SIAM J. CONTROL OPTIM.
Vol. 40, No. 6, pp. 1888–1904
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A UNIFYING INTEGRAL ISS FRAMEWORK FOR STABILITY OF NONLINEAR CASCADES*

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Abstract. We analyze nonlinear cascades in which the driven subsystem is integral input-to-state stable (ISS), and we characterize the admissible integral ISS gains for stability. This characterization makes use of the convergence speed of the driving subsystem and allows a larger class of gain functions when the convergence is faster. We show that our integral ISS gain characterization unifies different approaches in the literature which restrict the nonlinear growth of the driven subsystem and the convergence speed of the driving subsystem. The result is used to develop a new observer-based backstepping design in which the growth of the nonlinear damping terms is reduced.

Key words. nonlinear cascades, stabilization, integral input-to-state stability

AMS subject classifications. 93C10, 93D05, 93D15, 93D25

PII. S0363012901387987

1. Introduction.

Studies on the stabilization of cascade systems have paved the road to major advances in nonlinear control theory. Among these advances are several constructive design methods such as *backstepping* and *forwarding*, which are based on recursive applications of cascade designs (see, e.g., Sepulchre, Janković, and Kokotović [17]), and the discovery of structural obstacles to stabilization such as the *peaking phenomenon* (see Sussmann and Kokotović [23]).

One of the main motivations for the stabilization of cascades came from the linear-nonlinear cascade

$$\dot{x} = f(x, z), \tag{1}$$

$$\dot{z} = Az + Bu \tag{2}$$

resulting from input-output linearization. Because global asymptotic stability (GAS) of the x-subsystem $\dot{x} = f(x, 0)$ is not sufficient to achieve GAS of the whole cascade with z-feedback $u = Kz$, alternative designs which employ x-feedback were developed, such as the *feedback passivation* design of Kokotović and Sussmann [8]. To achieve GAS by z-feedback, Sontag [18], Seibert and Suarez [16], Mazenc and Praly [11], Janković, Sepulchre, and Kokotović [6], and Panteley and Loría [12, 13] studied general cascades in the which the z-subsystem is nonlinear and derived conditions for the x- and z-subsystems that ensure stability of the cascade. Among these results, a particularly useful one is the input-to-state stability (ISS) condition in [19], which states that if the x-subsystem is ISS with input z and the z-subsystem is GAS, then the cascade is GAS. This result has been widely used for nonlinear designs based on the *normal form* (1)–(2), in which the *zero dynamics* (1) is ISS. Other results, such

*Received by the editors April 12, 2001; accepted for publication (in revised form) October 24, 2001; published electronically March 5, 2002.
http://www.siam.org/journals/sicon/40-6/38798.html

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as [11] and [6], make less restrictive assumptions than ISS for the x-subsystem but restrict the z-subsystem to be locally exponentially stable (LES).

The integral version of ISS (iISS), recently introduced in [20], requires that an energy norm of the input be bounded to ensure boundedness of the states. As shown in [20], iISS is less restrictive than ISS because, in an iISS system, a bounded input may lead to unbounded solutions if its energy norm is infinite.

In this paper, we analyze the stability of nonlinear cascades in which the x-subsystem is iISS and the z-subsystem is GAS. The admissible iISS gains for stability are characterized from the speed of convergence of the z-subsystem. When the convergence is fast, the iISS gain function of the x-subsystem is allowed to be "steep" at zero. We show that this trade-off between slower convergence and steeper iISS gain encompasses and unifies several results in the literature. In particular, if the x-subsystem is ISS, then the slope of its iISS gain function is very gentle at zero and tolerates every GAS z-subsystem no matter how slow its convergence is. On the other hand, if the convergence is exponential, that is, if the z-subsystem is LES, then the cascade is stable for a large class of iISS gains. This class includes all iISS x-subsystems that are affine in the input z. Thus, for systems like (1)–(2), where a control law can be designed to render the z-subsystem GAS and LES, the iISS of the x-subsystem ensures GAS of the cascade.

In section 2, we define the new concepts used in the paper and present lemmas which are preliminary to our main results. In section 3, we present our main result, Theorem 1, which characterizes the admissible iISS gains from the speed of convergence of the z-subsystem. We show that several results in the literature are special cases of Theorem 1, including those that restrict the x-subsystem to be ISS (Corollary 1) and those that restrict the z-subsystem to be LES (Corollary 2). In section 4, we show that Corollary 2 restricts the nonlinear growth of the interconnection term and illustrate with an example that violating this growth condition leads to instability of the cascade.

The second main contribution of the paper is an output-feedback application of our cascade result. Due to the absence of a separation principle, it is necessary to design control laws that guarantee robustness against the observer error. We present a design which renders the system iISS with respect to the observer error and hence ensures robustness when the error is exponentially decaying. The advantage of our design over the observer-based backstepping scheme of Kanellakopoulos, Kokotović, and Morse [7] is that we employ "weak" nonlinear damping terms which grow slower than those in [7] and result in a "softer" control law. The main features of the design are discussed and illustrated in an example in section 5. The general design procedure and its stability proof are given in section 6.

2. Definitions and preliminary lemmas. In this section, we give definitions and present lemmas that will be used in the rest of the paper. The proofs are given in section 7.

We first recall standard definitions: $\mathcal{K}$ is the class of functions $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which are zero at zero, strictly increasing, and continuous. $\mathcal{K}_\infty$ is the subset of class-$\mathcal{K}$ functions that are unbounded. $\mathcal{L}$ is the set of functions $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which are continuous, decreasing, and converging to zero as their argument tends to $+\infty$. $\mathcal{KL}$ is the class of functions $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which are class-$\mathcal{K}$ in the first argument and class-$\mathcal{L}$ in the second argument.

Definition 1. *We say that the function $\mu(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is class-$\mathcal{K}_o$ if it is class-$\mathcal{K}$ and $\mathcal{O}(s)$ near $s = 0$; that is, for all $s \in [0, 1]$, $\mu(s) \leq ks$ for some $k > 0$.*

DEFINITION 2 (see [20]). *The system*

$$\dot{x} = f(x, z) \tag{3}$$

is said to be iISS with input z if there exist a class-$\mathcal{K}_\infty$ function $\omega(\cdot)$, a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$, and a class-$\mathcal{K}$ iISS gain $\mu(\cdot)$ such that, for all $t \geq 0$,

$$\omega(|x(t)|) \leq \beta(|x(0)|, t) + \int_0^t \mu(|z(\tau)|)d\tau. \tag{4}$$

Lemma 1 computes an iISS gain $\mu(\cdot)$ from the derivative of an *iISS Lyapunov function*.

LEMMA 1.

(i) *If there exists a C^1, positive definite, radially unbounded function $V(x)$ satisfying*

$$\frac{\partial V}{\partial x} f(x, z) \leq -\rho(|x|) + \mu(|z|) \tag{5}$$

for some positive definite function $\rho(\cdot)$, then the system (3) *with input z is iISS with gain $\mu(\cdot)$.*

(ii) *If there exists a C^1, positive definite, radially unbounded function $V(x)$ satisfying*

$$\frac{\partial V}{\partial x} f(x, z) \leq \sigma(|z|) \tag{6}$$

for some class-$\mathcal{K}$ function $\sigma(\cdot)$, and if the system (3) *is GAS when $z \equiv 0$, then there exists a class-$\mathcal{K}_o$ function $\theta_o(\cdot)$ such that* (3) *is iISS with gain $\mu(\cdot) = \sigma(\cdot) + \theta_o(\cdot)$.*

The following lemma proves that if (3) is affine in the input z, then $\sigma(\cdot)$ in (6) is class-$\mathcal{K}_o$, which means that the iISS gain $\mu(\cdot) = \sigma(\cdot) + \theta_o(\cdot)$ is also class-$\mathcal{K}_o$.

LEMMA 2. *If the input-affine system*

$$\dot{x} = f(x) + g(x)z \tag{7}$$

is iISS, then it is also iISS with a class-$\mathcal{K}_o$ gain $\mu(\cdot)$.

It is known [20] that ISS implies iISS. We further show that ISS allows us to select the iISS gain $\mu(\cdot)$ in (4) to match any desired class-$\mathcal{K}$ function $\tilde{\mu}(\cdot)$ locally.

LEMMA 3. *Suppose the system* (3) *is ISS. Then, for any class-$\mathcal{K}$ function $\tilde{\mu}(\cdot)$, it is iISS with a gain $\mu(\cdot)$ satisfying $\mu(s) = \tilde{\mu}(s)$ for all $s \in [0, 1]$.*

It is proved in [20, Proposition 7] that, for a GAS system $\dot{z} = q(z)$, the solutions $z(t)$ satisfy

$$|z(t)| \leq \alpha(e^{-kt}\gamma(|z(0)|)) \tag{8}$$

for some constant $k > 0$ and class-$\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\gamma(\cdot)$. The following definition classifies GAS systems using the function $\alpha(\cdot)$ as an index of their speed of convergence to zero.

DEFINITION 3. *Given a class-$\mathcal{K}_\infty$ function $\alpha(\cdot)$, we say that the system $\dot{z} = q(z)$ is GAS(α) if there exist a class-$\mathcal{K}_\infty$ function $\gamma(\cdot)$ and a positive constant $k > 0$ such that* (8) *holds for all $z(0)$.*

Thus, for the identity function $\alpha(\cdot) = I(\cdot)$, GAS(I) consists of systems in which the convergence is exponential. We next show that $\alpha(\cdot)$ is determined by the local speed of convergence.

LEMMA 4. *If the equilibrium $z = 0$ of $\dot{z} = q(z)$ is GAS and if there exist a constant $\epsilon > 0$ and a $\mathcal{K}_\infty$ function $\tilde{\gamma}(\cdot)$ such that*

$$|z(0)| \leq \epsilon \quad \Rightarrow \quad |z(t)| \leq \alpha(e^{-kt}\tilde{\gamma}(|z(0)|))\,, \tag{9}$$

then there exists a class-$\mathcal{K}_\infty$ function $\gamma(s) = \mathcal{O}(\tilde{\gamma}(s))$ near $s = 0$ such that (8) *holds for all $z(0)$; that is, $\dot{z} = q(z)$ is GAS(α).*

The following definition will be used in our cascade result to characterize the admissible iISS gains from the speed of convergence $\alpha(\cdot)$ of the GAS(α) driving subsystem.

DEFINITION 4. *Given a class-$\mathcal{K}$ $\alpha(\cdot)$, we say that the function $\mu(\cdot)$ is class-$\mathcal{H}_\alpha$ if it is class-$\mathcal{K}$ and satisfies*

$$\int_0^1 \frac{(\mu \circ \alpha)(s)}{s} ds < \infty. \tag{10}$$

In particular, for the identity function $\alpha(\cdot) = I(\cdot)$, class-$\mathcal{H}_I$ is defined by

$$\int_0^1 \frac{\mu(s)}{s} ds < \infty. \tag{11}$$

Thus $\mu(s)$ is class-$\mathcal{H}_I$ if it is class-$\mathcal{K}_o$ or if $\mu(s) \leq s^p$ for some $p > 0$, such as $\mu(s) = \sqrt{s}$.

3. Main results.

We consider the cascade

$$\dot{x} = f(x, z), \tag{12}$$

$$\dot{z} = q(z), \tag{13}$$

where $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}^{n_z}$, and $f(\cdot,\cdot)$ and $q(\cdot)$ are locally Lipschitz and satisfy $f(0,0) = 0$, $q(0) = 0$. The stability properties to be analyzed are with respect to the origin $(x, z) = (0, 0)$, which is an equilibrium for (12)–(13).

Our main stability result characterizes the admissible iISS gains for the x-subsystem from the speed of convergence of the z-subsystem.

THEOREM 1. *If the z-subsystem* (13) *is GAS(α) as in* (8) *and the x-subsystem with input z is iISS with a class-$\mathcal{H}_\alpha$ iISS gain $\mu(\cdot)$ as in* (10), *then the cascade* (12)–(13) *is GAS.*

Proof. We note from (8) that

$$\int_0^\infty \mu(|z(\tau)|)d\tau \leq \int_0^\infty (\mu \circ \alpha)(\gamma(|z(0)|)e^{-k\tau})d\tau = \frac{1}{k}\int_0^{\gamma(|z(0)|)} \frac{(\mu \circ \alpha)(s)}{s} ds\,, \tag{14}$$

where $s := \gamma(|z(0)|)e^{-k\tau}$. From (10),

$$\lambda(s') := \frac{1}{k}\int_0^{s'} \frac{(\mu \circ \alpha)(s)}{s} ds \tag{15}$$

exists for all $s' \geq 0$, and it is class-$\mathcal{K}$ because $\lambda(0) = 0$ and $\frac{(\mu \circ \alpha)(s)}{s} > 0$ for all $s > 0$. Thus, from (4),

$$\omega(|x(t)|) \leq \beta(|x(0)|, 0) + \lambda(\gamma(|z(0)|)), \tag{16}$$

which proves stability of the cascade (12)–(13). Because $\int_0^\infty \mu(|z(\tau)|)d\tau$ is bounded, (4) implies that $x(t) \to 0$ as $t \to \infty$, as proved in [20, Proposition 6]. ∎

If the x-subsystem with input z is ISS as in [19], then, from Lemma 3, it is also iISS with a class-$\mathcal{H}_\alpha$ gain. This means that no matter what the speed of convergence $\alpha(\cdot)$ is for the z-subsystem, the ISS x-subsystem satisfies the corresponding $\mathcal{H}_\alpha$ iISS gain condition of Theorem 1. Thus Theorem 1 encompasses the following well-known result.

COROLLARY 1. *If the z-subsystem* (13) *is GAS and the x-subsystem is ISS, then the cascade* (12)–(13) *is GAS.*

Another particular case of interest is when the z-subsystem is LES, that is, when (9) holds with $\alpha(\cdot) = I(\cdot)$ and $\tilde{\gamma}(s) = cs$ for some $c \geq 1$. From Lemma 4, there exist a class-$\mathcal{K}_o$ function $\gamma(\cdot)$ and a constant $k > 0$ such that, for all $z(0)$,

$$|z(t)| \leq e^{-kt}\gamma(|z(0)|)\,. \tag{17}$$

This means that the z-subsystem is GAS(I), and hence Theorem 1 requires that the iISS gain be class-$\mathcal{H}_I$.

COROLLARY 2. *If the z-subsystem* (13) *is GAS and LES and the x-subsystem is iISS with a class-$\mathcal{H}_I$ gain $\mu(\cdot)$ as in* (11)*, then the cascade* (12)–(13) *is GAS.*

We note from Lemma 2 that the class-$\mathcal{H}_I$ restriction of Corollary 2 is satisfied when the iISS x-subsystem is affine in z.

COROLLARY 3. *If the z-subsystem* (13) *is GAS and LES and the x-subsystem is iISS and affine in z, then the cascade* (12)–(13) *is GAS.*

Examples 1 and 2 illustrate that the LES and the class-$\mathcal{H}_I$ gain restrictions cannot be removed from Corollary 2.

Example 1. For the cascade system

$$\dot{x} = -\mathrm{sat}(x) + xz, \tag{18}$$

$$\dot{z} = -z^3\,, \tag{19}$$

where $\mathrm{sat}(x) := \mathrm{sgn}(x)\min\{1, |x|\}$, the x-subsystem with input z is iISS with a class-$\mathcal{K}_o$ (hence class-$\mathcal{H}_I$) gain, as verified from $V(x) = \frac{1}{2}\ln(1+x^2)$, which satisfies $\dot{V} \leq |z|$ as in Lemma 1. However, the cascade (18)–(19) has unbounded solutions because the z-subsystem is not LES. To prove this, we let $z(0) = 1$ so that $z(t) = \frac{1}{\sqrt{1+2t}}$, and we let $x(0) > 1$ so that, as long as $x(t) \geq 1$,

$$\dot{x} = \frac{1}{\sqrt{1+2t}}\,x - 1 \quad \Rightarrow \quad x(t) = e^{(\sqrt{1+2t}-1)}\left[x(0) - \int_0^t e^{(1-\sqrt{1+2\tau})}d\tau\right]. \tag{20}$$

Using the change of variables $s = -1 + \sqrt{1+2\tau}$, we obtain

$$\int_0^t e^{(1-\sqrt{1+2\tau})}d\tau \leq \int_0^\infty e^{(1-\sqrt{1+2\tau})}d\tau = \int_0^\infty e^{-s}(s+1)\,ds = 2; \tag{21}$$

thus, if $x(0) \geq 3$, then (20) implies $x(t) \geq e^{(\sqrt{1+2t}-1)}$. This means that $x(t) \geq 1$ for all $t \geq 0$ and $x(t) \to \infty$ as $t \to \infty$.

Example 2. In this example, we show that the class-$\mathcal{H}_I$ gain restriction cannot be removed from Corollary 2. Consider the locally Lipschitz system

$$\dot{x} = -m(x) + g^{-1}\left(g\left(e^{e^{\frac{x-1}{b}}-a}z^2\right)\mathrm{sat}(z^2)/2\right), \tag{22}$$

$$\dot{z} = -z/2, \tag{23}$$

where $g : [0, +\infty) \to [0, 1]$ is defined as $g(x) := e^{-\frac{1}{x}}$ for $x > 0$ and $g(0) = 0$, $m(x)$ is a locally Lipschitz function satisfying $m(x)x > 0$ for all $x \neq 0$,

$$m(x) = \frac{1}{2e^{\frac{|x|-1}{b}}} \qquad \forall\, x \geq 1 + b\ln(a)\,, \tag{24}$$

and $a, b > 0$ are constants to be specified. Using $V(x) = \ln(1 + x^2)$ and $|g(r)| \leq 1$ for all $r \in \mathbb{R}_{\geq 0}$, we obtain

$$\begin{aligned}\dot{V} &= -\frac{2m(x)x}{1+x^2} + \frac{2x}{1+x^2} g^{-1}\left(g\left(e^{e^{\frac{x-1}{b}}-a} z^2\right)\operatorname{sat}(z^2)/2\right)\\ &\leq -\frac{2m(x)x}{1+x^2} + g^{-1}(\operatorname{sat}(z^2)/2)\,,\end{aligned} \tag{25}$$

which, from Lemma 1, proves that the x-subsystem is iISS with input z. Moreover, the z-subsystem is exponentially stable as in Corollary 2. However, because the iISS gain is not class-$\mathcal{H}_I$, the cascade (22)–(23) admits the unbounded solution

$$x(t) = 1 + b\ln(a + t) \tag{26}$$

when $z(0) \in (0, 1)$, $a = \ln(2/z^2(0)g(z^2(0)))$, and $b = 1/2$. To see that (26) satisfies (22), note that the time derivative of $x(t)$ is $\dot{x}(t) = b/(t+a)$ and that the substitution in (22) of

$$m(x(t)) = \frac{1}{2(a+t)} = \frac{b}{a+t}, \tag{27}$$

$$\begin{aligned} g^{-1}\left(g\left(e^{e^{\frac{x(t)-1}{b}}-a} z^2(t)\right)\operatorname{sat}(z^2(t))/2\right) &= g^{-1}(g(e^t z^2(0)e^{-t})z^2(0)e^{-t}/2)\\ = g^{-1}(g(z^2(0))z^2(0)e^{-t}/2) &= \frac{1}{t + \ln(2/z^2(0)g(z^2(0)))} = \frac{2b}{a+t}\end{aligned} \tag{28}$$

indeed yields $\dot{x}(t) = b/(t + a)$.

Theorem 1 characterized the class of admissible iISS gains from the speed of convergence of the z-subsystem. It may appear that this class can be enlarged by a change of coordinates in which the z-subsystem converges faster as in Grüne, Sontag, and Wirth [5]. The following example illustrates that such an attempt fails because in the new coordinates the iISS gain of the x-subsystem becomes steeper.

Example 3. For the system (18)–(19), the change of coordinates

$$\tilde{z} = \Phi(z) := ze^{-\frac{1}{2z^2}} \tag{29}$$

ensures exponential convergence for the $\tilde{z}$-subsystem:

$$\dot{x} = -\operatorname{sat}(x) + x\Phi^{-1}(\tilde{z}), \tag{30}$$

$$\dot{\tilde{z}} = -(1 + [\Phi^{-1}(z)]^2)\tilde{z}\,. \tag{31}$$

Using $V(x) = \frac{1}{2}\ln(1 + x^2)$, we obtain

$$\dot{V} \leq -\frac{x\operatorname{sat}(x)}{1+x^2} + \Phi^{-1}(|\tilde{z}|)\,,$$

which, from Lemma 1, implies that the x-subsystem with input $\tilde{z}$ is iISS with gain $\Phi^{-1}(\cdot)$. The cascade (30)–(31) has unbounded solutions as proved in Example 1 because all derivatives of $\Phi(z)$ vanish at $z = 0$, and hence the inverse function $\Phi^{-1}(\cdot)$ is too steep at zero to satisfy the class-$\mathcal{H}_I$ condition of Corollary 2.

4. Growth restrictions on the interconnection term. It is well known that the nonlinear growth of the interconnection term $h(x,z) := f(x,z) - f(x,0)$ plays an important role for the stability of the cascade (12)–(13), rewritten here as

$$\dot{x} = f(x,0) + h(x,z), \tag{32}$$
$$\dot{z} = q(z)\,. \tag{33}$$

In this section, we show that the class-$\mathcal{H}_I$ gain condition of Corollary 2 imposes a restriction on the nonlinear growth of $h(x,z)$ in x. To this end, we consider the cascade (32)–(33) with $x \in \mathbb{R}$ and with $h(x,z)$ bounded by

$$|h(x,z)| \leq \gamma_1(|z|) + \gamma_2(|z|)\varphi(|x|). \tag{34}$$

We characterize the class-$\mathcal{K}$ functions $\varphi(\cdot)$ for which the x-subsystem satisfies the class-$\mathcal{H}_I$ gain condition of Corollary 2.

PROPOSITION 1. *Consider the cascade* (32)–(33)*, where* $x \in \mathbb{R}$*. If* $\dot{x} = f(x,0)$ *is GAS,* $\dot{z} = q(z)$ *is GAS and LES, and* $h(x,z)$ *satisfies* (34) *for some class-*$\mathcal{H}_I$ *functions* $\gamma_1(\cdot)$*,* $\gamma_2(\cdot)$*, and a class-*$\mathcal{K}$ *function* $\varphi(\cdot)$ *satisfying*

$$\int_1^\infty \frac{1}{\varphi(s)} ds = \infty\,, \tag{35}$$

then the origin is GAS.

Proof. To prove that the x-subsystem is iISS with a class-$\mathcal{H}_I$ gain, we let $V(x)$ be a smooth, positive definite function such that $V(x) = V(-x)$, and

$$x \geq 1 \quad \Rightarrow \quad V(x) = V(1) + \int_1^x \frac{1}{\varphi(s)} ds\,. \tag{36}$$

Because of (35), $V(x)$ is radially unbounded and satisfies

$$|x| \geq 1 \quad \Rightarrow \quad \left|\frac{\partial V}{\partial x}\right| = \frac{1}{\varphi(|x|)}\,. \tag{37}$$

Thus, if $|x| \geq 1$, (34) and (37) yield

$$\frac{\partial V}{\partial x}[f(x,0) + h(x,z)] \leq \frac{1}{\varphi(1)}\gamma_1(|z|) + \gamma_2(|z|) =: \gamma_5(|z|)\,. \tag{38}$$

If $|x| \leq 1$, then $\left|\frac{\partial V}{\partial x}\right| \leq b$ for some positive constant b, and

$$\frac{\partial V}{\partial x}[f(x,0) + h(x,z)] \leq b\gamma_1(|z|) + b\gamma_2(|z|)\varphi(1) := \gamma_6(|z|)\,. \tag{39}$$

Because $\dot{V} \leq \max\{\gamma_5(|z|), \gamma_6(|z|)\}$, the x-subsystem is iISS with a class-$\mathcal{H}_I$ gain as in Corollary 2, and hence the cascade (32)–(33) is GAS. □

It is important to note that the growth condition (35) encompasses functions that grow faster than linear, such as $\varphi(|x|) = |x|\ln(|x|)$. On the other hand, (35) disallows $\varphi(|x|) = |x|^2$, $\varphi(|x|) = |x|^3$, etc. Growth conditions similar to (35) have been derived by Mazenc and Praly [11] and, more recently, by Panteley and Loría [13]. Proposition 1 gives a simple iISS interpretation of their more involved Lyapunov arguments. We finally show that (35) is tight and cannot be relaxed.

Example 4. The cascade

$$\dot{x} = -\mathrm{sat}(x) + \varphi(x)z, \tag{40}$$

$$\dot{z} = -z \tag{41}$$

exhibits finite escape time when the class-$\mathcal{K}$ function $\varphi(x)$ fails to satisfy (35), that is,

$$\int_1^\infty \frac{1}{\varphi(s)} ds < \infty \,. \tag{42}$$

To prove this, we let $V(x)$ be as in (36) and note from (42) that there exists a constant $V_\infty > 0$ such that $V(x) < V_\infty$ for all $x \in \mathbb{R}$, and $V(x) \to V_\infty$ as $x \to \infty$. Moreover, from (36),

$$x(t) \geq 1 \quad \Rightarrow \quad \dot{V} = -\frac{1}{\varphi(x(t))} + z(t) \geq -\frac{1}{\varphi(1)} + z(0)e^{-t} \,. \tag{43}$$

If $z(0) > 1/\varphi(1)$, then we can find $T > 0$ such that

$$\phi(t) := \int_0^t \left(-\frac{1}{\varphi(1)} + z(0)e^{-t} \right) dt = -\frac{t}{\varphi(1)} + z(0)(1 - e^{-t}) \tag{44}$$

satisfies $\phi(t) > 0$ for all $t \in (0, T]$. Thus, if $x(0)$ is such that $x(0) \geq 1$ and $V_\infty - \phi(T) \leq V(x(0)) < V_\infty$, then it follows from (43) that

$$V(x(T)) \geq V(x(0)) + \phi(T) \geq V_\infty \,,$$

which proves that $x(T)$ is not defined.

5. Application to output-feedback design. One of the major difficulties in nonlinear output-feedback design is the absence of a separation principle. Even when a nonlinear observer is available, it may be necessary to redesign the control law to make it robust against the observer error. One such design is the observer-based backstepping scheme of Kanellakopoulos, Kokotović, and Morse [7], further extended by Praly and Jiang [14], which makes use of *nonlinear damping* terms to render the system ISS with respect to the observer error. A shortcoming of this design, pointed out by several authors, is the rapid growth of the nonlinearities in the control law due to nonlinear damping terms. Such nonlinearities in the control law make the implementation difficult and are harmful in the presence of unmodeled actuator dynamics, saturation, etc. Efforts to reduce the growth of nonlinear damping terms are restricted to a result for Euler–Lagrange systems, Aamo et al. [1], and an adaptive backstepping design in Krstić, Kanellakopoulos, and Kokotović [10, section 5.8], where stability is achieved with the help of a strengthened parameter identifier.

We now give a systematic procedure to reduce the growth of nonlinear damping terms. Our main idea is to render the system iISS against the observer error. Because iISS is less restrictive than ISS, it is achieved with a "weak" form of nonlinear damping. Closed-loop stability is then established using Corollary 2 because the observer error is exponentially decaying. This exponential decay condition is satisfied by most observers used in backstepping, including Krener and Isidori [9], Arcak and Kokotović [4, 3], and Praly and Kanellakopoulos [15]. We wish to emphasize that our iISS design is not of "certainty-equivalence" type because, as in the ISS design of [7], the design of the controller makes use of the observer equations.

To make the main features of our design more apparent, we first illustrate it in an example. The general design procedure and its stability proof are given in the next section.

Example 5. For the system

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1^3,\\ \dot{x}_2 &= u + x_2 - x_2^3,\\ y &= x_1,\end{aligned} \tag{45}$$

the problem is to stabilize the origin $x = 0$ by output-feedback. The following observer, designed as in [3], ensures exponential convergence of the estimates $\hat{x}_1$ and $\hat{x}_2$ to the true states:

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + G_1(y, \hat{x}_1) := \hat{x}_2 + y^3 - 2(\hat{x}_1 - y),\\ \dot{\hat{x}}_2 &= u + G_2(y, \hat{x}_1, \hat{x}_2) := u + \hat{x}_2 - (\hat{x}_2 - 1.5(\hat{x}_1 - y))^3 - 3(\hat{x}_1 - y).\end{aligned} \tag{46}$$

To incorporate this observer in feedback control we employ the observer-based backstepping procedure of Kanellakopoulos, Kokotović, and Morse [7]. Defining the observer error $z_2 := x_2 - \hat{x}_2$ and letting $\chi_1 := x_1$, we rewrite the first equation of (45) as

$$\dot{\chi}_1 = \hat{x}_2 + \chi_1^3 + z_2\,. \tag{47}$$

For $\hat{x}_2$ we design the virtual control law

$$\alpha_1(\chi_1) = -c_1\chi_1 - \chi_1^3\,, \quad c_1 > 0, \tag{48}$$

which results in

$$\dot{\chi}_1 = -c_1\chi_1 + \chi_2 + z_2\,, \tag{49}$$

where $\chi_2 = \hat{x}_2 - \alpha_1(\chi_1)$. Differentiating χ_2 with respect to time, we obtain

$$\dot{\chi}_2 = u + G_2(y, \hat{x}_1, \hat{x}_2) - \frac{\partial \alpha_1}{\partial \chi_1}(\chi_1^3 + \hat{x}_2) - \frac{\partial \alpha_1}{\partial \chi_1} z_2 \tag{50}$$

and note that the control law

$$u = -(c_2 + \delta(\chi_1))\chi_2 - \chi_1 - G_2(y, \hat{x}_1, \hat{x}_2) + \frac{\partial \alpha_1}{\partial \chi_1}(\chi_1^3 + \hat{x}_2) \tag{51}$$

yields

$$\dot{\chi}_2 = -(c_2 + \delta(\chi_1))\chi_2 - \chi_1 - \frac{\partial \alpha_1}{\partial \chi_1} z_2\,. \tag{52}$$

In the ISS design of Kanellakopoulos, Kokotović, and Morse [7], the *nonlinear damping* term is

$$\delta(\chi_1) = \delta_{\mathrm{ISS}}(\chi_1) = \left(\frac{\partial \alpha_1}{\partial \chi_1}\right)^2 = (c_1 + 3\chi_1^2)^2, \tag{53}$$

whose growth in χ_1 is quartic.

To design a "softer" $\delta(\chi_1)$, let us pursue an iISS design with the help of the Lyapunov function $U(\chi_1,\chi_2) = \frac{1}{2}\chi_1^2 + \frac{1}{2}\chi_2^2$. From (47) and (50),

$$\dot{U} = -c_1\chi_1^2 + \chi_1 z_2 - c_2\chi_2^2 - \delta(\chi_1)\chi_2^2 + \chi_2(c_1 + 3\chi_1^2)z_2, \tag{54}$$

and hence, using the inequalities

$$3\chi_2\chi_1^2 z_2 \le \chi_2^2\chi_1^2 + \frac{9}{4}\chi_1^2 z_2^2, \quad \chi_1 z_2 \le \frac{c_1}{2}\chi_1^2 + \frac{1}{2c_1}z_2^2, \quad c_1\chi_2 z_2 \le \frac{c_2}{2}\chi_2^2 + \frac{c_1^2}{2c_2}z_2^2, \tag{55}$$

we can find positive constants k_1, k_2, k_3 such that

$$\dot{U} \le -k_1 U - \delta(\chi_1)\chi_2^2 + \chi_1^2\chi_2^2 + k_2 U z_2^2 + k_3 z_2^2 . \tag{56}$$

Unlike the iISS Lyapunov function (5) in Lemma 1, the inequality (56) contains the product of U and the disturbance z_2^2, which means that $U(\chi)$ cannot be an iISS Lyapunov function. However, the new C^1 function $V(\chi) := \ln(1 + U(\chi))$ results in

$$\dot{V} = \frac{\dot{U}}{1+U} , \tag{57}$$

which means that the product $k_2 U z_2^2$ in (56) is now

$$k_2 \frac{U}{1+U} z_2^2 \le k_2 z_2^2, \tag{58}$$

and hence

$$\dot{V} \le -k_1 \frac{U}{1+U} + (k_2 + k_3)z_2^2 + (\chi_1^2 - \delta(\chi_1))\frac{\chi_2^2}{1+U} . \tag{59}$$

Because the first two terms on the right-hand side are as in the iISS Lyapunov function (5), the choice

$$\delta(\chi_1) = \delta_{\text{iISS}}(\chi_1) = \chi_1^2 \tag{60}$$

eliminates the third term and ensures iISS as in Lemma 1. It follows from Corollary 3 that our iISS design ensures GAS of the closed-loop system (45)–(46) because the observer error z is exponentially decaying and the χ-subsystem is affine in z_2.

Unlike the quartic $\delta_{\text{ISS}}(\chi_1)$, the "weak" nonlinear damping term $\delta_{\text{iISS}}(\chi_1)$ is only quadratic. This reduction in the nonlinear growth is more pronounced for higher relative degree systems studied in the next section, which require several steps of observer-based backstepping.

6. Observer-based backstepping with weak nonlinear damping. We now generalize the above design to the system

$$\begin{aligned} y &= x_1, \\ \dot{x}_1 &= x_2 + g_1(x_1), \\ \dot{x}_2 &= x_3 + g_2(x_1, x_2), \\ &\cdots \\ \dot{x}_r &= p(\xi, u) + g_r(x), \\ \dot{\xi} &= q(\xi, x, u) , \end{aligned} \tag{61}$$

where $x := (x_1, \ldots, x_r)^T$ and the functions $g_1, \ldots, g_r, p$ and q are smooth and vanish when their arguments are zero; that is, the origin $(x, \xi) = (0, 0)$ is an equilibrium when $u = 0$.

We assume the availability of a global observer of the form

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + G_1(y, \hat{x}_1), \\ \dot{\hat{x}}_2 &= \hat{x}_3 + G_2(y, \hat{x}_1, \hat{x}_2), \\ &\cdots \\ \dot{\hat{x}}_r &= p(\hat{\xi}, u) + G_r(y, \hat{x}), \\ \dot{\hat{\xi}} &= Q(\hat{\xi}, \hat{x}, u, y)\,. \end{aligned} \tag{62}$$

ASSUMPTION 1. *The observer* (62) *guarantees exponential convergence of the state estimates to the true states; that is, there exist a constant* $k > 0$ *and a class-*$\mathcal{K}$ *function* $\gamma(\cdot)$ *such that, for every input* u *and for every initial condition* $x(0)$, $\xi(0)$, $\hat{x}(0)$, $\hat{\xi}(0)$, *the observer error* $z = (x^T, \xi^T)^T - (\hat{x}^T, \hat{\xi}^T)^T$ *satisfies*

$$|z(t)| \leq e^{-kt}\gamma(|z(0)|) \tag{63}$$

for all t *in the maximal interval of existence* $[0, t_f)$ *of* (61)–(62).

Observer designs satisfying Assumption 1 are being reported at an increasing rate [15, 4, 3]. Our next assumption is that the function $p(\xi, u)$ is invertible in u.

ASSUMPTION 2. *There exists a function* $\pi(\cdot, \cdot)$ *satisfying*

$$v = p(\xi, u) \quad \Leftrightarrow \quad u = \pi(\xi, v). \tag{64}$$

Finally, the zero dynamics of the system (61),

$$\dot{\xi} = q(\xi, 0, \pi(\xi, 0)), \tag{65}$$

satisfy the following robust stability assumption.

ASSUMPTION 3. *The zero dynamics* (65), *perturbed by* v_0, v_1, *and* v_2,

$$\dot{\xi} = q(\xi, v_0, \pi(\xi - v_1, v_2)), \tag{66}$$

are ISS with input (v_0, v_1, v_2).

The class of systems defined by (61) and Assumptions 1–3 encompasses the one studied in [7] and [10] for observer-based backstepping. We now present our new design, which renders the system (61) iISS with respect to the observer error z.

Step 1. Defining

$$\chi_1 := y \tag{67}$$

and using $x_2 = \hat{x}_2 + z_2$, we rewrite the first equation of (61) as

$$\dot{\chi}_1 = \hat{x}_2 + g_1(\chi_1) + z_2\,. \tag{68}$$

For $\hat{x}_2$, we design the *virtual* control law

$$\alpha_1(\chi_1) = -c_1\chi_1 - g_1(\chi_1), \quad c_1 > 0\,, \tag{69}$$

and obtain

$$\dot{\chi}_1 = -c_1\chi_1 + \chi_2 + z_2, \tag{70}$$

where

$$\chi_2 := \hat{x}_2 - \alpha_1(\chi_1)\,. \tag{71}$$

Step 2. From (62) and (68), χ_2 is governed by

$$\dot{\chi}_2 = \hat{x}_3 + G_2(\chi_1, \hat{x}_1, \hat{x}_2) - \phi_1(\chi_1)(\hat{x}_2 + g_1(\chi_1)) - \phi_1(\chi_1)z_2\,, \tag{72}$$

where

$$\phi_1(\chi_1) := \frac{\partial \alpha_1}{\partial \chi_1}\,. \tag{73}$$

Because $\phi_1(\chi_1)$ is a smooth function, we can rewrite it as

$$\phi_1(\chi_1) = \phi_{10} + \chi_1\Phi_1(\chi_1), \tag{74}$$

where

$$\phi_{10} = \phi_1(0)\,, \quad \Phi_1(\chi_1) = \int_0^1 \frac{\partial \phi_1(X_1)}{\partial X_1}\bigg|_{X_1 = s\chi_1} ds\,. \tag{75}$$

For $\hat{x}_3$, we design the virtual control law

$$\alpha_2(\chi_1, \hat{x}_1, \hat{x}_2) = -[c_2 + d_2\Phi_1^2(\chi_1)]\chi_2 - \chi_1 - G_2(\chi_1, \hat{x}_1, \hat{x}_2) + \phi_1(\chi_1)(\hat{x}_2 + g_1(\chi_1)), \tag{76}$$

which results in

$$\dot{\chi}_2 = -\chi_1 - [c_2 + d_2\Phi_1^2(\chi_1)]\chi_2 + \chi_3 - [\phi_{10} + \chi_1\Phi_1(\chi_1)]z_2\,, \tag{77}$$

where

$$\chi_3 := \hat{x}_3 - \alpha_2(\chi_1, \hat{x}_1, \hat{x}_2)\,. \tag{78}$$

Step i $(3 \le i \le r)$. For

$$\chi_i := \hat{x}_i - \alpha_{i-1}(\chi_1, \hat{x}_1, \ldots, \hat{x}_{i-1})\,, \tag{79}$$

we obtain

$$\begin{aligned} \dot{\chi}_i = {} & \hat{x}_{i+1} + G_i(\chi_1, \hat{x}_1, \ldots, \hat{x}_i) - \phi_{i-1}(\chi_1, \hat{x}_1, \ldots, \hat{x}_{i-1})[\hat{x}_2 + g_1(\chi_1) + z_2] \\ & - \frac{\partial \alpha_{i-1}}{\partial \hat{x}_1}(\hat{x}_2 + G_1(y, \hat{x}_1)) - \cdots - \frac{\partial \alpha_{i-1}}{\partial \hat{x}_{i-1}}(\hat{x}_i + G_{i-1}(y, \ldots, \hat{x}_{i-1})), \end{aligned} \tag{80}$$

where $\hat{x}_{r+1} := p(\hat{\xi}, u)$ and

$$\phi_{i-1}(\chi_1, \hat{x}_1, \ldots, \hat{x}_{i-1}) := \frac{\partial \alpha_{i-1}}{\partial \chi_1}\,. \tag{81}$$

To factor ϕ_{i-1} as in (74) in Step 2, we first rewrite it as a function of $(\chi_1, \ldots, \chi_{i-1}, z_1)$, where $z_1 = x_1 - \hat{x}_1$:

$$\phi_{i-1}(\chi_1, \hat{x}_1, \ldots, \hat{x}_{i-1}) = \tilde{\phi}_{i-1}(\chi_1, \ldots, \chi_{i-1}, z_1)\,. \tag{82}$$

Next, defining $\phi_{i-1,0}(z_1) = \tilde{\phi}_{i-1}(0,\ldots,0,z_1)$ and

$$\Phi_{i-1}(\chi_1,\ldots,\chi_{i-1},z_1) = \int_0^1 \left(\frac{\partial \tilde{\phi}_{i-1}(X_1,\ldots,X_{i-1},z_1)}{\partial(X_1,\ldots,X_{i-1})} \right)^T \Bigg|_{(X_1,\ldots,X_{i-1})=s(\chi_1,\ldots,\chi_{i-1})} ds\,, \tag{83}$$

we get

$$\phi_{i-1}(\chi_1,\hat{x}_1,\ldots,\hat{x}_{i-1}) = \phi_{i-1,0}(z_1) + [\chi_1 \ldots \chi_{i-1}]\, \Phi_{i-1}(\chi_1,\ldots,\chi_{i-1},z_1)\,. \tag{84}$$

The virtual control law for $\hat{x}_{i+1}$ is

$$\begin{aligned}\alpha_i(\chi_1,\hat{x}_1,\ldots,\hat{x}_i) = {} & -\chi_{i-1} - [c_i + d_i\Phi_{i-1}^T\Phi_{i-1}]\chi_i - G_i(\chi_1,\hat{x}_1,\ldots,\hat{x}_i) \\ & + \frac{\partial \alpha_{i-1}}{\partial \hat{x}_1}(\hat{x}_2 + G_1(y,\hat{x}_1)) + \cdots + \frac{\partial \alpha_{i-1}}{\partial \hat{x}_{i-1}}(\hat{x}_i + G_{i-1}(y,\ldots,\hat{x}_{i-1})) \\ & + \phi_{i-1}(\chi_1,\hat{x}_1,\ldots,\hat{x}_{i-1})[\hat{x}_2 + g_1(\chi_1)]\,. \end{aligned} \tag{85}$$

If $i < r$, we define

$$\chi_{i+1} = \hat{x}_{i+1} - \alpha_i(\chi_1,\hat{x}_1,\ldots,\hat{x}_i) \tag{86}$$

and proceed with step $i+1$. The control law obtained by r steps of backstepping is

$$u = \pi(\hat{\xi}, \alpha_r(\chi_1,\hat{x}_1,\ldots,\hat{x}_r))\,, \tag{87}$$

where the function $\pi(\cdot,\cdot)$ is as in (64).

The resulting closed-loop system consists of the exponentially converging observer error z driving the (χ,ξ)-subsystem

$$\dot{\xi} = q(\xi, v_0(\chi,z), \pi(\xi - v_1(z), v_2(\chi,z))), \tag{88}$$

$$\begin{aligned}\dot{\chi}_1 &= -c_1\chi_1 + \chi_2 + z_2, \\ &\cdots \\ \dot{\chi}_i &= -\chi_{i-1} - [c_i + d_i\Phi_{i-1}^T\Phi_{i-1}]\chi_i + \chi_{i+1} - (\phi_{i-1,0}(z_1) + [\chi_1\cdots\chi_{i-1}]\Phi_{i-1})z_2, \quad (89)\\ &\cdots \\ \dot{\chi}_r &= -\chi_{r-1} - [c_r + d_r\Phi_{r-1}^T\Phi_{r-1}]\chi_r - (\phi_{r-1,0}(z_1) + [\chi_1\cdots\chi_{r-1}]\Phi_{r-1})z_2\,,\end{aligned}$$

where the functions $v_0(\chi,z) = x$, $v_2(\chi,z) = \alpha_r(\chi_1,\hat{x}_1,\ldots,\hat{x}_r)$, and $v_1(z) = \xi - \hat{\xi}$ vanish at $(\chi,z) = (0,0)$.

THEOREM 2. *If Assumptions* 1–3 *hold, then the control law* (87) *guarantees GAS of the closed-loop system* (61)–(62).

Proof. We first prove that the χ-subsystem (89) is iISS with input z. To this end, we note that the function $U = \frac{1}{2}\sum_{i=1}^r \chi_i^2$ satisfies

$$\dot{U} = \left(\sum_{i=1}^r -c_i\chi_i^2 - \chi_i\phi_{i-1,0}(z_1)z_2 \right) + \left(\sum_{i=2}^r -d_i\Phi_{i-1}^T\Phi_{i-1}\chi_i^2 - z_2[\chi_1\cdots\chi_{i-1}]\Phi_{i-1}\chi_i \right)\,, \tag{90}$$

where $\phi_{00}(z_1) = -1$ and $\phi_{10}(z_1) = \phi_{10}$ as in (74). Using the inequalities

$$-\chi_i\phi_{i-1,0}(z_1)z_2 \le \frac{c_i}{2}\chi_i^2 + \frac{1}{2c_i}\phi_{i-1,0}^2(z_1)z_2^2, \tag{91}$$

$$-z_2[\chi_1\cdots\chi_{i-1}]\Phi_{i-1}\chi_i \le d_i\Phi_{i-1}^T\Phi_{i-1}\chi_i^2 + \frac{1}{4d_i}[\chi_1\cdots\chi_{i-1}]\,[\chi_1\cdots\chi_{i-1}]^T z_2^2\,, \tag{92}$$

we obtain

$$\dot{U} \leq -\sum_{i=1}^{r} \frac{c_i}{2}\chi_i^2 + \sum_{i=1}^{r} \frac{1}{2c_i}\phi_{i-1,0}^2(z_1)z_2^2 + \sum_{i=2}^{r} \frac{1}{4d_i}(\chi_1^2 + \cdots + \chi_{i-1}^2)z_2^2 \tag{93}$$

$$\leq -cU + \frac{1}{2c}\left(\sum_{i=1}^{r} \phi_{i-1,0}^2(z_1)\right) z_2^2 + \frac{r}{2d}Uz_2^2\,, \tag{94}$$

where $c := \min_{1\leq i\leq r} c_i$, $d := \min_{2\leq i\leq r} d_i$. Thus

$$V(\chi) := \ln(1 + U(\chi)) \tag{95}$$

satisfies

$$\dot{V} \leq -c\frac{U}{1+U} + \left(\frac{r}{2d} + \frac{1}{2c}\sum_{i=1}^{r} \phi_{i-1,0}^2(z_1)\right) z_2^2\,, \tag{96}$$

which, from Lemma 1, proves that the χ-subsystem (89) with input z is iISS with a class-$\mathcal{K}_o$ gain. Moreover, the observer error z satisfies the exponential decay condition (63) for all $t \in [0, t_f)$. Thus, letting $T < t_f$, we can show from Corollary 2 that there exists a class-$\mathcal{KL}$ function $\beta_1(\cdot,\cdot)$ such that, for all $t \in [0, T]$,

$$|(\chi(t), z(t))| \leq \beta_2(|(\chi(0), z(0))|, t)\,. \tag{97}$$

Next, we note from Assumption 3 that the ξ-subsystem (88) is ISS with input (χ, z). In view of Corollary 1, this means that a class-$\mathcal{KL}$ function $\beta_3(\cdot,\cdot)$ exists such that, for all $t \in [0, T]$,

$$|(\xi(t), \chi(t), z(t))| \leq \beta_3(|(\xi(0), \chi(0), z(0))|, t)\,. \tag{98}$$

Finally, we note that $|(\xi(T), \chi(T), z(T))|$ is bounded by $\beta_3(|(\xi(0), \chi(0), z(0))|, 0)$, which is independent of T. This means that $t_f = \infty$, and hence (98) holds for all $t \geq 0$, which proves GAS of the closed-loop system (61)–(62). ◻

7. Proofs of lemmas.

Proof of Lemma 1. Part (i) is proved in [2]. To prove part (ii), we modify [2, Proposition II.5], which is proved for a class-$\mathcal{K}$ function $\theta(\cdot)$, and show that it actually holds with a class-$\mathcal{K}_o$ function $\theta_o(\cdot)$.

LEMMA 5. *The system $\dot{x} = f(x, 0)$ is GAS iff there exist a smooth semiproper*[1] *function $W(x)$, a class-$\mathcal{K}_o$ function $\theta_o(\cdot)$, and a continuous positive definite function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that*

$$\frac{\partial W}{\partial x} f(x, z) \leq -\rho(|x|) + \theta_o(|z|). \tag{99}$$

The proof is given at the end of this section. In passing we emphasize that, in general, Lemma 5 does not hold with a *proper* (radially unbounded) $W(x)$.

To complete the proof of Lemma 1, we define $V_1(x) = V(x) + W(x)$, where $V(x)$ and $W(x)$ are as in (6) and (99), respectively, and obtain

$$\dot{V}_1 \leq -\rho(|x|) + \sigma(|z|) + \theta_o(|z|)\,, \tag{100}$$

which is (5) with $\mu(|z|) := \sigma(|z|) + \theta_o(|z|)$. ◻

[1] A positive definite function $W(x)$ is called *semiproper* if there exist a class-$\mathcal{K}$ function $\pi(\cdot)$ and a radially unbounded positive definite function $W_0(x)$ such that $W(x) = \pi(W_0(x))$. Thus $W(x)$ may not be radially unbounded.

Proof of Lemma 2. Because (7) is iISS with input z, it follows from [2] that there exists an *iISS Lyapunov function* $V(x)$ satisfying

$$L_f V(x) + L_g V(x) z := \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x) z \leq -\rho(|x|) + \tilde{\sigma}(|z|) \tag{101}$$

for some positive definite function $\rho(\cdot)$ and some class-$\mathcal{K}$ function $\tilde{\sigma}(\cdot)$. To prove that $\mu(\cdot)$ in (4) is class-$\mathcal{K}_o$, we first show that

$$L_f V(x) + |L_g V(x)| \leq \tilde{\sigma}(1)\,. \tag{102}$$

If $L_g V(x) \neq 0$, then (102) follows by evaluating (101) at $z = \frac{1}{|L_g V(x)|} L_g V(x)^T$. If $L_g V(x) = 0$, then (102) holds because $L_f V(x) \leq -\rho(|x|)$ from (101). Next, we note that $L_f V(x) \leq L_f V(x)|z|$ when $|z| \leq 1$ and obtain

$$|z| \leq 1 \quad \Rightarrow \quad L_f V(x) + L_g V(x) z \leq L_f V(x)|z| + |L_g V|\,|z| = (L_f V + |L_g V|)|z| \leq \tilde{\sigma}(1)|z|\,. \tag{103}$$

Inequalities (101) and (103) imply

$$L_f V(x) + L_g V(x) z \leq \sigma(|z|), \tag{104}$$

where

$$\sigma(|z|) := \begin{cases} \tilde{\sigma}(1)|z| & \text{if} \quad |z| \leq 1, \\ \tilde{\sigma}(|z|) & \text{if} \quad |z| > 1\,. \end{cases} \tag{105}$$

Because $\sigma(\cdot)$ is class-$\mathcal{K}_o$, it follows from Lemma 1 that the system (7) is iISS with a class-$\mathcal{K}_o$ gain $\mu(\cdot)$. □

Proof of Lemma 3. Because of ISS, it follows from [22] that there exists an *ISS Lyapunov function* $V(x)$ satisfying

$$\dot{V} \leq -\rho(|x|) + \sigma(|z|) \tag{106}$$

for some class-$\mathcal{K}_\infty$ functions $\rho(\cdot)$ and $\sigma(\cdot)$. Letting $\mu(\cdot)$ be a class-$\mathcal{K}$ function such that $\mu(s) = \tilde{\mu}(s)$ when $s \in [0,1]$ and $\mu(s) = \sigma(s)$ when $s \geq 2$ and applying the *changing supply functions lemma* [21], we can find another ISS Lyapunov function $\tilde{V}(x)$ satisfying

$$\dot{\tilde{V}} \leq -\tilde{\rho}(|x|) + \mu(|z|) \tag{107}$$

for some class-$\mathcal{K}$ function $\tilde{\rho}(\cdot)$. From Lemma 1, this implies iISS with gain $\mu(\cdot)$. □

Proof of Lemma 4. Because of GAS, there exists a $T^* > 0$ such that $|z(T^*)| \leq \epsilon$, and hence, for all $t \geq T^*$,

$$|z(t)| \leq \alpha\left(e^{-k(t-T^*)}\tilde{\gamma}(\epsilon)\right) = \alpha\left(e^{kT^*}e^{-kt}\tilde{\gamma}(\epsilon)\right). \tag{108}$$

Again, from GAS, there exists a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that

$$|z(t)| \leq \alpha(\beta(|z(0)|, t)); \tag{109}$$

thus, for all $t \in [0, T^*]$,

$$|z(t)| \leq \alpha\left((\beta(|z(0)|, t)e^{kt})e^{-kt}\right) \leq \alpha((\beta(|z(0)|, 0)e^{kT^*})e^{-kt})\,. \tag{110}$$

Choosing $T^* = T(|z(0)|)$ such that $T(|z(0)|) = 0$ for $|z(0)| \in [0, \epsilon]$ and $T(|z(0)|)$ is a continuous, strictly increasing function for $|z(0)| \geq \epsilon$, we conclude from (9), (108), and (110) that (8) holds with

$$\gamma(s) = \begin{cases} \max\left\{1, \frac{\beta(\epsilon,0)}{\tilde{\gamma}(\epsilon)}\right\} \tilde{\gamma}(s) & \text{if} \quad s \leq \epsilon, \\ \max\{\tilde{\gamma}(\epsilon), \beta(s,0)\} e^{kT(s)} & \text{if} \quad s > \epsilon. \end{cases} \tag{111}$$ □

Proof of Lemma 5. The result follows by using Lemma 6 instead of [2, Corollary IV.5] and modifying the proof of [2, Proposition II.5] accordingly. For Lemma 6, we define a function $\sigma_- : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ to be class-$\mathcal{K}_-$ if it is continuous and strictly increasing but, unlike a class-$\mathcal{K}$ function, not necessarily zero at zero.

Lemma 6. *If $\gamma : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ is such that $\gamma(\cdot, s)$ is class-$\mathcal{K}$ for each $s \in \mathbb{R}_{\geq 0}$ and $\gamma(r, \cdot)$ is class-$\mathcal{K}_o$ for each $r \in \mathbb{R}_{\geq 0}$, then there exist a class-$\mathcal{K}_-$ function $\sigma_-(\cdot)$ and a class-$\mathcal{K}_o$ function $\sigma_o(\cdot)$, such that*

$$\gamma(r, s) \leq \sigma_-(r)\sigma_o(s)\,. \tag{112}$$

Proof. Because both $\gamma(\cdot, s)$ and $\gamma(r, \cdot)$ are class-$\mathcal{K}$, it follows from [2, Corollary IV.5] that there exists a class-$\mathcal{K}$ function $\sigma_1(\cdot)$ such that

$$\gamma(r, s) \leq \sigma_1(r)\sigma_1(s)\,. \tag{113}$$

Because $\gamma(r, s) = \mathcal{O}(s)$ for all $r \geq 0$, we can find a class-$\mathcal{K}_-$ function $\sigma_2(\cdot)$ such that, for all $s \leq 1$,

$$\gamma(r, s) \leq \sigma_2(r)s\,. \tag{114}$$

The inequalities (113) and (114) imply $\gamma(r, s) \leq \sigma_-(r)\tilde{\sigma}_o(s)$, where $\sigma_-(r) := \max\{\sigma_1(r), \sigma_2(r)\}$ and

$$\tilde{\sigma}_o(s) := \begin{cases} s, & s \leq 1, \\ \sigma_1(s), & s > 1\,. \end{cases} \tag{115}$$

Thus (112) follows by finding a continuous upper-bound $\sigma_o(\cdot)$ on $\tilde{\sigma}_o(\cdot)$. □

8. Conclusion. We have studied the stability of nonlinear cascades and showed that a trade-off exists between slower convergence for the driving subsystem and steeper iISS gain for the driven subsystem. This approach unifies several results in the literature, obtained by restricting the speed of convergence of the driving subsystem and the nonlinear growth of the driven subsystem. We have studied the connection between these growth conditions and the iISS gain and have proved that our iISS gain restriction leads to a less restrictive condition than the linear growth assumption common in the literature. The cascade result has been used to develop a new observer-based backstepping design which reduces the growth of nonlinear damping terms. It would be of interest to extend our cascade result to feedback interconnections, where small-gain formulations of iISS can be pursued.

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