CONTINUOUS STABILIZERS AND HIGH-GAIN FEEDBACK

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ABSTRACT

A controller is shown to exist, universal for the family of all systems of fixed dimension n, and m controls, which stabilizes those systems that are stabilizable, if certain gains are large enough. The controller parameters are continuous, in fact polynomial, functions of the entries of the plant. As a consequence, a result is proved on polynomial stabilization of families of systems.

1. Introduction.

This work continues the investigation of synthesis problems for parametrized families of systems. There are two main motivations for this line of research. The first is the expectation that parametrized controllers should prove useful in shifting the computational effort to "offline" preprocessing in situations in which the precise values of some system parameters are not known in advance but can be determined on-line. The second motivation is purely mathematical: it is natural to ask whether the constructions in control theory can be made "continuous" or "algebraic" in system parameters.

Consider, for any fixed positive integers n,m, the set of all possible continuous-time systems

$$\dot{x}(t) = Ax(t) + Bu(t) \ , \qquad (1.1)$$

for A an $n\times n$ and B an $n\times m$ real matrix. We know that, if a given pair (A,B) is stabilizable --that is, all uncontrollable eigenvalues of A have negative real part,-- then there exists a feedback matrix K = K(A,B) such that A-BK is Hurwitz (has all eigenvalues with negative real part). This construction is continuous, in fact smooth, on the stabilizable pairs (A,B), because a suitable K(A,B) can be found via the solution of a well-posed quadratic optimization problem; see for instance [D] for a discussion of this point. What is not known is if a stabilizing K(A,B) can be computed in a more algebraic fashion (the optimization argument depends on the implicit function theorem). We shall prove in this paper that this can indeed be done provided that *dynamic* feedback be allowed (we define "algebraic" precisely later).

Another natural question, which turns out to be related to the previous one about algebraic dependency, is whether it is possible to give a more general construction of "nice" K(A,B), for *arbitrary* (not necessarily stabilizable) pairs (A,B) with fixed (n,m), which results in a Hurwitz matrix A-BK(A,B) whenever the pair (A,B) happens to be stabilizable. Such questions are of interest in adaptive control. Posed in this way, the answer is negative even in the case n=1,m=1: as $a\to 1$ and $b\to 0$ the limit k(a,b) cannot be finite, since 1-0k(1,0) = 1 is not Hurwitz. A more plausible variation is suggested by a result in [S1] that says that there is a K(A,B) depending polynomially on arbitrary (A,B) with the property that, if

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(A,B) happens to be controllable, then A-γBK(A,B) is Hurwitz whenever the multiplicative gain γ is large enough. Moreover, an estimate on how large is "large enough" is given explicitely by the condition that $\gamma > \rho$(A,B), where ρ is a rational function with no poles at reachable (A,B). For instance, for n=m=1 we may choose k(a,b):= b; then $a-\gamma bk = a-\gamma b^2$ is negative whenever $\gamma > a/b^2$. Note that $b \neq 0$ is precisely the condition that characterizes controllability in this case.

We don't know if the above result can be generalized to work with stabilizable families (and $n,m \neq 1$). But we present here a variation of it which states essentially that the same is true provided that dynamic feedback is used. (And multiple gains are allowed.) As an easy consequence of this result and through the application of a theorem of Hormander ([H]), we conclude the above mentioned fact on algebraic dependency.

The paper [S3] presents an introductory survey to the general topic of control of parametrized families of systems, and should be consulted for other results and for a large list of references. (A sketch of a proof of the algebraic dependency result was given in an appendix to that paper. The proof here, though having many elements in common with that, is considerably simpler, mainly because the real algebraic material is left out of the main proof and appears only at the end through Hormander's theorem. Further, the results are stronger here, in that explicit multiplicative gains are constructed. On the other hand, discrete time systems are not treated here, and the reader is refered to [S3] for the appropriate generalizations.)

2. Definitions and Statement of Main Result.

It is worth giving some of the needed definitions and intermediate results in somewhat more generality than needed for the main results of this paper, since the proofs will be exactly the same, and the lemmas proved are of interest in themselves. The more general context is that of "systems over rings".

An (n,m) (free) *system* Σ over a commutative ring R is given by a pair of matrices A, B with $A \in R^{n \times n}$ and $B \in R^{n \times m}$. We shall be especially interested in two particular cases, "classical" *real* systems, for which R = $\Re$ = reals, and (polynomial) *families*, where $R = \Re[\lambda] = \Re[\lambda_1,\cdots,\lambda_r]$, the polynomial ring over the reals in the variables $\lambda = (\lambda_1,\cdots,\lambda_r)$, and r is an integer, the *number of parameters*. For any system (A,B) we consider its associated *controllability matrix* C = C(A,B); this is defined in block form as

$$C(A,B) := [B,AB,\cdots,A^{n-1}B] \in R^{n \times mn} .$$

By the Cayley-Hamilton theorem, the column module $\mathcal{C}$(A,B) of C(A,B) is A-invariant. Equivalently, there exists a matrix $D = D(A,B) \in R^{nm \times nm}$ such that

$$AC = CD .$$

(For a very readable and complete introduction to linear algebra over commutative rings, see [M].)

A very special system will be of interest, to which the intermediate lemmas will be applied in order to conclude the main result. For fixed (n,m), $R^{[n,m]}$ denotes the real polynomial ring in n(n+m) indeterminates, $R^{[n,m]} = \Re[\alpha,\beta]$, where $\alpha = (\alpha_{11},\cdots,\alpha_{nn})$ and $\beta = (\beta_{11},\cdots,\beta_{nm})$. The *universal* (n,m) system $\Sigma^{[n,m]}$ is the system over $R^{[n,m]}$ for which $(A^{[n,m]})_{ij} = \alpha_{ij}$ and $(B^{[n,m]})_{ij} = \beta_{ij}$. Any (n,m) real system (A,B) can be obtained by evaluating the entries of $A^{[n,m]}$ and $B^{[n,m]}$ at appropriate real numbers. If $a = (a_{11},\cdots,a_{nn})$ and $b = (b_{11},\cdots,b_{nm})$, we let $\Sigma^{[n,m]}(a,b)$, or just $\Sigma(a,b)$ denote the system obtained from the evaluations $\alpha_{ij} := a_{ij}$ and $\beta_{ij} := b_{ij}$. The corresponding pair of matrices is denoted by A(a) and B(b) respectively, to emphasize the fact that we are viewing the particular system as obtained by evaluation of the entries of the universal system at the vectors a and b respectively.

There are various abstract notions of stability for systems over rings, which generalize the standard one for real systems. See for instance [HS], [KS], [E]. One such notion is as follows. Assume given a multiplicatively closed subset S of the polynomial ring R[z] which consists entirely of monic (leading coefficient =1) polynomials and which contains at least one polynomial of positive degree. We call S a set of *Hurwitz* polynomials. With this definition, a linear map ϕ:M$\rightarrow$M, where M is an R-module, is *Hurwitz* iff there exists a Hurwitz polynomial p(z) which annihilates it: p(ϕ)=0. An $n\times n$ matrix A over R is Hurwitz if any (and hence all) linear maps it represents are Hurwitz, that is, if there is a Hurwitz p(z) with p(A)=0 as a matrix. When R = $\Re$, we take S = all monic polynomials with no zeroes in the closed right-half plane. In the case of families, we take S to be the set of all polynomials p(λ,z) in $\Re[\lambda][z] = \Re[\lambda_1,\cdots,\lambda_r][z]$ monic in z and such that p(l,z) is Hurwitz for all $l\in\Re^r$. That is, all "pointwise Hurwitz" polynomials.

(This definition of stability for linear maps over rings is slightly different from the usual one --see above references,-- where one asks that the characteristic polynomial of ϕ be itself Hurwitz. With this new approach, however, the definition of stabilizability becomes much more natural than in previous work. In any case, for the case of interest R = $\Re[\lambda]$, and A(λ) is a matrix, a pointwise argument with minimal polynomials shows that A(λ) is Hurwitz --in the sense defined above for families-- precisely when its characteristic polynomial is, or equivalently, iff A(l) is a classical Hurwitz matrix for each $l\in\Re^r$.)

Fix now a system (A,B) over R. Consider the controllability module $\mathcal{C}$(A,B) $\subseteq R^n$. Since $\mathcal{C}$ is A-invariant, there is a well-defined linear mapping

$$A_f : R^n/\mathcal{C} \rightarrow R^n/\mathcal{C}$$

induced by A. The subscript "f" is intended to indicate that A_f corresponds to the "free" dynamics of Σ, the part not influenced by controls. This can be made explicit ("Kalman decomposition") when $\mathcal{C}$ and $R^n/\mathcal{C}$ are free. (This happens, for all systems (A,B), if --and only if-- R is a field.) There is in that case a $T\in Gl(R,n)$ such that

$$T^{-1}C = \begin{pmatrix} C_1 \\ 0 \end{pmatrix} ,$$

where C_1 is a matrix of size $s\times nm$, s = rank of $\mathcal{C}$. For any such T, there are decompositions

$$T^{-1}AT = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

$$T^{-1}B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

where B_1 is $s\times m$, and where A_3 is an $(n-s)\times(n-s)$ matrix representing A_f. For the universal system $\Sigma^{[n,m]}$, we denote the mapping A_f corresponding to each specialization Σ(a,b) as $A(a,b)_f$. The 'b' serves to emphasize that this map depends on B(b) as well as on A(a).

The system (A,B) is called ("globally") *stabilizable* if A_f is Hurwitz. (As a convention, in the "completely controllable" case, in which $\mathcal{C}=R^n$, --so that A_f acts on a trivial module,-- we define (A,B) to be stabilizable.) For real systems, this is well-known to be equivalent to the existence of a matrix K such that A-BK is stable; for more general rings this is equivalent to the existence of a dynamic stabilizer over the ring (see below). For the case of a family (A(λ),B(λ)), i.e., a system over $\Re[\lambda_1,\cdots,\lambda_r]$, it is natural to also define (A,B) to be *pointwise stabilizable* if (A(l),B(l)) is stabilizable for each $l\in\Re^r$. It will follow from the material in this paper that global and pointwise stabilizability coincide for families.

For real systems, one extends results from controllable to stabilizable systems by decomposing (A,B) as above via the change of basis T, and noting that (A_1,B_1) is controllable. When dealing with rings, and in particular with the universal systems $\Sigma^{[n,m]}$, this cannot be done. For instance, for m=n=1, (α,β) is such that $R^1/\mathcal{C} = \Re[\alpha,\beta]/(\beta)$ is not a free $\Re[\alpha,\beta]$-module. More geometrically, the problem is that the reachability matrix does not have constant rank as (A,B) ranges over all possible (n,m)-systems, that is, $\mathcal{C}$ does not define a vector bundle over $\Re^{n(n+m)}$.

A dynamic controller for the system in equation (1.1) consists of a system of the same type, whose inputs are the states x(t) of (1.1) and whose output is the input u(t). Thus there are in that case a pair of equations

$$\dot{z}(t) = Fz(t) + Gx(t), \quad u(t) = Hz(t) + Jx(t), \tag{2.1}$$

where z(t) is for each t a vector of size k (=dimension of controller) and F,G,H,J are matrices of appropriate sizes. Equivalently, we may write the closed-loop equations (1.1)+(2.1) as the result of starting with the *k-th extension* of Σ:

$$\Sigma^k = (A^k,B^k) := \left(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix}\right),$$

(where I is a $k\times k$ identity matrix, so that this is an (n+k,m+k)-system) and applying feedback

$$K = \begin{pmatrix} J & H \\ G & F \end{pmatrix}$$

to Σ^k. Thus it is reasonable to define a dynamic feedback controller for the system (A,B) (over any ring R) as simply the specification of an integer k and an $(m+k)\times(n+k)$ matrix K over R. For families, this will correspond to the specification of a (polynomially parametrized) family of real systems as in (2.1).*

Consider now the universal system $\Sigma^{[n,m]}$, and let $R := R^{[n,m]}[\varepsilon]$, where ε is a new indeterminate (to be used to control stability margins). Assume we are given nonnegative integers κ,μ, matrices $K_o,\cdots,K_\mu$ in $R^{(m+\kappa)\times(n+\kappa)}$, such that the first m rows of K_o are identically zero, and elements ψ, and ϕ_i, θ_i, $i=1,\cdots,\mu$, in R. For each set of t positive real numbers $g_1,\cdots,g_t$, $0\le t\le\mu$, we introduce the parametric feedback

$$K_{g_1\cdots g_t} := K_o + g_1\theta_1K_1 + \cdots + g_t\theta_tK_t .$$

Finally, for any (n,m)-system $\Sigma(a,b)$, where $a = (a_{11},\cdots,a_{nn})$ and $b = (b_{11},\cdots,b_{nm})$, and any $e\in\Re$, we let

$$\sigma = \sigma(a,b,e) := \min\{i,\ 0\le i\le\mu,\ \theta_j(a,b,e)=0 \text{ for all } j>i\}$$

(so $\sigma=0$ if all $\theta_i(a,b,e)$ vanish, and $\sigma=\mu$ if they are all nonzero). Pick any such (a,b,e), and assume first that $\sigma>0$. Consider the set $\mathcal{G}(a,b,e)$ consisting of all those positive reals $g_1,\cdots,g_\sigma$ such that

$$g_\sigma\theta_\sigma^2(a,b,e) > \psi(a,b,e) + g_1\theta_1(a,b,e)\phi_1(a,b,e) + \cdots + g_{\sigma-1}\theta_{\sigma-1}(a,b,e)\phi_{\sigma-1}(a,b,e) .$$

Note that this is a "high-gain set" in the sense that $r(g_1,\cdots,g_\sigma)$ is in $\mathcal{G}(a,b,e)$ if $(g_1,\cdots,g_\sigma)$ is. If instead $\sigma=0$, we let $\mathcal{G}(a,b,e)$ be arbitrary. We shall be interested in the the closed loop characteristic polynomial

$$\chi_{cl} := \text{char.poly}\,\{A^\kappa(a) - B^\kappa(b)K_{g_1\cdots g_\sigma}(a,b,e)\}. \tag{2.2}$$

(When $\sigma=0$, χ_{cl} reduces to $A^\kappa(a)-B^\kappa(b)K_o(a,b)$.) This is the characteristic polynomial of the composite system

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*Mixing terminologies from algebraic topology and control theory, the dynamic stabilization problem is obtained by "stabilizing" --in the K-theoretic sense-- the static stabilization problem.

$$\dot{x} = Ax + B(g_1 K_1 + \cdots + g_\sigma K_\sigma)z$$
$$\dot{z} = Fz + Gx,$$

where F, G, and the K_i are obtained from the above data and have entries over R (and we omit the arguments a,b,e). The main result is:

Theorem A. For any n,m, there exist data as above such that, for each (a,b,e) and each $(g_1,\cdots,g_\sigma)$ in $\mathcal{G}$(a,b,e), χ_{cl} splits as a product $\chi_s \chi_f$, where χ_s has all roots with real part $\leq$ -e and χ_f is the characteristic polynomial of $A(a,b)_f$. Further, if $\sigma = \sigma(a,b,e)$ and $A^\kappa(a) - B^\kappa(b) K_{g_1 \cdots g_{\sigma-1}}(a,b,e)$ is already Hurwitz, and e>0, then (2.2) is Hurwitz for *arbitrary* positive g_σ.n

In particular if Σ(a,b) is stabilizable and e>0, the matrix in (2.2) is Hurwitz if the gains g_i are large enough.

The proof will give (rather impractical) μ = n, κ = n^2, and σ (independent of e) = dimension of the pointwise controllability subspace $\mathcal{C}$(a,b). It would be an interesting question to know if smaller κ,μ can be used.

We shall apply Theorem A in establishing the following.

Theorem B. Let Σ = (A,B) be a system over $\Re[\lambda] = \Re[\lambda_1,\cdots,\lambda_r]$ (that is, a polynomially parametrized family of real systems). Let Σ(l) = (A(l),B(l)) be the system obtained when substituting $\lambda = l \in \Re^r$. If Σ(l) is stabilizable for each $l \in \Re^r$, then there exist an integer κ and a matrix $K \in R^{(m+\kappa)\times(n+\kappa)}$ (that is, a polynomially parametrized dynamic feedback law) such that

$$A(l)^\kappa - B(l)^\kappa K(l)$$

is Hurwitz for all $l \in \Re^r$.

The following local-global principle is basically a restatement of the above:

Theorem C. A family (A,B) is stabilizable iff it is pointwise stabilizable.

3. Some Results on Systems over Rings.

We need a lemma on pseudoinverses of matrices over rings, which generalizes the result in [S2]. This is exactly as in [S3], but since the construction is so central to all that follows, we include the (short) proof here. We let R be an arbitrary commutative ring.

Let $C = (c_{ij})$ be an $n\times q$ matrix over R. For any positive $r \le \min\{n,q\}$, we denote by $I_r(C)$ the ideal of R generated by all the $r\times r$ minors of C. In general, we let $C(\alpha,\beta)$, where α and β are ordered sets of indices for rows and columns respectively, denote the minor obtained from the rows/columns indexed by α, β. Thus $I_r(C)$ is the set of all linear combinations, with coefficients in R, of the $C(\alpha,\beta)$ with α and β ordered index sets of cardinality r. If $\alpha = (\alpha_1,\cdots,\alpha_r)$ and ν is an integer, we write "$\nu\in\alpha$" to indicate that there is an index k such that $\alpha_k = \nu$; this index k is then denoted by $\alpha[\nu]$. If $\nu\in\alpha$, $\alpha\backslash\{\nu\}$ denotes the (r-1)-tuple obtained by deleting ν; if $\nu\notin\alpha$, $\alpha\cup\{\nu\}$ is the (r+1)-tuple obtained by inserting ν in the appropiate position of α. Finally, we also let $C(\{\},\{\}):= 1$ for the empty sets of indices, and $I_s(C):= \{0\}$ if s is larger than $\min\{n,q\}$.

Lemma 3.1: ([S3]) Let C be as above, and let θ be an arbitrary element of $I_r(C)$. Then there exists a matrix H over R such that

$$CHC = \theta C + L$$

for some matrix L all whose entries are in $I_{r+1}(C)$.

Proof: Let $\theta = -\sum m_{\alpha,\beta}C(\alpha,\beta)$ be an expression in terms of the generators of $I_r(C)$ (we will omit summation indices when clear from the context). Then, define $H := (h_{ij})$, where

$$h_{ji} := \sum (-1)^{\alpha[i]+\beta[j]+1}C(\alpha\backslash\{i\},\beta\backslash\{j\})m_{\alpha,\beta} \tag{3.1}$$

with the sum over all ordered index sets α and β of cardinality r for which $i\in\alpha$ and $j\in\beta$. We must prove that, for each indices ν, μ, $(CHC)_{\nu\mu} = \theta c_{\nu\mu} + l$, with l in $I_{r+1}(C)$. This is done exactly as in [B] (which deals essentially with the case $\theta = 1$). First note that, for any such ν, μ, and any fixed index sets as above α, β,

$$\sum (-1)^{\alpha[i]+\beta[j]+1}c_{\nu j}c_{i\mu}C(\alpha\backslash\{i\},\beta\backslash\{j\}) + c_{\nu\mu}C(\alpha,\beta) = l, \tag{3.2}$$

(sum over all $i\in\alpha$ and $j\in\beta$) with l in $I_{r+1}(C)$. This is proved as follows. Let $l := \det(C)$, where C is obtained by adjoining row ν and column μ to the matrix corresponding to α and β. Thus either $\det(C) = 0$ (if $\nu\in\alpha$ or $\mu\in\beta$) or $\det(C) = \pm C(\alpha\cup\{\nu\},\beta\cup\{\mu\})$, so that l is in $I_{r+1}(C)$ as required. The formula now follows by expanding first in terms of the last row and then the last column. Now just calculate $(CHC)_{\nu\mu} = \sum_{i,j}c_{\nu j}h_{ji}c_{i\mu}$. Substituting 3.1 into h_{ji}, and using property 3.2, this equals $\theta c_{\nu\mu} + l\sum m_{\alpha,\beta}$.n

Lemma 3.2: Let $\Sigma = (A,B)$ be an (n,m)-system over R, and let $C = C(A,B)$. Pick $\theta_1,\cdots,\theta_n$ in R such that $\theta_i\in I_i(C)$ for each i. Then, there are matrices $H_1,\cdots,H_n$ in $R^{mn\times n}$ with the following property. Let $\gamma_1,\cdots,\gamma_n$ be indeterminates over R, and let

$$G(\gamma_1,\cdots,\gamma_n) := A - C\sum_{i=1}^{n}\gamma_i\theta_iH_i$$

(a matrix over $R[\gamma_1,\cdots,\gamma_n]$). Let **F** be an algebraically closed field and π: $R[\gamma_1,\cdots,\gamma_n]\to$**F** a ring homomorphism. A superscript π in a matrix will denote evaluation of all entries by π. Assume that $\mathrm{rank}C^\pi = \sigma>0$. Then, the characteristic polynomial of $G(\gamma_1,\cdots,\gamma_\sigma,0,\cdots,0)^\pi$ factors as

$$\chi_f\chi_s ,$$

where χ_f is the characteristic polynomial of $(A^\pi)_f$ and where each root of χ_s is of the form

$\rho - \gamma_\sigma^\pi(\theta_\sigma^\pi)^2$,

ρ = eigenvalue of $G(\gamma_1,\cdots,\gamma_{\sigma-1},0,\cdots,0)^\pi$.

Proof: We apply lemma (3.1) n times, using always the same matrix C but with each of the possible $\theta = \theta_i$. There result n matrices H_i, with

$$CH_iC = \theta_iC + L_i, \quad L_i \text{ with entries in } I_{i+1}(C) .$$

(Thus L_n=0.) Let $E_i := CH_i$ for each i=1,⋯,n. Then,

$$E_i^2 = \theta_iE_i + N_i, \quad N_i \text{ with entries in } I_{i+1}(C) .$$

Consider now a homomorphism π such that $\mathrm{rank}(C^\pi)=\sigma>0$. Then, $I_j(C^\pi)=0$ for $j>\sigma$, so N_j^π and θ_j^π vanish for such j. In particular,

$$(E_\sigma^\pi)^2 = \theta_\sigma^\pi E_\sigma^\pi .$$

Thus E_σ^π is annihilated by

$$z(z-\theta_\sigma^\pi) .$$

If $\theta_\sigma^\pi \neq 0$, the minimal polynomial of E_σ^π is either $z(z-\theta_\sigma^\pi)$, z, or $z-\theta_\sigma^\pi$.

We let $E=E_\sigma$, $\theta=\theta_\sigma$, and $\gamma=\gamma_\sigma$. Further, we drop from now on the superscripts π; thus A will denote A^π, θ denotes θ_σ^π, and so forth. This will cause no confusion, since all further arguments are over the given field **F**. Assume first that $\theta\neq0$. It follows then from the form of the minimal polynomial of E that there is a $T\in Gl(\mathbf{F},n)$ such that

$$E_1 := T^{-1}ET = \begin{pmatrix}\theta I & 0\\ 0 & 0\end{pmatrix} .$$

If the minimal polynomial is $z-\theta$, the 0 blocks are not there. If instead the minimal polynomial is z, the θI block is empty (but we prove later that this case cannot happen). Let

$$L = \sum_{i=1}^{a} \gamma_i\theta_iH_i$$

where a=σ-1, (evaluated by π), and H be H_σ (evaluated). Since

$$E = CH \quad \text{and } C = (1/\theta)EC ,$$

it follows that E and C have the same column space, so rankE = σ. Thus the block θI in E_1 is $\sigma\times\sigma$. Denote

$$C_1 := T^{-1}C, \quad A_1 := T^{-1}AT, \quad LT = (L_1,L_2) , \tag{3.3}$$

where L_1 is $nm\times\sigma$ and L_2 is $nm\times(n-\sigma)$. Then, the equality $E_1C_1 = \theta C_1$ implies that C_1 has the partitioned form

$$\begin{pmatrix}C_2\\ 0\end{pmatrix} , \tag{3.4}$$

where C_2 is of size $\sigma\times nm$. Thus C_2 has rank σ. Finally, partition A_1 as

$$\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix} , \tag{3.5}$$

where A_{11} is $\sigma\times\sigma$. From the A-invariance of C, we can write

$$AC = CD ,$$

from where it follows that $A_1C_1 = C_1D$, and hence $A_{21}C_2 = 0$. Since C_2 has rank σ and A_{21} is $\sigma\times\sigma$, we conclude that $A_{21}=0$; thus we are in the standard case discussed in the introduction where A_{22} represents A_f. Note that then

$$T^{-1}(A-CL-\gamma\theta E)T = \begin{pmatrix} A_{11}-C_2L_1-\gamma\theta^2 I & A_{12}-C_2L_2 \\ 0 & A_{22} \end{pmatrix}, \qquad (3.6)$$

so the characteristic polynomial of the desired $G(\gamma_1,\cdots,\gamma_\sigma,0,\cdots,0) = A-CL-\gamma\theta E$ splits as that of A_f and of $A_{11}-C_2L_1-\gamma\theta I$. The eigenvalues of the latter matrix are translates by $\gamma\theta^2$ of those of $A_{11}-C_2L_1$, which is in turn by formula (3.6) (when $\gamma=0$) a matrix whose eigenvalues are among the eigenvalues of $A-CL = G(\gamma_1,\cdots,\gamma_{\sigma-1},0,\cdots,0)$.

If instead $\theta=0$, the statement to be proved is simply that χ_f divides A-CL. But we may always find an invertible T such that, with (3.3), the forms (3.4) and (3.5) hold, and $A_{21}=0$. Thus χ_f is also then a factor. This completes the proof.n

Partition now each matrix H_i in the form

$$\begin{matrix} H_{i1} \\ \cdot \\ \cdot \\ H_{in} \end{matrix}$$

where each block H_{ij} is of size $m\times n$. Thus

$$G(\gamma_1,\cdots,\gamma_n) = A - \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_i\theta_i A^{j-1}BH_{ij} .$$

Note that, for each positive j,

$$A^jB = (zI-A)U_j + BV_j , \qquad (3.7)$$

for suitable U_j, V_j over R[z] (z = indeterminate over R). This is easy to prove, by induction on j, using that

$$A^jB = (zI-A)(-A^{j-1}B) + z(A^{j-1}B) .$$

Let $\Gamma := zI-G$. Then,

$$\Gamma = (zI-A)(I+\sum_{j=1}^{n} U_j(z)X_j) + B(\sum_{j=1}^{n} V_j(z)X_j) , \qquad (3.8)$$

where

$$X_j := \sum_{i=1}^{n} \theta_i\gamma_i H_{ij}.$$

Let $\Delta(z)$ *be any* fixed polynomial in $R[z,\varepsilon]$, where ε is yet another indeterminate, Δ monic and of degree at least 1 in z. (We shall think of $R[z,\varepsilon]$ as polynomials in z with coefficients in $R[\varepsilon]$.) For the main theorem, we shall use

$$\Delta(z) := z+\varepsilon , \qquad (3.9)$$

but the argument to follow is more general (and will be used later in proving results over arbitrary rings). Let χ denote the characteristic polynomial of A. Since this is monic, there is a well defined division of polynomials by χ, and in particular there are polynomial matrices $T_j(z)$, $j=1,\cdots,n$, and $S_j(z)$, $j=1,\cdots,n$, such that

$$\Delta^n V_j(z) = \chi(z)T_j(z) + S_j(z)$$

and degree $S_j \leq n-1$. (All polynomials are over $R[\varepsilon]$.) Let $\Delta'(z) := \Delta^n(z)\Gamma$. It follows that

$$\Delta'(z) = (zI-A)[\Delta^n(I+\textstyle\sum_{j=1}^{n} U_j(z)X_j)] + \\ + \chi(z)B[\textstyle\sum_{j=1}^{n} T_j(z)X_j] + B[\textstyle\sum_{j=1}^{n} S_j(z)X_j] \ . \quad (3.10)$$

Since $\chi(z)B = (zI-A)\mathrm{cof}(zI-A)B$ ("cof" = matrix of cofactors), it follows from (3.10), by collecting the first two terms, that

$$\Delta'(z) = (zI-A)Q(z) - BW(z) \ , \quad (3.11)$$

where $W(z) := -\sum_{j=1}^{n} S_j(z)X_j$ is a polynomial matrix (in z) of degree at most n-1. Comparing leading coefficients in z, since Δ' is monic of degree $1+\deg\Delta(z)$, it follows that Q(z) is also monic, of degree $n' = n.\deg\Delta(z)$. The argument used in passing from (3.8) to (3.11), with Δ independent of ε, proves also a general fact, which we state here for future reference:

Proposition 3.3: Assume that (A,B) is an (n,m)-system over R, and that U(z), V(z) are matrices over R[z] of sizes $n\times n$ and $m\times n$ respectively, such that

$$\Gamma := (zI-A)U(z) + BV(z)$$

is monic. Let Δ be any monic polynomial in R[z] of degree at least 1. Then, there exist polynomial matrices Q(z), W(z), where Q(z) is monic and of strictly larger degree than W(z), such that, with $\Delta' := \Delta^n\Gamma$,

$$\Delta'(z) = (zI-A)Q(z) - BW(z) \ .n$$

(This result provides the essential step in the proof of the result given in [E]; the author learned the above simple proof from Malo Hautus.) For our main result, we must now obtain a realization of $Q^{-1}W$ that preserves linearity in the gains γ. For simplicity, from now on we do take Δ as in equation (3.9), so that n'=n, and we may write

$$Q(z) = z^n I - \textstyle\sum_{i=0}^{n-1} z^i Q_{i+1}$$

and for each $k=1,\cdots,n$,

$$Q_k = R_k + \textstyle\sum_{i=1}^{n} \gamma_i\theta_i R_{ki} \ ,$$

where the matrices R_i and R_{ki} are all $n\times n$ matrices over $R[\varepsilon]$. Similarly,

$$W(z) = \textstyle\sum_{i=0}^{n-1} z^i W_{i+1} \ ,$$
$$W_k = \textstyle\sum_{i=1}^{n} \gamma_i\theta_i S_{ki} \ ,$$

where the S_{ki} are $m\times n$ matrices over $R[\varepsilon]$. We now apply the lemma in the appendix, with the above Q and with $P(z) := BW(z)$ and k=n. (And, "R" in the appendix is $R[\varepsilon]$.) We can interpret the conclusions in term of the extended system Σ^κ, where $\kappa = n^2$. Consider the matrix

$$K := -\begin{pmatrix} 0 & W_1 & W_2 & W_3 & . & . & . & W_n \\ 0 & 0 & I & 0 & . & . & . & 0 \\ 0 & 0 & 0 & I & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & I \\ I & Q_1 & Q_2 & Q_3 & . & . & . & Q_n \end{pmatrix} .$$

Then,

$$A^\kappa - B^\kappa K \quad (3.12)$$

equals M in the appendix, and hence has characteristic polynomial

$$\det((z+\varepsilon)^n(zI-G)) = (z+\varepsilon)^{n^2}\det(zI-G)\ .$$

Furthermore, there is an expression

$$K = K(\gamma_1,\cdots,\gamma_n) = K_o + \sum_{i=1}^{n}\gamma_i\theta_i K_i\ , \tag{3.13}$$

where the K_i are (n^2+m) by (n^2+n) matrices over $R[\varepsilon]$ and the first m rows of K_o are identically zero. From lemma 3.2 we may then conclude:

Proposition 3.4: Let $\Sigma = (A,B)$ be an (n,m)-system over R, and let $C = C(A,B)$. Pick $\theta_1,\cdots,\theta_n$ in R such that $\theta_i \in I_i(C)$ for each i. Let $\gamma_1,\cdots,\gamma_n$ and ε be indeterminates over R, and $\kappa=n^2$. For the extended system Σ^κ, there exists then a matrix K over $R[\gamma_1,\cdots,\gamma_n,\varepsilon]$ of the form in equation (3.13), such that the following holds. Let π: $R[\gamma_1,\cdots,\gamma_n,\varepsilon]\to\mathbf{F}$ be any homomorphism onto an algebraically closed field such that C^π has rank σ, $0\le\sigma\le n$. Then, the characteristic polynomial of (3.12) splits as a product $\chi_s\chi_f$, where χ_f is the characteristic polynomial of $(A^\pi)_f$, and where each root of χ_s either equals $-\pi(\varepsilon)$ or is of the form

$$\rho - \gamma_\sigma^\pi(\theta_\sigma^\pi)^2\ ,$$

where ρ is an eigenvalue of

$$(A^\kappa)^\pi - (B^\kappa)^\pi K(\gamma_1,\cdots,\gamma_{\sigma-1},0,\cdots,0)^\pi$$

(if $\sigma=0$, all roots of χ_s equal $-\pi(\varepsilon)$). n

(When $\sigma=0$ then $C^\pi=0$, and all θ_i evaluate to zero, so the matrix $G(\gamma_1,\cdots,\gamma_n)^\pi$ reduces to $A^\pi = (A^\pi)_f$. Thus all zeroes of χ_s in fact equal $-\pi(\varepsilon)$ in that case.)

To prove Theorem A, we choose now $R = R^{[n,m]}$, and

$$\theta_i := \text{sum of the squares of all } i\times i \text{ minors of } C\ .$$

We apply the above proposition, so there result matrices K_i as there. The number μ in the statement of theorem A will be n, and κ there is n^2. We pick

$$\psi := \varepsilon + 2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}^{\ 2} + \sum_{i=1}^{n}\sum_{j=1}^{n}(BK_o)_{ij}^{\ 2}\ ,$$
$$\phi_l := 1 + \sum_{i=1}^{n}\sum_{j=1}^{n}(BK_l)_{ij}^{\ 2}\ .$$

Let $\Sigma(a,b) = (A(a),B(b))$ be choosen so that $C=C(a,b)$ has rank σ, where $0\le\sigma\le n$. Pick any real e and positive $g_1,\cdots,g_n$. Let π be homomorphism into $\mathbf{C}$ induced by the evaluation of α, β, ε, $\gamma_1,\cdots,\gamma_n$ into a, b, e, and $g_1,\cdots,g_n$ respectively. Assume first that $\sigma=0$. Then $C=0$, so by the proposition the desired characteristic polynomial factors as the characteristic polynomial of A times one having all roots = -e. And, since $C=0$, $A=A_f$, so the result follows. So assume that $\sigma>0$, and that $g_1,\cdots,g_n$ are arbitrary, with $g_\sigma>0$. By proposition 3.4, the characteristic polynomial of $A^\kappa(a) - B^\kappa(b)K_{g_1\cdots g_\sigma}(a,b)$ splits as the product of the characteristic polynomial of $A(a,b)_f$ and of a polynomial χ_s each of whose roots either equals -e or is of the form $\rho - g_\sigma\theta_\sigma^2(a,b,e)$, where ρ is an eigenvalue of

$$F := A(a) - B(b)K_o(a,b,e) - \sum_{i=1}^{q} g_i\theta_i(a,b,e)B(b)K_i(a,b,e)$$

(where $q=\sigma-1$) Let ρ be any such eigenvalue. Then its real part is less than $|\rho|$, which is dominated by the spectral radius of F, and hence by the norm of F induced by Euclidean norm in $\mathfrak{R}^n$; thus

$$\mathrm{Re}\,\rho \le \|A(a)\| + \|B(b)K_o\| + \sum_{i=1}^{q} g_i\theta_i(a,b,e)\|BK_i(a,b,e)\|$$
$$\le 2 + \psi(a,b,e) + \sum_{i=1}^{q} g_i\theta_i(a,b,e)\phi_i(a,b,e) - e \,.$$

where $q=\sigma-1$. It follows that

$$\mathrm{Re}(\rho - g_\sigma\theta_\sigma^2(a,b,e)) \le \{\psi(a,b,e)+\sum_{i=1}^{q} g_i\theta_i(a,b,e)\phi_i(a,b,e) - g_\sigma\theta_\sigma^2(a,b,e)\} - e \qquad (3.14)$$

where $q=\sigma-1$. If now $g_1,\cdots,g_\sigma$ satisfy

$$g_\sigma\theta_\sigma(a,b,e)^2 > \psi(a,b,e) + \sum_{i=1}^{q} \theta_i(a,b,e)g_i\phi_i(a,b,e) \,,$$

where $q=\sigma-1$, then the expression in (3.14) is less than $-e$, as desired. Finally, assume that $e>0$ and that g_σ is arbitrary but that all eigenvalues ρ of F have negative real part. Then χ_s has all zeroes equal to $-e<0$ or of the form ρ-(positive), so all such zeroes again have negative real part. This completes the proof of the main theorem.n

4. Proof of Theorem B.

In this section we prove the result on polynomially parametrized families. This will be an easy consequence of Theorem A once that we establish a result in real algebraic geometry. Recall that a *semialgebraic* subset of $\Re^r$ is one that can be defined by a first-order formula in the theory of real-closed fields. By a *rational function defined on* F, where F is a semialgebraic set of $\Re^r$, we shall mean a rational function in r variables which has no poles in F. The main fact that we need is as follows.

Proposition 4.1: Given a closed semialgebraic subset F of $\Re^r$ and a rational function ζ defined on F, there exists a polynomial $p\in\Re[\lambda_1,\cdots,\lambda_r]$ such that $p(l)>\zeta(l)$ whenever $l\in F$ and $p(l)>0$ for all $l\in\Re^r$.

Proof: Let $\zeta = \phi/\theta$, where θ has no zeroes on F. Without loss, we assume that $\theta>0$ on F (otherwise use $-\phi$). Also, we may assume that F is nonempty, otherwise the result is trivial. Consider the following subset E of $\Re^2$:

$$E := \{(x,y) \text{ s.t. if } l\in\Re^r \text{ is such that } \|l\|^2\le x \text{ and } l\in F \text{ then } y\theta(l)>\phi(l)\}.$$

This is again a semialgebraic set. Now let $f: \Re\to\Re$ be the function defined by

$$f(x) := \inf\{y \text{ s.t. } (x,y)\in E\} \,,$$

with $f(x)=+\infty$ if the set is empty. Then (see [H], pages 367-368,) the function f is semialgebraic, and hence if it is finite for large positive x then f has the form

$$f(x) = Ax^\alpha(1+o(1)) \text{ as } x\to\infty \,, \qquad (4.1)$$

where α is rational. Let $x>0$ be such that $C_x := F\cap B_x$ is nonempty, where B_x is the ball of radius x. Since ζ is continuous on the compact set C_x, it is in particular bounded there. So

$$f(x) = \sup\{\zeta(l),\ l\in C_x\}$$

is finite, and f(x) has the form (4.1) for large x. Let q(x) be a polynomial such that q(x)>f(x) for all large x; such a q exists because of (4.1). Since f is a nondecreasing function for x>0, there is a constant c such that q'(x):= c+q(x) is larger than f(x) for all positive x. Finally, choose

$$p(\lambda) := q'(\|\lambda\|^2).$$

This is a polynomial, and it dominates ζ on F by construction. If p is not everywhere positive, just replace it by p^2+1, which is positive and dominates p.n

We now complete the proof of theorem B. It is slightly easier to prove the theorem if we use the explicit construction of the θ_i's, but we prefer to obtain it as a corollary of theorem A. In this way we emphasize that B follows from the existence of a "high-gain theorem" plus the above algebraic-geometric fact. Possible improvements in theorem A (smaller μ and κ, for instance) will then give improvements in B.

Pick n,m, and let κ, μ, K_i's, etc., be as concluded by theorem A. Now assume that Σ is a pointwise stabilizable (n,m)-system over $\Re[\lambda] = \Re[\lambda_1,\cdots,\lambda_r]$, and denote by $a_{ij}(\lambda)$ and $b_{ij}(\lambda)$ the entries of A and B respectively, as polynomials in the variables λ. Evaluate the entries of the K_i, ϕ_i, θ_i, and ψ, at $\alpha_{ij} := a_{ij}(\lambda)$, $\beta_{ij} := b_{ij}(\lambda)$, and $\varepsilon := 1$. There result polynomial matrices and polynomials in $\Re[\lambda]$, which we denote again by K_i, etc. Define the function

$$\sigma: \Re^r \to \text{nonnegative integers}, \; \sigma(l) := \sigma(a(l), b(l), 1).$$

Then, whenever $\sigma(l) = \sigma$ and $g_1,\cdots,g_\sigma$ satisfy

$$g_\sigma \theta_\sigma^2(l) > \psi(l) + g_1\theta_1(l)\phi_1(l) + \cdots + g_{\sigma-1}\theta_{\sigma-1}(l)\phi_{\sigma-1}(l) \,,$$

it follows that

$$L_{g_1,\ldots,g_j}(l) := A(l)^\kappa - B(l)^\kappa(K_0(l) + g_1\theta_1K_1(l) + \cdots + g_j\theta_jK_j(l))$$

is Hurwitz for j=σ. (When $\sigma(l)=0$, only K_0 appears.) This is true because the vanishing of θ_i for $i>\sigma$ means that $L_{g_1,\ldots,g_\sigma}$ coincides with the closed-loop matrix in (2.2), and stabilizability of $\Sigma(l)$ means that $A(l)_f$ is stable. Further, it also follows from theorem A that $L_{g_1,\ldots,g_j}(l)$ is Hurwitz, for j=σ, if g_σ is arbitrary positive but $L_{g_1,\ldots,g_j}(l)$, j=σ-1, is known to be Hurwitz.

Claim: There are polynomials p_j, j=1,$\cdots$,μ, such that, if l is such that $\sigma(l)=j>0$, then $L_{p_1(l),\ldots,p_j(l)}$ is Hurwitz.

Theorem B follows from this: for any l, if $\sigma(l)=0$ then stability follows from theorem A, independently of the choice of the p_i's; for $\sigma(l)>0$ the conclusion follows from the claim.

We prove the claim by induction on j. Assume that $p_1,\cdots,p_{j-1}$ have been constructed, such that if $\sigma(l)=i\leq j-1$ then $L_{p_1(l),\ldots,p_i(l)}$ is Hurwitz (no assumption when j=1). Consider the set

$$F_j := \{l \text{ s.t. } L_{p_1(l),\ldots,p_{j-1}(l)} \text{ is not Hurwitz and } \theta_i(l)=0 \text{ for } i>j\} \,.$$

This is a closed semialgebraic set, because Hurwitz matrices form an open semialgebraic set. We claim that if l is in F_j then $\theta_j(l)\neq 0$. Otherwise, $L_{p_1(l),\ldots,p_{j-1}(l)}$ coincides with $L_{p_1(l),\ldots,p_i(l)}$, with $\sigma(l)=i<j$, and this contradicts the inductive hypothesis. (When j=1, only K_0 appears, and this matrix equals $A^\kappa(l)-B^\kappa(l)K_0(l)$, which is Hurwitz since σ=0.) Thus, by proposition (4.1), there is a polynomial p_j such that

$$p_j(l)\theta_j^2(l) > \psi(l) + p_1(l)\theta_1(l)\phi_1(l) + \cdots + p_{j-1}(l)\theta_{j-1}(l)\phi_{j-1}(l) \,, \tag{4.2}$$

whenever l is in F_j. Assume now that $\sigma(l)=j$. If l is in F_j, then by (4.2) it follows that $L_{p_1(l),\ldots,p_j(l)}$ is indeed Hurwitz. If not in F_j, then $L_{p_1(l),\ldots,p_{j-1}(l)}$ must be Hurwitz, so since p_σ is always positive, again $L_{p_1(l),\ldots,p_j(l)}$ is Hurwitz. This completes the proof of the claim and hence of theorem B.n

5. Complements on Stabilizability.

In this section, we include some remarks concerning stabilizability of systems over arbitrary commutative rings, and in particular show why theorem C is just a restatement of B. It will also follow that our definition of stabilizability coincides with the usual one (see e.g. [HS], [KS], [E]). Let R be a fixed commutative ring, with a given Hurwitz set S. Also, Σ = (A,B) is a fixed (n,m)-system over R.

We shall say that a polynomial $\Psi \in R[z]$ is *assignable* for Σ iff there exist polynomial matrices $Q \in R[z]^{n\times n}$ and $P \in R[z]^{m\times n}$ such that

$$(zI-A)Q(z) + BP(z) = \Psi(z)I \,. \tag{5.1}$$

Assume that Ψ is like this, and let $P = \sum P_i z^i$. Let C:= C(A,B). For each i>0 we may write

$$BP_i z^i = (zI-A)M_i(z) + A^i BL_i \,,$$

for suitable polynomial M_i and constant L_i. This follows by iterating on the formula

$$BPz = (zI-A)BP + ABP \,.$$

The argument can be reversed --using equation (3.7). Thus, Ψ is assignable iff there exist Q(z) as before and $L \in R^{nm\times n}$ (constant, not polynomial) such that

$$(zI-A)Q(z) + CL = \Psi(z)I \,. \tag{5.2}$$

This last equation implies that $\Psi(A) = CL$. (Basically, by evaluation of both sides at z:= A. But the argument is slightly more subtle, because of the noncommutativity of the matrix ring, and it depends on the fact that $\Psi(z)I$ commutes with A. See [G], "generalized Bezout’s theorem," IV.3, theorem 1.) Conversely, if $\Psi(z)$ is arbitrary, we may divide on the left by the monic polynomial (zI-A), thus if $\Psi(A) = CL$ then (5.2) holds for some Q(z). So, Ψ is assignable iff the image of $\Psi(A)$ is included in C, or, by definition of A_f:

Proposition 5.1: Ψ is assignable if and only if it annihilates A_f. n

The usual definition of stabilizability for systems over rings is in terms of assignability of Hurwitz polynomials; by the proposition, it coincides with the definition which we use. It is a result of Emre (see [E]) that this implies the existence of dynamic stabilizers; we shall prove this now, using facts already derived.

Assume that Ψ is assignable and Hurwitz. Pick any $\Delta \in S$ of positive degree. Then, proposition 3.3 applies, and we may write

$$\Delta^n(z)\Psi(z)I = (zI-A)Q(z) + BP(z) \,,$$

with Q monic and larger degree than P. By the lemma in the Appendix, there are then an integer κ and a matrix K such that $A^\kappa - B^\kappa K$ has characteristic polynomial $(\Delta^n\Psi)^n$, and hence is Hurwitz.

Conversely, assume that there exist κ,K like that. Let Ψ be a Hurwitz polynomial annihilating $A^\kappa - B^\kappa K$. Then Ψ also annihilates A_f. Indeed, if C^κ denotes $C(A^\kappa, B^\kappa)$, then

$$\Psi(A^\kappa - B^\kappa K) = \Psi(A^\kappa) + C^\kappa L$$

and the form of A^κ, B^κ imply that $\Psi(A)+CL' = 0$ for some L’. We conclude then:

Theorem D. The following statements are equivalent, for any fixed R,S, and Σ:

i. Σ is stabilizable (i.e., A_f is Hurwicz).
ii. There is an assignable Hurwitz polynomial.
iii. There are κ,K such that $A^\kappa - B^\kappa K$ is Hurwitz (Σ is dynamically stabilizable).n

Theorem C then follows from B and D. From the last condition in D, it follows that stabilizability implies pointwise stabilizability for families. And the converse is proved by theorem B, using again the last characterization.

6. Appendix.

The following lemma is needed in the text.

Lemma 6.1: Let R be a commutative ring, n,k positive integers, and consider the following n(k+1) by n(k+1) matrix over R (each block of size n×n):

$$M = \begin{pmatrix} A & P_1 & P_2 & P_3 & . & . & . & P_k \\ 0 & 0 & I & 0 & . & . & . & 0 \\ 0 & 0 & 0 & I & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & I \\ I & Q_1 & Q_2 & Q_3 & . & . & . & Q_k \end{pmatrix} .$$

Then, the characteristic polynomial of M equals the determinant of

$$N(z) = (zI-A)Q(z) - P(z) ,$$

where

$$Q(z) := z^k I - \sum_{i=0}^{k-1} z^i Q_{i+1} , \text{ and}$$
$$P(z) := \sum_{i=0}^{k-1} z^i P_{i+1} .$$

Proof: Consider the matrix zI-M. It is enough to prove that there is an unimodular (det=1) matrix E over R[z] such that

$$E(zI-M) = \begin{pmatrix} I & * & * & . & . & . & 0 \\ 0 & N & 0 & . & . & . & 0 \\ 0 & * & I & 0 & . & . & 0 \\ 0 & * & 0 & I & 0 & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & * & * & * & * & . & I \end{pmatrix} . \qquad (6.1)$$

The matrix zI-M has k+1 block rows each consisting of n rows; when we write "row i", we shall mean "i-th block of rows", and row operations will be by blocks. Now operate as follows. In the order i=3,···,k, do

row_i := row_i + z.row_(i-1) .

Thus the i-th block of zI-M becomes

$$[0,z^{i-1},0,\cdots,0,-I,0,\cdots,0] ,$$

with the -I in block position i+1, for i=2,···,k. Now do

row_1 := row_1 + (zI-A)row_(k+1) ,

so row 1 now looks as

$$[0,-P_1-(zI-A)Q_1,\cdots,-P_{k-1}-(zI-A)Q_{k-1},-P_k-(zI-A)Q_k+z(zI-A)] .$$

Operating now again on row 1,

row_1 := row_1 + $[-P_k-(zI-A)Q_k+z(zI-A)]$row_k ,

results in the block (1,k+1) being zero. Finally, apply the operations

row_1 := row_1 + $[-P_i - (zI-A)Q_i]$row_i ,

in the order i = k-1,···,2. There results a matrix

$$0 \quad N(z) \quad 0 \quad . \quad . \quad . \quad 0$$

$$
\begin{array}{ccccccc}
0 & zI & -I & 0 & . & . & 0 \\
0 & z^2 I & 0 & -I & . & . & 0 \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
-I & * & * & * & . & . & *
\end{array}
$$

Multiply now rows 2,···,k+1 by -1 and do kn exchanges to bring block row k+1 to block row 1. This results in the desired form (6.1).n

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