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# Further Facts about Input to State Stabilization

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*Abstract*—**Previous results about input to state stabilizability are shown to hold even for systems which are not linear in controls, provided that a more general type of feedback be allowed. Applications to certain stabilization problems and coprime factorizations, as well as comparisons to other results on input to state stability, are also briefly discussed.**

## I. Introduction

In a previous paper [3] we studied the problem of when a system on $\mathbb{R}^n$,

$$\dot{x} = f(x) + G(x)u \tag{1}$$

with $f$ and the entries of the $n \times m$ matrix $G$ being smooth, can be made input to state stable (*ISS*) in a rather strong sense to be reviewed below. Our main result there was that this system is *smoothly input to state stabilizable*, that is, there exists a smooth (i.e., infinitely differentiable) map $K: \mathbb{R}^n \to \mathbb{R}^m$ with $K(0) = 0$ and such that under the control law $u = K(x) + v$ the new system

$$\dot{x} = (f(x) + G(x)K(x)) + G(x)u$$

is ISS, *if and only if* the system (1) is *smoothly stabilizable*, that is, there exists an (in general different) $K$ so that

$$\dot{x} = f(x) + G(x)K(x)$$

is globally asymptotically stable (*GAS*). The necessity is a trivial consequence of our definition of ISS, which implies GAS, but the converse is somewhat harder to establish. (It is based essentially on using a new feedback $u = K(x) - (\nabla V \cdot G)' + v$, where $V$ is a suitably chosen Lyapunov function.)

It is natural to ask if the same result can be proved for the more general system

$$\dot{x} = f(x, u) \tag{2}$$

which is not necessarily linear in $u$. More precisely, we assume that $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, that $f$ is a differentiable (much less is needed) function from $\mathbb{R}^{n+m}$ into $\mathbb{R}^n$, and that 0 is an equilibrium point for the system, $f(0, 0) = 0$. Unfortunately, the result does not generalize. As a counterexample, let $m = n = 1$ and consider the system

$$\dot{x} = -x + u^2x^2.$$

We claim that, for no possible feedback law $K$, can there hold for

$$\dot{x} = -x + (K(x) + u)^2x^2 \tag{3}$$

that: a) for $u \equiv 0$ the system is GAS, and b) for $u \equiv 1$ and initial condition $x(0) = 4$ the solution remains bounded. Indeed, property a) implies that $|K(x)| < 1/\sqrt{x}$ for all $x > 0$, but then this implies that the right-hand side of (3) is positive for all $x \geq 4$, so the solution with $x(0) = 4$ diverges to $+\infty$. So not even a very weak notion of input to state stability can be obtained for this example.

However, if one allows instead more general feedback laws, of the type

$$u = K(x) + G(x)v \tag{4}$$

where $G$ is an $n \times n$ matrix of smooth functions invertible for all $x$, not necessarily the identity, then the theorem *can* be proved for the larger class of systems. This more general class of feedback laws is of some interest—for instance, it is the class used in most modern "geometric" nonlinear control—although it is not appropriate for solving the type of problem (Bezout-type coprime factorizations) that was of interest in [3]. One may weaken the concept of coprimeness, however, and then the generalized theorem becomes applicable in this area as well; this will be discussed later.

In this note we establish the generalized result, for (2) under feedback laws (4). As an application, we give an alternative proof of a "folk" fact about dynamic extensions and a result which shows that GAS by itself is sufficient to guarantee a notion of ISS with "small controls." Applications to coprime factorizations and the local stabilization of cascaded systems are also given. As a corollary we show, using only elementary techniques, that *any cascade* of locally asymptotically stable systems is again asymptotically stable; no assumptions need to be made, as would be the case with center-manifold type of arguments. See [8] for a Lyapunov-theoretic proof of this, and [5] for a global version as well as further results.

It should be remarked that global smoothness of control laws is in general a restrictive requirement; see [4] for a discussion of this point and references to the literature.

## II. Definitions and Statement of Results

We recall the basic terminology from [3]. The function $\gamma: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{K}$ if it is continuous strictly increasing and satisfies $\gamma(0) = 0$; it is *of class* $\mathcal{K}_\infty$ if in addition $\gamma(s) \to \infty$ as $s \to \infty$. If $\gamma$ is of class $\mathcal{K}_\infty$ then the inverse function $\gamma^{-1}$ is well defined and is again of class $\mathcal{K}_\infty$. A function $\beta: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{KL}$ if for each fixed $t$ the mapping $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s$ it is decreasing to zero on $t$ as $t \to \infty$. We use single bars $|\xi|$ to denote Euclidean norm of states and controls, and use $\|u\| := \text{ess.sup.}\{|u(t)|, t \geq 0\}$ for measurable essentially bounded controls.

The system (2) is *globally asymptotically stable* (*GAS*) if there exists a function $\beta(s, t)$ of class $\mathcal{KL}$ such that, with the control $u \equiv 0$, given any initial state $\xi_0$ the solution exists for all $t \geq 0$ and it satisfies the estimate

$$|x(t)| \leq \beta(|\xi_0|, t).$$

The system is *input to state stable* (*ISS*) if there is a function $\beta$ of class $\mathcal{KL}$ and there exists a function $\gamma$ of class $\mathcal{K}$ such that for each measurable essentially bounded control $u(\cdot)$ and each initial state $\xi_0$, the solution exists for each $t \geq 0$ and furthermore it satisfies

$$|x(t)| \leq \beta(|\xi_0|, t) + \gamma(\|u\|). \tag{5}$$

Since $\gamma(0) = 0$, an ISS system is necessarily GAS; the latter is equivalent to the usual notion of asymptotic stability ("$\epsilon - \delta$" stability plus attractivity). The former says basically that for bounded initial state and control, a bounded trajectory results, and further (since $\beta$ decays) that eventually the state is bounded by a function of the control alone (and this bound is small if the control is small). This is much stronger that asking GAS plus "bounded-point bounded-state" stability. To see this, take the system

$$\dot{x} = (\sin^2 u - 1)x.$$

This is "BIBS," and with control $u \equiv 0$ one has $\dot{x} = -x$ which is globally stable. However, with $u \equiv \pi/2$ trajectories do *not* become ulti-

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mately bounded by a constant independent of $\xi_0$. Since we prove mainly positive results, choosing such a strong notion of stability makes the results more informative. (The one negative example given earlier, dealing with restrictive types of feedback laws, was such that not even the weakest possible version of these properties holds.)

We shall say that the system (2) is *weakly input to state stabilizable* if there exists a continuously differentiable map $K: \mathbb{R}^n \to \mathbb{R}^m$ with $K(0) = 0$ and an $n \times n$ matrix $G$ of continuously differentiable functions, invertible for each $x$, such that under the control law (4) (and replacing $v$ by $u$) the system

$$\dot{x} = f(x, K(x) + G(x)u) \tag{6}$$

is ISS. We say that the system is *stabilizable* if there is a differentiable $K$ such that this new system (with $G \equiv 0$) is GAS. The first property implies the second. The next result shows they are equivalent. The terminology "weak" is used to avoid confusion with the notion in [3], where $G =$ identity is required.

*Theorem 1:* For (2) under control laws of the type (4), stabilizability implies weak input to state stabilizability.

To prove this theorem, we may assume that the original system is already GAS, and we look for a $G$ so that $\dot{x} = f(x, G(x)u)$ is ISS. The idea of the proof can be easily understood in the case of one-dimensional systems ($n = 1$). Since the system is GAS, it is necessary in that case that

$$xf(x, 0) < 0$$

for all nonzero $x$. Thus, by continuity also $xf(x, u) < 0$ for small $u$. So it is only necessary to multiply controls by a very small number in order to have states approach the origin. The same idea works in higher dimensions, using Lyapunov functions, but working out the details takes some effort. The proof is given in the next section; it is far less constructive than the result given in [3].

The one-dimensional case also suggests that as long as the controls remain small, ISS may hold. This idea gives rise to our other result.

*Theorem 2:* If the system (2) is GAS, then there exist a function $\beta$ of class $\mathcal{KL}$, a function $\gamma$ of class $\mathcal{K}$, and a continuous function $\sigma: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $\sigma(s) \neq 0$ if $s \neq 0$, such that, for each initial state $\xi_0$ and for each measurable essentially bounded control $u(\cdot)$ for which

$$\|u\| \leq \sigma(|\xi_0|),$$

the solution of (2) exists for each $t \geq 0$ and it satisfies (5).

This theorem is not at all surprising, and it is probably a particular case of some result on "total stability," but we have not found it stated in this form, and in any case it is very easy to establish once that we have the proof of the first result. In the case of exponential stability, where a Lyapunov function of quadratic growth exists (see, e.g., [7] and [1]), it is much easier to prove.

## III. Proof of Main Results

We first prove a technical lemma needed later.

*Lemma 3.1:* Assume that $\delta: \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous mapping such that $\delta(s, 0) < 0$ for all $s > 0$. Then, there exist

- a smooth function $g: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$;
- a function $\alpha_4$ of class $\mathcal{K}_\infty$; and
- a continuous function $\theta: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, with $\theta(s) \neq 0$ for every $s \neq 0$ and decreasing on $[1, +\infty)$;

such that

1) $g(s) \neq$ for all $s$, and $g(s) \equiv 1$ for $s \in [0, 1]$;
2) for each pair $(s, r) \in \mathbb{R}^2_{\geq 0}$ for which $r < \theta(s)$, necessarily $\delta(s, r) < 0$; and
3) for each pair $(s, r) \in \mathbb{R}^2_{\geq 0}$ for which $\alpha_4(r) < s$ and each $v \leq g(s)r$, necessarily $\delta(s, v) < 0$.

*Proof:* First observe that, for each closed and bounded interval $I \subseteq \mathbb{R}_{\geq 0}$, there exists some $r_0 > 0$ such that

$$s \in I \Rightarrow \delta(s, r) < 0 \qquad \text{for all } r \in [0, r_0]. \tag{7}$$

This follows from the uniform continuity of $\delta$ on (for instance) the interval $I \times [0, 1]$. Now consider the intervals $[1, 2]$, $[2, 3], \cdots$, and pick for each of these an $r_i > 0$, $i = 1, 2, \cdots$, so that (7) holds; without loss of generality, take $r_1 < 1$ and assume that the $r_i$ decrease with $i$. Similarly, choose a sequence of decreasing positive numbers $r_i'$, with $r_1' < r_1$, corresponding to the intervals $[1/2, 1]$, $[1/3, 1/2]$, $[1/4, 1/3], \cdots$. We also assume (changing the $r_i$'s if necessary) that $r_1' > r_2$.

Now define the piecewise constant function $\rho$ as follows: on each interval of the form $[k, k+1)$ (integer $k > 0$), let $\rho \equiv r_k$, and on each interval $[1/(k+1), 1/k)$ let $\rho \equiv r_k'$. By construction, it holds that, for each $(s, r) \in \mathbb{R}^2_{\geq 0}$,

$$r < \rho(s) \Rightarrow \delta(s, r) < 0. \tag{8}$$

We next define $g$, $\alpha_4$, $\theta$.

Let $g$ be any smooth function, never zero, which is identically 1 on the interval $[0, 1]$ and which satisfies the inequalities

$$g(s) < \frac{\rho(s)}{s} \qquad \text{if } s \geq 2 \tag{9}$$

$$g(s) \leq 1 \qquad \text{for all } s. \tag{10}$$

Such a $g$ can always be defined: simply pick a function on each interval $[k, k+1]$, $k \geq 2$ which has all derivatives equal to zero at the endpoints and so that $r_k/(k+1) = g(k) \leq g(s) \leq g(k+1)$ on each such interval, pick a similar function on $[1, 2]$ with values $\leq 1$ and $g(1) = 1$, and then patch these together with $g \equiv 1$ on $[0, 1]$. Clearly this $g$ satisfies the conditions in the first conclusion of the lemma.

Let $\alpha$ be any function of class $\mathcal{K}_\infty$ which satisfies

$$\alpha(s) < \rho(s) \qquad \text{if } s \leq 2 \tag{11}$$

$$\alpha(s) < s \qquad \text{for all } s \geq 2. \tag{12}$$

For instance, one may take $\alpha$ piecewise linear, with linear values on each interval of the type $[1/(k+1), 1/k]$, as well as on $[1, 2]$, and interpolating $\alpha(1/k) = r_k'/2$, $\alpha(2) = r_1/2$, and linear on $[2, +\infty)$ with $\alpha(s) = s + r_1/2 - 2$ there. Let then $\alpha_4 := \alpha^{-1}$.

Finally, let $\theta$ be defined just as $\alpha$ on $[0, 1]$ but piecewise linear interpolating $\alpha(k) = r_k/2$ for $k = 2, 3, \cdots$. Since $\theta(s) < \rho(s)$ for all $s$, the second property in the statement is a consequence of (8).

We must prove that the third property in the statement of the lemma holds. Pick then a pair $(s, r) \in \mathbb{R}^2_{\geq 0}$ for which $\alpha_4(r) < s$, and any $v < g(s)r$. Note that $r < \alpha(s)$ for such pairs, so $v \leq g(s)\alpha(s)$. In case $s \leq 2$, (11) together with (10) imply that

$$v \leq \rho(s)$$

while for $s \geq 2$ the same conclusion follows from (9) and (12). Then $\delta$ must be negative, again by (8). ∎

Assume that the system $\dot{x} = f(x, u)$ is GAS. As in [3], we know that there exists a Lyapunov function for the system $\dot{x} = f(x, 0)$, that is, there is a smooth function $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1$, $\alpha_2$, $\alpha_3$ of class $\mathcal{K}_\infty$ such that, for each $\xi \in \mathbb{R}^n$,

$$\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|) \tag{13}$$

and

$$\nabla V(\xi) \cdot f(\xi, 0) + \alpha_3(|\xi|) < 0. \tag{14}$$

With respect to these functions we have the following lemma.

*Lemma 3.2:* There exist a function $\alpha_4 \in \mathcal{K}_\infty$ and an $m \times m$ matrix $G$ of infinitely differentiable functions on $\mathbb{R}^n$, $G(x)$ invertible for each $x$, such that for each $\xi_0 \in \mathbb{R}^n$ and each essentially bounded measurable $u(\cdot)$, the following property holds for the solution of

$$\dot{x} = f(x, G(x)u) \tag{15}$$

with $x(0) = \xi_0$: for each $t \geq 0$ for which the solution is defined,

$$|x(t)| > \alpha_4(\|u\|) \Rightarrow \nabla V(x(t)) \cdot f(x(t), G(x(t))u(t)) + \alpha_3(|x(t)|) < 0. \tag{16}$$

(To be more precise, this holds for almost all $t$, since $u$ is merely assumed to be measurable.)

*Proof:* We define the (continuous) function $\delta: \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$:

$$\delta(s, r) := \max_{|\xi|=s, |\mu|=r} \nabla V(\xi) \cdot f(\xi, \mu) + \alpha_3(|\xi|).$$

Note that for $s > 0$, it holds that $\delta(s, 0) < 0$, because of (14) and compactness of the sphere of radius $s$. Apply Lemma 3.1 to obtain functions $\alpha_4$ and $g$, and let

$$G(\xi) := g(|\xi|) \cdot I$$

which is smooth because $g$ is smooth and constant near 0. We must prove that (16) holds. So assume that $|x(t)| > \alpha_4(\|u\|) \geq \alpha_4(|u(t)|)$, and let $s := |x(t)|$ and $r := |u(t)|$. By definition of $\delta$,

$$\nabla V(x(t)) \cdot f(x(t), G(x(t))u(t)) + \alpha_3(|x(t)|) \leq \delta(s, v)$$

where

$$v = |G(x(t))u(t)| \leq g(s)r.$$

Since $s > \alpha_4(r)$, $\delta$ must be negative because of the third conclusion in Lemma 3.1. ■

The proof of Theorem 1 is now completed precisely as that of the main result in [3]: the above lemma gives (36) in that paper, and the rest of the proof goes through word by word (with the same notation). ■

We now prove Theorem 2. Let $\delta$ be as above, and assume that $\alpha_4$ and $\theta$ are as in Lemma 3.1. The functions $\beta$ and $\gamma$ constructed in [3] from $\alpha_1$ to $\alpha_4$, together with the function $\sigma$ to be constructed below, satisfy the conclusion of the theorem. To see this, we first establish two facts.

Assume that $x(\cdot)$ is a solution of (2) with initial condition $x_0$ and with a control $u$ which satisfies the inequalities

$$\|u\| \leq \begin{cases} \alpha_4^{-1}(\alpha_2^{-1}(\alpha_1(1))) & \text{(i)} \\ \theta(\alpha_1^{-1}(\alpha_2(|x_0|))) & \text{(ii)} \\ \theta(|x_0|) & \text{(iii)}. \end{cases}$$

Let $S$ be the closed set

$$S := \{\xi \in \mathbb{R}^n | V(\xi) \leq \alpha_2(\alpha_4(\|u\|))\}$$

and let

$$\Delta := \{(\xi, v) \in \mathbb{R}^{n+m} | \nabla V(\xi) \cdot f(\xi, v) + \alpha_3(|\xi|) \geq 0\}$$

which is also closed.

*Fact 3.3:* If for some $t$ it holds that $V(x(t)) < V(x_0)$ and that $(x(t), \mu) \in \Delta$ for some $\mu$ so that $\mu \leq \|u\|$, then $x(t) \in S$.

*Proof:* From (13) and the first assumption, it follows that

$$|x(t)| < \alpha_1^{-1}(\alpha_2(|x_0|)).$$

Assume that if it were the case that $|x(t)| > 1$, then it would follow (because $\theta$ decreases on $[1, +\infty)$) that $\theta(|x(t)|) > \theta(\alpha_1^{-1}(\alpha_2(|x_0|)))$, which is greater than $\|u\|$ because of (ii) above, and hence

$$\theta(|x(t)|) > |\mu|$$

which implies, because of the second conclusion in Lemma 3.1, that $(x(t), \mu) \notin \Delta$, which is a contradiction. Thus, $|x(t)| \leq 1$.

If it were the case that

$$\alpha_4(|u|) < |x(t)|, \tag{17}$$

then $|x(t)| \leq 1$ would again imply the contradiction $(x(t), \mu) \notin \Delta$ (since $g \equiv 1$ for $s \in [0, 1]$).

So we need only prove that if $x(t) \notin S$, then (17) must hold. In other words, that

$$V(x(t)) > \alpha_2(\alpha_4(\|u\|))$$

implies (17). But this is clear from (13) and the fact that $\alpha_2$ is strictly increasing. ■

*Fact 3.4:* For the above trajectory, the set $S$ is invariant; that is, if $x(t_0) \in S$ for some $t_0$, then $x(t) \in S$ for all $t \geq t_0$.

*Proof:* Let $\xi := x(t_0)$ be like this. Note that

$$|\xi| \leq \alpha_1^{-1}(V(\xi)) \leq \alpha_1^{-1}(\alpha_2(\alpha_4(\|u\|))) \leq 1$$

by definition of $S$ and because of (i) above. Without loss of generality, if the fact is false, we may assume that there is a sequence $t_k \to t_0^+$ such that $\xi_k := x(t_k) \notin S$ (otherwise pick a larger $t_0$). So $V(\xi_k) > \alpha_2(\alpha_4(\|u\|))$ for each $k$, and by continuity of $V$ it follows that $V(\xi) \geq \alpha_2(\alpha_4(\|u\|))$ and therefore that

$$|\xi| > \alpha_4(\|u\|).$$

Again, because of the third conclusion in Lemma 3.1, this essentially implies that $(x(t), u(t)) \notin \Delta$, so $dV(x(t))/dt$ is negative at $t_0$. But then $V(\xi_k) < V(\xi)$ for all large $k$, contradicting the fact that $\xi_k := x(t_k) \notin S$, and the proof is complete. (The precise argument is a bit more subtle: it may happen that the absolutely continuous function $V(x(\cdot))$ is not differentiable at $t_0$. However, the fact that $(x(t), \mu) \notin \Delta$ for all $\mu \leq \|u\|$ means that, by continuity of $x(\cdot)$ and openness of the complement of $\Delta$, this derivative is negative for almost all $t$ in a neighborhood of $t_0$.) ■

We now complete the proof of Theorem 2. We define

$$\sigma(s) := \min\{\alpha_4^{-1}(\alpha_2^{-1}(\alpha_1(1))), \theta(\alpha_1^{-1}(\alpha_2(s))), \theta(s)\}.$$

The same proof [discussion after (36)] as in [3] works again, for controls satisfying $\|u\| \leq \sigma(|\xi_0|)$. It is only necessary to show that for each such trajectory either

$$(x(t), u(t)) \notin \Delta \qquad \text{for almost all } t \tag{18}$$

or there is some $t_0$ such that

$$(x(t), u(t)) \notin \Delta$$

for almost all $t < t_0$ and $x(t_0) \in S$ (and hence also $x(t) \in S$ for all $t \geq t_0$, by Fact 3.4). So assume that (18) would not hold. Let

$$t_0 := \sup\{\tau | (x(t), u(t)) \notin \Delta \qquad \text{for almost all } t < \tau\} < \infty.$$

Then also $(x(t), u(t)) \notin \Delta$ for almost all $t < t_0$, so $V(x(t)) < V(x_0)$. (Note also that, because $\sigma(s) < \theta(s)$ for all $s$, it holds that $t_0 > 0$.) From the definition of $t_0$, continuity of $x(\cdot)$, and closedness of $\Delta$, it follows that there is some $\mu < \|u\|$ such that $(x(t), \mu) \in \Delta$. The desired conclusion follows then from Fact 3.3. ■

## IV. Coprime Factorizations

We refer the reader to [3] for the definitions of input/output stable (IOS) operators, and the basic properties of this concept. We now modify Definition 4.1 (more precisely, its equivalent formulation via Lemma 4.2) in the above reference, to weaken the concept of coprimeness. See, e.g., [2] for this more general definition, which is not of a "Bezout" type as the one given in [3].

*Definition 4.1:* The I/O operator $P: \mathfrak{D}(P) \to L^p_{\infty,e}$ *admits a weakly coprime right factorization* if there exist IOS operators $A: L^m_{\infty,e} \times L^p_{\infty,e} \to L^m_{\infty,e}$, $N: L^m_{\infty,e} \to L^p_{\infty,e}$, and $D: L^m_{\infty,e} \to L^m_{\infty,e}$, such that $D$ is causally invertible, $\mathfrak{D}(D^{-1}) = \mathfrak{D}(P)$, $P = ND^{-1}$ and, if $I$ denotes the identity in $L^m_{\infty,e}$,

$$A \circ \begin{pmatrix} D \\ N \end{pmatrix} = I.$$

*Theorem 3:* If (2) is stabilizable, then its input to state mapping admits a weak coprime right factorization.

*Proof:* We first use Theorem 1 to obtain $G$ and $k$. Now, let $P$ be the operator defined as follows:

$$P\begin{pmatrix} v(\cdot) \\ x(\cdot) \end{pmatrix}(t) := G^{-1}(x(t))[-k(x(t)) + v(t)].$$

Since it is memoryless, and $G$ and $k$ are continuous everywhere, it follows that $P$ is IOS. The operators $N$ and $D$ are chosen precisely as in [3], that is, $N$ is the input to state mapping of the closed-loop system (6), and $D$ is the resulting mapping $u \mapsto k(x) + G(x)u$, which is itself stable (for the same reason that $P$ is) and invertible. ∎

## V. Further Remarks

In [7] (see also, e.g., [1]), it is proved that under certain hypotheses, GAS implies ISS (or more precisely, a notion of bounded-input bounded-state stability). The hypotheses are basically that the map $f$ be globally Lipschitz on $x$, uniformly in $u$ (if the system has the form $\dot{x} = f(x) + G(x)u$ this is ensured provided that the control term $G(x)$ is bounded) and that stability be *exponential*. Our first remark is that no such result holds if one has only nonexponential stability. To illustrate this point, it is sufficient to consider the system

$$\dot{x} = -\tanh x + u.$$

This is GAS, and $f(x, u)$ is globally Lipschitz, but with $u \equiv 1$ one has that all trajectories diverge to $+\infty$.

It was remarked in [3] that a cascade composition of what are there called "input/output stable" maps is again of that kind. In terms of state-space notions only, this translates into the fact that cascading a GAS system with an ISS one gives a GAS system. More precisely, consider the composite system

$$\dot{x} = f(x, z) \tag{19}$$

$$\dot{z} = g(z) \tag{20}$$

where the second is GAS and the first is ISS (with $z$ as control). The claim is that the composite system is GAS. Indeed, if $\beta_1$ and $\gamma$ are as in the definition of ISS for the first system, and $\beta_2$ is as in the definition of GAS for the second, then for each trajectory necessarily

$$|x(t)| + |z(t)| \le \beta(|x(0)| + |z(0)|, t)$$

where

$$\beta(s, t) := \beta_2(s, t) + \beta_1\left(\beta_1\left(s, \frac{t}{2}\right) + \gamma\left(\beta_2\left(s, \frac{t}{2}\right)\right), \frac{t}{2}\right) + \gamma\left(\beta_2\left(s, \frac{t}{2}\right)\right)$$

is again of class $\mathcal{KL}$.

If the first system is just GAS rather than ISS, there is no reason for the cascade to be again GAS. One possibility is to require less than ISS, for instance, a "bounded-input bounded-state" condition in addition to GAS which, as remarked earlier, is weaker (see [5]). Another possibility is to give only a local result, as provided below. However, it is often possible to apply feedback so that this cascade does become stable. This is based on the above discussion, as follows.

The proof of Theorem 1 gives some $G(x)$ so that $\dot{x} = f(x, G(x)u)$ is ISS. Changing coordinates

$$(x, z) \mapsto (x, y), \qquad y := G(x)^{-1}z$$

there results a new composite system

$$\dot{x} = f(x, G(x)y)$$
$$\dot{y} = \tilde{g}(x, y)$$

where the first system is now ISS. If it were possible to modify $\tilde{g}$ in some manner so that it becomes independent of $x$ and the second system is GAS, then the above argument gives that the composite system (even in the original coordinates) is GAS. This program can be carried out in the following situation. Assume that $z$ has dimension 1 and one considers the cascade

$$\dot{x} = f(x, z)$$
$$\dot{z} = u$$

(a "dynamic extension" of the first system). Then the above change of coordinates results in a second equation of the type

$$\dot{z} = h(x, z)u + k(x, z)$$

where $h$ is always nonzero. Applying the feedback law

$$u = \frac{1}{h(x, z)}(-k(x, z) - z)$$

results then in a composite GAS system. We thus recover the result (see, e.g., [5] for an application and references) that if a system is smoothly stabilizable, then adding an integrator does not change this property. ∎

Finally, we give an application of Theorem 2 to show that, for each compact subset of the state space, small enough controls tending to zero give rise to trajectories that also converge to zero. This is a local version of the corresponding global result given in [3], with basically the same proof.

*Corollary 5.1:* With the notations in Theorem 2, for each real number $k > 0$, the following property holds. For each control $u \to 0$ such that $\|u\| < \sigma(k)$ and each initial $|x_0| < k$, the solution of (2) exists for all $t > 0$ and it satisfies $\lim_{t \to 0} x(t) = 0$.

*Proof:* Let $l := \beta(k, 0) + \gamma(\sigma(k)) \le k$, and pick any $u$ and $x(\cdot)$ as above. We claim that $|x(t)| \le l$ for all $t$ (and in particular, then, the trajectory is defined globally). Indeed, if the trajectory ever exits the ball of radius $k$, then there is a first $T$ so that $|x(T)| = k$. Consider in that case the restriction of $u$ to times $t \ge T$,

$$v := u|_{[T, +\infty)} \tag{21}$$

which has norm again $< \sigma(k)$. Then, by Theorem 2,

$$|x(t)| \le \beta(|x(T)|, t - T) + \gamma(\|v\|) \tag{22}$$

which is less than $l$ for all $t$.

Pick any $\epsilon > 0$; we wish to show that $|x(t)| < \epsilon$ for all large $t$. Let $c > 0$ be the minimum of $\sigma$ on the interval $[\epsilon, l]$. Since $u(t) \to 0$, there is some $t_0$ such that $\gamma(\|v\|) < \epsilon$ and also $\|v\| < c$ for all restrictions $v$ as in (21) and all $T \ge t_0$. If it happens that $|x(t)| < \epsilon$ for all $t \ge t_0$, then there is nothing left to prove. Otherwise, there is some $T \ge t_0$ so that $\epsilon \le |x(T)| \le l$. For this $T$, (22) holds. Since the first term tends to zero and the second is less than $\epsilon$, it follows that $|x(t)| < \epsilon$ for large $t$, as desired. ∎

Together with the argument used in proving the stability of the cascade composition (19), (20), it follows then that, if (19) is GAS when $z \equiv 0$ and (20) is locally asymptotically stable (*LAS*), then the composed system (19), (20) is also LAS. Indeed, we may fix any $k$, and then pick a $0 < \delta < \sigma(k)$ for which all trajectories of (20) starting at every $|z_0| < \delta$ necessarily satisfy $z(t) \to 0$ and $|z(t)| < \sigma(k)$. Then the corollary guarantees that $x$ also goes to zero, and moreover that it remains bounded by $l = l(k)$, which is small for small $k$. In fact (we owe this observation to H. Sussmann), it is enough to suppose that the first system (19) is itself *locally* stable because one can just restrict the system to the domain of attraction of the origin (which is diffeomorphic to Euclidean space). We conclude the following fact, proved using very elementary techniques (see also [8]).

*Corollary 5.2:* If (19) is LAS when $z \equiv 0$, and if (20) is also LAS, then the composite system (19), (20) also is. ∎

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