

Characterizing Innovations Realizations for Random Processes

CARLA A. SCHWARTZ† and BRADLEY W. DICKINSON‡

Department of Electrical Engineering and Computer Science, Princeton University, Princeton, NJ 08544

and

EDUARDO D. SONTAG‡

Department of Mathematics, Rutgers University, New Brunswick, NJ 08903

(Accepted for publication August 4 1983)

In this paper we are concerned with the theory of second order (linear) innovations for discrete random processes. We show that of existence of a finite dimensional linear filter realizing the mapping from a discrete random process $\{y_t\}_{t=0}^{\infty}$ to its innovations is equivalent to a certain semiseparable structure of the covariance sequence of the process. We also show that existence of a finite dimensional realization (linear or nonlinear) of the mapping from a process to its innovations implies that the process have this semiseparable covariance sequence property. In particular, for a stationary random process, the spectral density function must be rational.

1. INTRODUCTION

The innovations filter for a discrete-time (m -vector) random process, $\{y_t\}_{t=0}^{\infty}$, is a causal whitening filter which performs the Gram-Schmidt orthogonalization of the process, $\{y_t\}_{t=0}^{\infty}$, producing an uncorrelated process, $\{v_t\}_{t=0}^{\infty}$, such that the two processes generate

†Work supported by the National Science Foundation through grant ECS82-05772.

‡Work supported by U.S. Air Force through grant AFOSR-80-0196.

the nested set of same linear spaces; i.e.

$$\overline{sp\{y_t\}_{t=0}^s} = \overline{sp\{v_t\}_{t=0}^s}, \quad s \geq 0, \quad (1)$$

where $\overline{sp\{\cdot\}}$ denotes closed linear span of the components. The input/output maps of innovations filters are intrinsically linear, being determined by the orthogonality principle of linear least squares estimation [1, chapter 3]. However, even for a stationary process, the input/output map is generally time varying.

We assume that $\{y_t\}_{t=0}^{\infty}$ is a full rank, zero mean, second order process so that $R_{k,l} = E[y_k y_l']$ is finite for all nonnegative integers k and l . Informally, we may regard the matrix $R = [R_{i,j}]_{i,j=0}^{\infty}$ as a positive definite symmetric linear operator. However, since R may be an unbounded operator, specification of its domain is needed to make R well defined [2]. Analogously, it is necessary to introduce a nondegeneracy condition on the covariance in order for the innovations filter for the process to be well defined. Following Rissanen and Barbosa [3] we assume that for each N , there exists a positive constant α , independent of N , such that $\alpha I_{N \times N} \leq R_N = [R_{i,j}]_{i,j=0}^N$. This is equivalent to existence of a constant $\beta > 0$ such that $\beta I \leq R$, in the sense that $R - \beta I$ is positive definite, which is stronger than simply assuming that R is positive definite. In the terminology of [2], R is *lower semibounded*.

In this paper we will consider the characterization of discrete-time random processes which admit finite dimensional realizations of the linear, time varying input/output map associated with their innovations representations. In section 2 we show that a process has a linear finite dimensional realization of its innovations filter if and only if its covariance sequence is *semiseparable*. This property of a covariance sequence concerns its representation in the following way: there exist matrix functions $M(\cdot)$, $\Phi(\cdot, \cdot)$, and $N(\cdot)$, $m \times n$, $n \times n$, and $n \times m$ respectively, for some fixed n , such that

$$R_{k,l} = M(k)\Phi(k,l)N(l)U(k-l) + N'(k)\Phi'(k,l)M'(l)U(l-k-1) \\ \times \Phi(k,l) = \Phi(k,r)\Phi(r,l) \quad l \leq r \leq k \quad (2)$$

where $R_{k,l} = E\{y_k y_l'\}$, an $m \times m$ matrix, $U(\cdot)$ is the unit step function, and $'$ denotes transpose.

Our result differs from those in [4, (Theorems 1 and 2 with corollaries)] in two important ways. First, we do not assume that the function $\Phi(\cdot, \cdot)$ appearing in (2) is invertible. Second, we deal with innovations representations on the infinite time interval $0 \leq t \leq \infty$, and in order to show that a process with covariance satisfying (2) can be obtained as the output of a finite dimensional linear time varying state space model, we must consider the infinite matrix R , and conditions for existence of its Cholesky factorization as HH' , where H is lower triangular with a bounded inverse (in a suitable sense if H is unbounded).

These differences are important ones. The first one is motivated by noting that in the stationary case, $\Phi(i,j) = A^{i-j}$ for some matrix A , whose characteristic values determine the poles of the spectral density of the stationary process. We don't wish to exclude processes for which A is not invertible, such as moving average processes. The second difference highlights the importance of the technical assumption that R is bounded below, as described above. When necessary properties (positive definiteness of R_N) for existence of the Cholesky factorization of R_N as $H_N H_N'$, where H_N is an invertible and lower triangular finite matrix, are assumed in [4], only the finite matrix R_N for fixed N is considered. This is used to show existence of a lumped state space model for the process, a condition necessary in the hypotheses of the theorems in [4], which, along with (2), implies existence of a finite dimensional linear state space innovations filter realization for the process. In contrast, we consider the infinite matrix R , in order to show existence of the innovations filter for the entire sequence $\{y_t\}_{t=0}^{\infty}$. The lower semibounded property of R is sufficient for this purpose. In the usual case that the innovations process covariance matrix is a bounded operator with bounded inverse, our condition is equivalent to the bounded input-bounded output stability of the prediction filter for the process [3, 5].

In section 3 we consider a more general class of finite dimensional state space realizations for the innovations filter of discrete random processes, a class of smooth, time varying nonlinear realizations, in order to see what covariance structures are associated with processes whose innovations representations are realized by this type of model. We show that existence of a finite dimensional discrete time innovations realization in this class of nonlinear models, implies the existence of a finite dimensional *linear* realization, and so the

semiseparable covariance structure described above. If, in addition, the process is stationary, then the existence of any finite dimensional discrete time innovations realization, which may be nonlinear and/or time varying, implies that the process has a rational spectral density function. Hence, for finite dimensional models, we need not consider a more general class of state space models than linear ones.

2. LINEAR STATE SPACE MODELS FOR INNOVATIONS REPRESENTATIONS

Let $\{y_i\}_{i=0}^{\infty}$ satisfy the assumptions described in section 1, and let $y_{i|j}$ be the best linear estimate of y_i given y_0, \dots, y_j . Our first result shows that the semiseparable structure (2) characterizes processes with linear, finite dimensional innovations realizations.

THEOREM 1 *There exists a finite dimensional, time varying, invertible, causal linear system taking y_0, y_1, y_2, \dots into the innovations process v_0, v_1, \dots*

$$v_0 = y_0 \quad (3a)$$

$$v_i = y_i - y_{i|i-1}, \quad i \geq 1 \quad (3b)$$

if and only if (2) holds.

Proof ← Assuming $R_{k,l}$ satisfies (2) and represents a positive definite $m \times m$ covariance sequence for $k, l \geq 0$, then there exists a causal impulse matrix $h_{i,j}$, such that a system with $\{h_{i,j}\}$ as impulse response, has covariance sequence $E(z_k z_l') = R_{k,l}$ at the output, (for output sequence z_t), when the system is excited by white noise. In other words,

$$R_{i,j} = \sum_{k=0}^{\infty} h_{i,k} h_{j,k}' = \sum_{k=0}^{\min(i,j)} h_{i,k} h_{j,k}' \quad (4)$$

since $h_{i,j} = 0, i < j$. This result is true since, by the assumption $\beta I \leq R$, the infinite positive definite matrix R can be factored as $R = HH'$, where H is a lower triangular $\infty \times \infty$ block matrix with $m \times m$ block entries, $h_{i,j}$. (Taking $h_{i,i}$ to be lower triangular with positive

diagonal elements, H is unique.) It follows that H is invertible, and its lower triangular inverse, V , is a bounded linear operator, satisfying $H = RV'$. If V takes the form

$$V' = \begin{bmatrix} v'_{0,0} & v'_{1,0} & \dots & v'_{k,0} & \dots \\ 0 & v'_{1,1} & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & v'_{k,k} & \dots \\ \dots & \dots & \dots & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \quad (5)$$

then $h_{i,j} = \sum_{k=0}^{j-1} R_{i,k} v'_{j,k}$. Using the causality of h and the form of V' , given above,

$$\begin{aligned} h_{i,j} &= M(i) \Phi(i,j) \left[\sum_{k=0}^j \Phi(j,k) N(k) v'_{j,k} \right] U(i-j) \\ &= M(i) \Phi(i,j) L(j) U(i-j) \end{aligned} \quad (6)$$

where

$$L(j) \triangleq \sum_{k=0}^j \Phi(j,k) N(k) v'_{j,k}$$

and therefore, $h_{i,j}$ is the impulse response sequence for a finite dimensional time varying system:

$$X_{k+1} = \Phi(k+1, k) X_k + L(k+1) u_{k+1} \quad (7)$$

$$y_k = M(k) X_k. \quad (8)$$

This model, along with the following conditions,

$$X_0 = x_0 \quad E[x_0] = 0.$$

$$E[x_0 u_t'] = 0; \quad E[x_0 x_0'] = \Pi_0 \text{ and } h_{0,0} \triangleq M(0) \Pi_0^{1/2},$$

provides a model for the process y_t , where $\{u_t\}$ is a zero mean, unit variance white noise, $E(y_k y_l') = R_{k,l}$ and $\Pi_0^{1/2}$ is the lower triangular

Cholesky factor of Π_0 , which exists by the positive definiteness of Π_0 .

Having shown that $\{y_t\}_{t=0}^\infty$ can be obtained as the output of a finite dimensional linear model, we can now appeal to theorem 1 and its corollary in [4], to infer existence of a finite dimensional, causal, causally invertible time varying linear innovations filter for the process $\{y_t\}_{t=0}^\infty$. The hypotheses of this theorem which yields a linear innovations model for the process, are only that the process, y_t , have a positive definite covariance satisfying (2), and that y_t arise from some lumped model, as above; theorem 1 of [4] does not require invertibility of $\Phi(i, j)$.

Suppose there exists a finite dimensional causal and causally invertible linear system taking $\{y_t\}_{t=0}^\infty$ to $\{v_t\}_{t=0}^\infty$; $\{v_t\}_{t=0}^\infty$ represents the Gram-Schmidt orthogonalization of $\{y_t\}_{t=0}^\infty$. Then there exist maps $V_k: (y_0 \cdots y_k) \rightarrow v_k$ and $H_k: (v_0 \cdots v_k) \rightarrow y_k$ defined by

$$v_k = \sum_{j=0}^k v_{k,j} y_j \quad y_k = \sum_{j=0}^k h_{k,j} v_j \quad (9)$$

and since $\{h_{k,j}\}$ is the input/output sequence for a time varying finite dimensional linear system, there exist matrices $M(i)$, $\Phi(i, j)$, and $L(i)$ of appropriate dimension so that

$$h_{k,j} = M(k)\Phi(k, j)L(j)U(k-j). \quad (10)$$

So, for $k \geq l$,

$$E(y_k y_l') = R_{k,l} \quad (11)$$

$$\begin{aligned} &= M(k) \left[\sum_{j=0}^l \Phi(k, j)L(j)\sigma_j^2 L(j)\Phi'(l, j) \right] M'(l) \\ &= M(k)\Phi(k, l) \left[\sum_{j=0}^l \Phi(l, j)L(j)\sigma_j^2 L(j)\Phi'(l, j)M'(l) \right] \\ &= M(k)\Phi(k, l)N(l) \end{aligned}$$

$$N(l) \stackrel{\Delta}{=} \sum_{j=0}^l \Phi(l, j)L(j)\sigma_j^2 L(j)\Phi'(l, j)M'(l) \quad \sigma_j^2 = E(v_j v_j') \quad (12)$$

and for $k < l$

$$\begin{aligned} R_{k,l} &= M(k) \left[\sum_{j=0}^k \Phi(k, j)L(j)\sigma_j^2 L(j)\Phi'(l, j) \right] M'(l) \\ &= M(k) \left[\sum_{j=0}^k \Phi(k, j)L(j)\sigma_j^2 L(j)\Phi'(k, j) \right] \Phi'(l, k)M'(l) \\ &= N'(k)\Phi'(l, k)M'(l) \end{aligned} \quad (13)$$

so therefore $R_{k,l}$ satisfies (2).

3. FINITE DIMENSIONAL INNOVATIONS REALIZATIONS FOR STATIONARY PROCESSES

The goal of this section is to show that even when time varying, non-linear realizations are considered, a stochastic process admits a finite dimensional innovations realization only when its covariance function takes the semiseparable form of (2). In addition, if the process is stationary, its spectral density function must be rational. Our proofs of these results are purely system-theoretic, relying on a fundamental property of realizations of linear time varying input/output maps.

We begin by introducing the class of nonlinear systems to be considered and some associated notation. We follow [6] in this regard. The space of input values, U , and output values, Y , may be taken to be differentiable manifolds; in our application $U = Y = \mathbf{R}^m$.

DEFINITION A system Σ is characterized by $(X, \{p_k\}_{k \geq 0}, \{q_k\}_{k \geq 0}, 0)$, where

X is a differentiable manifold of states,

0 , the initial state, is an element of X ,

$p_k: X \times U \rightarrow X$ is a continuously differentiable map for all $k \geq 0$,

$q_k: X \times U \rightarrow Y$ is a continuously differentiable map for all $k \geq 0$.

The state equations that comprise the system are:

$$x_{k+1} = p_k(x_k, u_k), \quad k \geq 0; \quad x_0 = 0 \quad (14)$$

$$y_k = q_k(x_k, u_k), \quad k \geq 0. \quad (15)$$

We say that Σ is finite dimensional if its associated state manifold, X , is finite dimensional.

Let $P_{t,j}: X \times U^j \rightarrow X$ denote the j -step state transition map for the system starting in state x at time $t \geq 0$. This is defined for $j \geq 0$ by

$$P_{t,0}(0) = x, \quad (16)$$

$$P_{t,1}(x, u) = p_t(x, u) \quad (17)$$

$$P_{t,j+1}(x, u_t, \dots, u_{t+j}) = p_{t+j}(P_{t,j}(x, u_t, \dots, u_{t+j-1}), u_{t+j}), \quad j \geq 0. \quad (18)$$

The corresponding output map is $Q_{t,j}: X \times U^j \rightarrow Y$, defined for $j \geq 1$ by

$$Q_{t,j}(x, u_t, \dots, u_{t+j-1}) = q_{t+j-1}(P_{t,j-1}(x, u_t, \dots, u_{t+j-2}), u_{t+j-1}), \quad j \geq 1. \quad (19)$$

For $k \geq 0$ we define $f_{k+1}(u_0, \dots, u_k) = Q_{0,k+1}(0, u_0, \dots, u_k)$. The set of these maps constitutes an external, or input/output description of the system Σ . We say that Σ is a realization of its input-output description.

Now let us turn to the use of these systems for generating innovations processes. Suppose we have a finite dimensional nonlinear system providing a model for the innovations of $\{y_t\}_{t=0}^{\infty}$ of the following form:

$$x_{t+1} = p_t(x_t, y_t)$$

$$v_t = q_t(x_t, y_t).$$

The input/output description of the system consists of linear maps,

i.e., v_t is a linear function of the input, so

$$Q_{0,t+1}(0, y_0, \dots, y_t) = q_t(x_t, y_t) = \sum_{s=0}^t v_{t,s} y_s,$$

we can write

$$v_t = v_{t,t} y_t + \sum_{s=0}^{t-1} v_{t,s} y_s$$

or

$$y_t = v_{t,t}^{-1} \left[v_t - \sum_{s=0}^{t-1} v_{t,s} y_s \right].$$

For $t \geq 1$, $x_t = P_{0,t}(0, y_0, \dots, y_{t-1})$ and q_t is linear in y_t , i.e. $\partial q_t(x_t, y_t) / \partial y_t = v_{t,t}$, a constant. Thus

$$q_t(x_t, y_t) = v_{t,t} y_t + g_t(x_t)$$

for some continuously differentiable function $g_t(\cdot)$, and so

$$y_t = v_{t,t}^{-1} [v_t - g_t(x_t)] \stackrel{\Delta}{=} s_t(v_t, x_t).$$

This equation may be substituted into the state equation, giving x_{t+1} in terms of x_t and v_t . Consequently we have shown that existence of a finite dimensional nonlinear innovations realization for a given discrete stochastic process implies the existence of a finite dimensional nonlinear modeling (shaping) filter realization for the process.

We are now ready to state and prove the main result of this section.

THEOREM 2 *Suppose there exists a finite dimensional nonlinear realization, having dimension n , for the linear time varying input/output map of the innovations representation for a random process, $\{y_t\}_{t=0}^{\infty}$, which satisfies the conditions described in the introduction. Then the covariance of $\{y_t\}_{t=0}^{\infty}$ is semiseparable.*

Proof By the argument above, we know that there is a finite dimensional system Σ which plays the role of a modeling (shaping)

filter, whose inputs are the innovations and whose outputs are the process $\{y_t\}_{t=0}^{\infty}$. Its input/output relationship is characterized by linear maps which take the form $y_k = f_{k+1}(u_0, \dots, u_k) = \sum_{j=0}^k h_{k,j} u_j$, where $\{h_{i,j}\}$ is related to the covariance through (4). For every $r \geq 0$, we define a map $F^{(r, \infty)}$ from the space of input values $U \times U \times \dots \times U = U^{r+1}$ to the infinite product space of output values Y^{∞} . Let $\omega_i^{(r)} = (u_{r+1}, \dots, u_{r+i})$, $1 \leq i \leq r < \infty$, be a fixed nested set of input strings, and let ω_0 be the empty string, where $u_j \in U$ for each j . For $1 \leq i < \infty$, the i th component of $F^{(r, \infty)}$, is given by

$$F_i = Q_{r+1,i}(P_{0,r+1}(0, u_0, \dots, u_r), u_r, \omega_{i-1}). \quad (20)$$

The superscripts on F_i and ω_{i-1} are deleted for convenience. Each component map is the composition of the state transition and output maps associated with the realization of the modeling filter. Thus $F^{(r, \infty)}$ factors through the map $P_{0,r+1}$ whose image is contained in the n -dimensional manifold X , and so

$$J(F^{(r, \infty)}) = \text{Jacobian } F^{(r, \infty)} = J([Q_{r+1,1}, \dots, Q_{r+1,r+1}, \dots]) J(P_{r+1,0})$$

has rank at most n . The components of $F^{(r, \infty)}$ may be expressed in terms of the input/output map of the modeling filter:

$$F_i = \sum_{j=0}^{r+1-i} h_{r+1-i,j} u_j. \quad (21)$$

Now the Jacobian can be evaluated, giving

$$J(F^{(r, \infty)}) = \begin{bmatrix} \frac{\partial F_1}{\partial u_r} & \frac{\partial F_1}{\partial u_{r-1}} & \dots & \frac{\partial F_1}{\partial u_0} \\ \frac{\partial F_2}{\partial u_r} & \frac{\partial F_2}{\partial u_{r-1}} & \dots & \frac{\partial F_2}{\partial u_0} \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix} = \begin{bmatrix} h_{r,r} & h_{r,r-1} & \dots & h_{r,0} \\ h_{r+1,r} & h_{r+1,r-1} & \dots & h_{r+1,0} \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix}. \quad (22)$$

The rank of $J(F^{(r, \infty)})$ is less than or equal to n for every r . We will

use a result of Fliess [7], independently obtained by Ferrer and Kamen [8], to infer that the sequence $\{h_{i,j}\}$ is the impulse response sequence of an n dimensional linear system. Then the semiseparable form, (2), of the covariance follows from Theorem 1, and the proof is complete.

The result proved by Fliess in [7] concerns realization of bilinear systems, and there it is briefly indicated how the result specializes to the case of linear systems. Since the linear case may be easily described without introducing the tools of noncommutative generating series, we will describe the construction of a linear realization here. We emphasize that the details may be obtained by a straightforward simplification of the construction for the bilinear case. In particular, the bounded rank condition on $J(F^{(r, \infty)})$ is exactly the bounded *homogeneous rank* condition on the Hankel matrix associated with the noncommutative generating series of [7].

An n dimensional linear realization is obtained from the series of matrices $J(F^{(r, \infty)})$, for $r \geq 0$, as follows: The state manifold, X , is a fixed (arbitrary) n dimensional vector space. Let C_r denote the span of the columns of $J(F^{(r, \infty)})$. For each r , define a 1-1 map $\varphi_r: C_r \rightarrow X$. This map is uniquely defined except in cases where the dimension of C_r is less than n . Define the linear map $\Psi(r): C_r \rightarrow C_{r+1}$, to be the upward shift operator on the columns of $J(F^{(r, \infty)})$; the $(i+1)$ th element of a column becomes the i th element of the image column, and the first element is discarded. Thus $\Psi(r)$ maps the k th column of $J(F^{(r, \infty)})$ to the $(k+1)$ th column in $J(F^{(r+1, \infty)})$. This defines $\Psi(r)$ for all $r \geq 0$. For $r > 0$, the second column of $J(F^{(r, \infty)})$ will be $\beta(r-1)$, a map from a scalar input into C_r in the obvious way. Take $\alpha(r)$ to be the mapping from C_r into a scalar output defined by the first component of a column. Finally, take $d(r)$ to be $h_{r,r}$. Then the maps $[A(\cdot), b(\cdot), c(\cdot)]$ are obtained as solutions of the equations

$$A(r)\varphi_r = \varphi_{r+1}\Psi(r), \quad r \geq 0$$

$$b(r-1) = \varphi_r \beta(r-1), \quad r \geq 1$$

$$c(r)\varphi_r = \alpha(r), \quad r \geq 0.$$

It is easily checked that $h_{i,j} = c(i)A(i-1)\dots A(j+1)b(j)$, $i > j+1$, and

$h_{i,j-1} = c(i)b(i-1)$. The following linear realization is obtained:

$$\begin{aligned}x_{r+1} &= A(r)x_r + b(r)u_r \\ y_r &= c(r)x_r + d(r)u_r.\end{aligned}$$

From Theorems 1 and 2, we have the immediate corollary, in which linearity is removed from the statement of Theorem 1.

COROLLARY 1 *There exists a finite dimensional, time varying, invertible, causal system mapping the process $\{y_t\}_{t=0}^{\infty}$ to its innovations if and only if the covariance of $\{y_t\}_{t=0}^{\infty}$ is semi-separable.*

By employing the factorization of the covariance matrix $R = HH'$, developed in the previous section, we may again write

$$h_{i,j} = \sum_{k=0}^j R_{i,k} v'_{j,k}, \quad (23)$$

where $V = H^{-1}$ is given by (5). Applying this relation to (22) gives

$$J(F^{(r,r)}) = \begin{bmatrix} R_{r,r} & \cdots & R_{r,0} \\ R_{r+1,r} & \cdots & R_{r+1,0} \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} v'_{r,r} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 \\ v'_{r,0} & \cdots & v'_{0,0} \end{bmatrix} \triangleq R_{Hank}(r) \tilde{V}_r \quad (24)$$

Since \tilde{V}_r is clearly invertible, the combination of Theorem 1 and the realizability results of [7,8] give the following result.

THEOREM 3 *A lower semibounded covariance is semiseparable if and only if for every $r \geq 0$ the matrices $R_{Hank}(r)$ have rank at most n .*

As an important special case, when $\{y_t\}_{t=0}^{\infty}$ is a stationary process, $R_{i,j}$ is a function only of $|i-j|$, and from (24) we have the following corollary of Theorem 2.

COROLLARY 2 *If a discrete stationary process $\{y_t\}_{t=0}^{\infty}$, whose spectral density function is bounded away from 0, has a finite dimensional nonlinear innovations filter realization, its spectral density function is rational.*

Proof In the stationary case, the matrix $R_{Hank}(r)$ in (24) is a (block) *Hankel* matrix, and because its rank is less than n for all r , the conclusion is a standard result of linear system theory [9], since the spectral density function is the z -transform of the covariance sequence, evaluated on the unit circle in the complex plane.

By applying Corollary 2, we may specialize Corollary 1 to the stationary case.

COROLLARY 3 *A discrete-time stationary process $\{y_t\}_{t=0}^{\infty}$ has a finite dimensional innovations representation if and only if it has a rational spectral density function.*

4. CONCLUDING REMARKS

The results described in this paper resolve the question of existence of finite dimensional innovations realizations for discrete-time random processes. Semiseparability is a necessary and sufficient condition and this, in turn, is equivalent to a sequence of finite rank conditions. For the stationary case, the condition is rationality of the power spectral density function. We have shown that linear realizations are the only ones that need to be considered when looking for finite dimensional innovations realizations.

We should comment briefly on the connection between our results and the theory of stochastic realization for stationary processes; the discrete-time theory may be found in references [10-12], for example. Stochastic realization theory includes the problem of obtaining the innovations process for $\{y_t\}_{t=-\infty}^{\infty}$. In this case, the elements of the state vector of the innovations model form a basis for the *predictor space*, the image of the orthogonal projection of the Hilbert space generated by $\{y_t\}_{t=s}^{\infty}$ onto the closed linear span of $\{y_t\}_{t=-\infty}^s$. The resulting innovations filter is time invariant because the process is stationary, and it follows directly from function theoretic arguments that the predictor space is finite dimensional if and only if the process has a rational spectral density [10].

References

- [1] E. Wong, *Stochastic Processes in Information and Dynamical Systems*, Kreiger, Huntington, NY, 1979.

- [2] F. Riesz and B. Sz. Nagy, *Functional Analysis*, Frederick Ungar, New York, 1955, Chapter VIII.
- [3] J. Rissanen and L. Barbosa, Properties of infinite covariance matrices and stability of optimum predictors, *Inform. Sci.* **1** (1969), 221-236.
- [4] M. R. Gevers and T. Kailath, An innovations approach to least-squares estimation. Part VI: Discrete-time innovations representations and recursive estimation, *IEEE Trans. Automat. Contr.* **AC-18** (1973), 588-600.
- [5] H. B. Aasnaes, *Estimation and Identification in Discrete-Time Systems Using the Innovations Approach*, Ph.D. dissertation, Stanford Univ., Stanford, CA, May, 1971.
- [6] B. W. Dickinson and E. D. Sontag, Dynamical realizations of sufficient sequences, submitted for publication.
- [7] M. Fliess, Un codage non commutatif pour certains systèmes échantillonnés non linéaires, *Infor. Contr.* **38** (1978), 264-287.
- [8] E. Kamen, Personal Communication, Univ. of Fla., Gainesville, Fla. 1983.
- [9] R. E. Kalman, P. L. Falb and M. A. Arbib, *Topics in Mathematical System Theory*, McGraw-Hill, New York, 1969, p. 301.
- [10] A. E. Frazho, On minimal splitting subspaces and stochastic realizations, *Siam J. Control and Optimization* **20** (1982), 553-562.
- [11] H. Akaike, Stochastic theory of minimal realization, *IEEE Trans. Automat. Contr.* **AC-19** (1974), 667-673.
- [12] A. Lindquist and M. Pavon, Markovian representation of discrete-time stationary stochastic vector processes, *Proc. 20th IEEE Conf. on Decision and Control* (1981), pp. 1345-1356.