u(s_1, \dots, s_\sigma; v_0, \dots, v_\sigma; \cdot)$ is in $\mathcal{A}$ whenever $(v_0, \dots, v_\sigma)$ is in $\mathcal{W}$. Consider H as a map restricted to $\mathcal{W}$, and restrict $\mathcal{V} = \mathcal{V}_\zeta$ so that β maps into $\mathcal{W}$. We apply Lemma 4.2 to obtain an ε, $\mathcal{V}'$ as there. Let $\mathcal{W}_\zeta := \mathcal{V}'$. For each $x \in \mathcal{V}_\xi$ and all large N,

$$H_{x,N}(v_0, \dots, v_\sigma) := \psi(T, x, p_N(s_1, \dots, s_\sigma; v_0, \dots, v_0; \cdot))$$

(notations as in § 3.2) is well-defined on $\mathcal{W}$, and $p_N(s_1, \dots, s_\sigma; v_0, \dots, v_\sigma; \cdot)$ can be nonsingularly applied to x. The $H_{x,N}$ converge uniformly to H as $x \to \xi$ and $N \to \infty$.

Thus there are a neighbourhood $\mathscr{W}_\xi$ of ξ and an N large enough that

$$\mathscr{W}_\zeta \subseteq H_{x,N}(\mathscr{W}) \tag{5.1}$$

and

$$p_N(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; \cdot\,) \text{ can be non-singularly applied to } x \tag{5.2}$$

for all $x \in \mathscr{W}_\xi$ and all $(v_0, \ldots, v_\sigma)$ in $\mathscr{W}$. By (5.1), if $x \in \mathscr{W}_\xi$ and $z \in \mathscr{W}_\zeta$ then z is in the image of $H_{x,N}$. Finally, let $\rho := 2N$ and

$$K := \left\{(\mu_0, \ldots, \mu_\sigma) \mid \quad \text{for some} \quad (v_0, \ldots, v_\sigma) \in \mathscr{W}, \quad p_N(s_1, \ldots, s_\sigma; v_0, \ldots, v_\sigma; t) = \sum_{i=0}^{2N} \mu_i t^{2N-i} \right\}$$

This set is compact, because the coefficients of p_N are continuous functions of the v_i—see (3.9). The result then follows from this construction.

We may now complete the proof of the theorem. Assume then that the system Ξ is sn-controllable, and apply the above proposition to it.

Fix ξ, ζ, and corresponding ρ, K, $\mathscr{W}_\xi$ and $\mathscr{W}_\zeta$. Choose also an open neighbourhood $\mathscr{V}_\xi$ of ξ with the property that its closure clos($\mathscr{V}_\xi$) is contained in $\mathscr{W}_\xi$. Similarly, pick an open neighbourhood $\mathscr{V}_\zeta$ of ζ with clos($\mathscr{V}_\zeta$) $\subseteq \mathscr{W}_\zeta$. Consider the mapping

$$\alpha : \mathbb{R}^{(\rho+1)m} \to \{L_\infty^m\}^{\rho+1}$$

$$\alpha(u_0 : \ldots : u_\rho) := (u_0 : u_0 t + u_1 : \ldots : u_0 t^\rho + \ldots + u_\rho)$$

This is (linear and) continuous. Let χ be the last component of this map,

$$\chi(u_0 : \ldots : u_\rho) := (u_0 t^\rho + \ldots + u_\rho)$$

Then, $\alpha(K)$ is a compact subset $\mathscr{K}$ of $\{L_\infty^m\}^{\rho+1}$, and $\mathscr{K}_\rho := \chi(K)$ is also compact. By the conclusions of Proposition 5.1, each control function $\omega \in \mathscr{K}_\rho$ is admissible for the system Ξ and can be applied to each $x \in \mathscr{W}_\xi$. Thus

$$[0, T] \times \text{clos}(\mathscr{V}_\xi) \times \mathscr{K}_\rho$$

is a compact subset of the domain of ψ. Consider now the mapping

$$(t, x, w) \mapsto (\alpha(w)(t), \psi(t, x, \chi(w)))$$

defined on this compact set. This is the composition of the continuous map

$$(t, x, w) \mapsto (t, \alpha(w), \psi(\,\cdot\,, x, \chi(w)))$$

into $[0, T] \times \mathscr{K} \times C^0([0, T], \mathbb{R}^n)$ with the continuous evaluation mapping

$$(t, \phi, \xi) \mapsto (\phi(t), \xi(t))$$

(defined for ϕ on the subspace of continuous functions in L_∞^m). Thus its image $\mathscr{O}_{\xi\zeta}$ consisting of all those $(v : y) \in \mathbb{R}^{(\rho+1)m} \times S_\Xi$ such that

$$\exists\, t \in [0, T], \quad x \in \text{clos}(\mathscr{V}_\xi), \quad w \in K^{\rho+1} \quad \text{with} \quad v = \alpha(w)(t) \quad \text{and} \quad y = \psi(t, x, \chi(w))$$

is also compact.

Let C be the compact set in the statement of the theorem. Now cover $C \times C$ by sets of the type $\mathscr{V}_\xi \times \mathscr{V}_\zeta$. Let $\mathscr{V}_i \times \mathscr{V}'_i$, $i = 1, \ldots, s$ be a finite subcover, and let subscripts 'i'

be used for the associated data as above. Write $\mathcal{O}_i$ instead of $\mathcal{O}_{\xi_i,\zeta_i}$, and let ρ be the largest of the ρ_i, T the largest of the T_i. Introduce

$$\Omega := \{\mathbb{R}^m\}^\rho \times U_\Xi = \mathbb{R}^{m(\rho+1)}$$

Finally, let

$$P(w_0 : \ldots : w_\rho) := (0 : w_0 : \ldots : w_{\rho-1})$$

$$Q(w_0 : \ldots : w_\rho) := w_\rho$$

These are the maps defining the open-loop control generator. For each $i = 1, \ldots, s$, let

$$\mathcal{O}'_i := \{(0 : \ldots : 0 : v : x) \in \Omega \times S_\Xi \mid (v : x) \in \mathcal{O}_i\}$$

where there are $\rho - \rho_i$ blocks of zeros. Each of these is a compact set, so that their union

$$\mathcal{O} := \cup\{\mathcal{O}_i, i = 1, \ldots, s\}$$

is also.

We now prove that this set $\mathcal{O}$ indeed satisfies the properties in the conclusions of the theorem. We first show that $\mathcal{O}$ is included in $ND(\Omega\downarrow\Xi)$. For this it is sufficient to show that each $\mathcal{O}'_i \subseteq ND(\Omega\downarrow\Xi)$. So fix i, and drop for the rest of this paragraph the indices i—all data will refer to this i. The elements of $\mathcal{O}'$ are of the form $(0 : \ldots : v : y)$ with

$$v = \alpha(w)(t) \quad \text{and} \quad y = \psi(t, x, \chi(w))$$

for

$$t \in [0, T], \quad w \in K^{\rho+1}, \quad x \in \operatorname{clos}(\mathcal{V}_\xi) \quad (\xi = \xi_i)$$

This is the same as the solution $(\omega(t), \xi(t))$ at time t of

$$\dot{\omega}_0(t) = 0$$
$$\vdots$$
$$\dot{\omega}_\rho(t) = \omega_{\rho-1}$$
$$\dot{\xi}(t) = f(\xi(t), \omega_\rho(t))$$

that starts at $\omega(0) = w$ and $\xi(0) = x$. As remarked earlier, $ND(\Omega\downarrow\Xi)$ is forward-invariant. Thus it is sufficient to establish that (w, x) is in $ND(\Omega\downarrow\Xi)$. But $x \in \mathcal{W}_\xi$, and $w = (u_0 : \ldots : u_\rho$ as all block components u_i in K. Thus (w, x) is indeed non-degenerate, by the conclusions of Proposition 5.1.

Finally, take any $(x, z) \in C$. Then there is some i such that (x, z) is in $\mathcal{V}_i \times \mathcal{V}'_i$, and hence is in $\mathcal{W}_i \times \mathcal{W}'_i$. Thus there is a sequence w in K such that $\chi(w)$ (ns-) steers x into z, and by construction, so that $(\alpha(w)(t) : \xi(t))$ is in $\mathcal{O}$ for all $t \in [0, T_i]$. □

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REFERENCES

BAUMANN, W. T., and RUGH, W. J., 1986, *I.E.E.E. Trans. autom. Control*, **31**, 40.

CHAMPETIER, C., MOUYON, P., and MAGNI, F. G., 1985, Pseudolinearization of nonlinear systems by dynamic precompensation. *I.E.E.E. Conf. on Decision and Control*, Fort Lauderdale, Florida, December 1985.

Elliot, D. L., 1971, *J. Diff. Eqns*, **10**, 364.
Grasse, K. A., 1981, *SIAM Jl Control Optim.*, **19**, 203.
Isidori, A., 1985, *Nonlinear Control Systems: An Introduction* (Berlin: Springer).
Lee, E. B., and Markus, L., 1968, *Foundations of Optimal Control Theory* (New York: Wiley).
Reboulet, C., and Champetier, C., 1984, *Int. J. Control*, **40**, 631.
Rugh, W. J., 1983, Linearization about constant operating points: an input-output viewpoint. *Proc. I.E.E.E. Conf. on Decision and Control*, San Antonio, Texas.
Sontag, E. D., 1983, Remarks on the preservation of various controllability properties under sampling. *Developpement et Utilisation d'Outils et Modèles Mathématiques en Automatique, Analyse de Systèmes et Traitement de Signal*, Coll. CNRS, RCP 567, Belle-Ile, France, pp. 623–637; 1986, Orbit theorems and sampling. *Algebraic And Geometric Methods in Nonlinear Control Theory*, edited by M. Fliess and M. Hazewinkel (Dordrecht: Reidel) pp. 441–483; 1987 a, Equilinearization: a simplified derivation and experimental results. *Proc. Conf. on Information Sci. and Systems*, Johns Hopkins University Press, Baltimore; 1987 b, *I.E.E.E. Trans. autom. Control*, **32**, 877.
Sussmann, H. J., 1976, *J. Diff. Eqns*, **20**, 292; 1979, *Ibid.*, **31**, 31.
Sussmann, H. J., and Jurdjevic, V., 1972, *J. Diff. Eqns*, **12**, 95.