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INPUT TO STATE STABILIZABILITY FOR PARAMETRIZED FAMILIES OF SYSTEMS

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SUMMARY

This paper studies various stability issues for parametrized families of systems, including problems of stabilization with respect to sets. The study of such families is motivated by robust control applications. A Lyapunov-theoretic necessary and sufficient characterization is obtained for a natural notion of robust uniform set stability; this characterization allows replacing ad hoc conditions found in the literature by more conceptual stability notions. We then use these techniques to establish a result linking state-space stability to 'input to state' (bounded-input bounded-state) stability. In addition, the preservation of stabilizability under certain types of cascade interconnections is analysed.

KEY WORDS nonlinear stability; robust control; stabilization with respect to sets; bounded-input/bounded-state; Lyapunov function techniques

1. INTRODUCTION

Questions of stability for parametrized families of nonlinear systems have long attracted much research attention; see for instance References 5, 4 and 2, and the references there. In this paper, we show how to characterize uniform stability precisely in terms of Lyapunov functions, and, moreover, how to do so when stability with respect to an invariant subset is of interest. (Our notion of robust uniform set stability for parametrized systems reduces to the one routinely used in the control literature when the set of interest is just an equilibrium.) A necessary and sufficient characterization is obtained, which allows the replacing of ad hoc conditions often found in the literature by more conceptual stability notions. A basic tool is provided by the converse Lyapunov theorem established in Reference 8.

In operator-theoretic studies of systems, the central interest is usually on a notion of 'input to state' stability (ISS) defined in terms of finiteness of gains (operator norms). As shown in Reference 12, and since explored by many other authors, a more natural notion for nonlinear systems is one in which the size of the state must satisfy a nonlinear estimate in terms of the initial state and the size of the control. In that reference, it was shown that, with this notion, a system is stabilizable with respect to an equilibrium if and only if it is input/state stabilizable.

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In Reference 8, this result was generalized to stability and ISS with respect to *sets*, using the above mentioned converse theorem. Here, we show how these results can be extended to deal with families of systems (and stability with respect to sets). The extension is fairly straightforward, but it is complicated by the fact that the Lyapunov function then depends on the parameters, which is not desirable in robust control applications. Thus we have to work harder in order to provide sufficient conditions under which a more uniform function exists.

One application of our techniques is in the study of the preservation of stability under cascades with integrators. This is the critical ingredient in the 'backstepping technique' currently popular; see for instance the textbook, Reference 13, Section 4.8, and references therein, for stabilization, and Reference 7 for many other issues. It is a natural extension of these studies to investigate the same question for parametrized families. Here the uncertainty caused by the presence of unknown parameters makes the problem far harder, and results for systems without parameters cannot be generalized in a straightforward manner to families. When studying stabilizability for parametrized nonlinear systems, much work classically imposed restrictive 'matching conditions' on uncertainties; see, for example, References 1, 4 and 5. In later work, such as for instance References 2, 11, 14 and 3, conditions were dropped and various local results were obtained. In their recent paper, Reference 6, Imura, Sugie, and Yoshikawa were able to give a general theorem on preservation of stability under cascades, for parametric families under no matching conditions. In the last section of our paper, we show how to apply our approach to generalize their result in various ways, especially relaxing their exponential stability conditions.

2. STABILITY OF PARAMETRIZED SYSTEMS (NO CONTROLS)

We start by giving a number of definitions that apply to parametrized families, first for systems with no controls and later with controls. These definitions apply in particular to systems that do not depend on parameters, of course. Some important facts and basic concepts about set stability for nonlinear systems without parameters are needed throughout the paper; we refer the reader to the Appendix and Reference 8 for these. In particular, the concepts of $\mathcal{K}_\infty$, $\mathcal{KL}$ and $\mathcal{K}$-functions, needed in the next definition, are recalled in the Appendix.

Consider a *parametrized family of systems* of the following type:

$$\dot{x} = f(x, \mu) \tag{1}$$

where the state $x(t) \in \mathbb{R}^n$ for all t, the parameter $\mu \in \mathbb{R}^l$, and f is a smooth map from $\mathbb{R}^{n+l}$ to $\mathbb{R}^n$. In particular, for each fixed μ, $f(\cdot, \mu)$ is a smooth vector field on $\mathbb{R}^n$. Here we assume that for each fixed parameter μ, the system (1) is complete and we use $x_\mu(t, x_0)$ to denote the solution at time $t \in \mathbb{R}$ of (1), with initial state $x(0) = x_0$ and parameter value μ. A system *with no parameters* is one in which f is independent of μ.

We say that a closed (not necessarily compact) subset $\mathcal{A}$ of $\mathbb{R}^n$ is an *invariant* set of (1) if it is an invariant set of $\dot{x} = f(x, \mu)$ for each fixed μ. Throughout this work we assume that the following property holds for $\mathcal{A}$:

$$\sup_{x \in \mathbb{R}^n} \{|x|_{\mathcal{A}}\} = \infty$$

where $|x|_{\mathcal{A}}$ denotes the standard point-to-set distance, i.e.,

$$|x|_{\mathcal{A}} \overset{\text{def}}{=} d(x, \mathcal{A}) = \inf_{z \in \mathcal{A}} d(x, z)$$

Note that when $\mathcal{A} = \{0\}$, $|x|_{\mathcal{A}} = |x|$ is the usual Euclidean norm.

Definition 2.1

Let $\mathscr{A}$ be a closed invariant set of (1). We say that (1) is *robustly uniformly globally asymptotically stable (RUGAS) with respect to the set* $\mathscr{A}$ if the following properties hold:

1. *Robust uniform stability*. There exists a $\mathscr{K}_\infty$-function $\delta(\cdot)$ such that for any $\varepsilon > 0$ given,
$$|x_\mu(t, x_0)|_{\mathscr{A}} < \varepsilon \text{ for all } \mu, \text{ all } t \geqslant 0$$
provided $|x_0|_{\mathscr{A}} < \delta(\varepsilon)$.
2. *Robust uniform attraction*. For any r, $\varepsilon > 0$, there exists a $T > 0$, such that
$$|x_\mu(t, x_0)|_{\mathscr{A}} < \varepsilon \text{ for all } \mu$$
whenever $|x_0|_{\mathscr{A}} < r$ and $t \geqslant T$.

For systems with no parameters, we simply say that the system is 'uniformly globally asymptotically stable' (UGAS) with respect to the set $\mathscr{A}$.

When $\mathscr{A}$ consists just of an equilibrium point, UGAS reduces to the usual notion of global asymptotic stability.

One can study the stability of a parametrized family of systems by studying the stability of the following augmented system (with no parameters):

$$\dot{x} = f(x, \mu), \ \dot{\mu} = 0 \tag{2}$$

in which both x and μ are treated as states. To make notations simpler, we use (x, μ) to denote the column vector

$$\begin{pmatrix} x \\ \mu \end{pmatrix},$$

and similarly, use $(f, 0)$ to denote

$$\begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Let $z = (x, \mu)$, $F = (f, 0)$; then the above system can be written as

$$\dot{z} = F(z) \tag{3}$$

Note that system (1) is RUGAS with respect to $\mathscr{A}$ if and only if system (3) is UGAS with respect to $\mathscr{A} \times \mathbb{R}^l$. Consequently, the stability of a parametrized family of systems with respect to an equilibrium x_0 is equivalent to the stability of its augmented system with respect to the invariant set $\{x_0\} \times \mathbb{R}^l$.

The following characterization of the RUGAS property follows from Proposition A.1 in the Appendix:

Proposition 2.2

The system (1) is RUGAS with respect to a closed, invariant set $\mathscr{A} \subseteq \mathbb{R}^n$ if and only if there exists a $\mathscr{KL}$-function β such that, given any initial state x_0, the solution $x_\mu(t, x_0)$ satisfies

$$|x_\mu(t, x_0)|_{\mathscr{A}} \leqslant \beta(|x_0|_{\mathscr{A}}, t)$$

for all μ and all $t \geqslant 0$.

It is well-known that Lyapunov functions provide a powerful tool for studying stabilizability of systems. We define Lyapunov functions for parametrized families of systems in the following way:

Definition 2.3

Let $\mathscr{A}$ be a nonempty, closed, invariant set of (1). A *smooth Lyapunov function for the parametrized family of systems* (1) with respect to $\mathscr{A}$ is a smooth function $V: \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$ satisfying, with the notation $V_\mu(\xi) \stackrel{\text{def}}{=} V(\xi, \mu)$:

1. there exist two $\mathscr{K}_\infty$-functions α_1 and α_2 such that
$$\alpha_1(|\xi|_{\mathscr{A}}) \leqslant V_\mu(\xi) \leqslant \alpha_2(|\xi|_{\mathscr{A}})$$
for any $\mu \in \mathbb{R}^l$ and any $\xi \in \mathbb{R}^n$;
2. there exists a continuous, positive definite function α_3 such that
$$\frac{\partial V_\mu}{\partial x} f(\xi, \mu) \leqslant -\alpha_3(|\xi|_{\mathscr{A}})$$

For $\mathscr{A} = \{0\}$, Lyapunov functions for parametrized families as in the above definition are routinely assumed in robust control studies; see for instance References 5, 4 and 2. Applying Theorem 3 in the Appendix for the augmented system (3) for families of systems (1), one obtains the following result:

Theorem 1

A parametrized family of systems (1) is RUGAS with respect to a nonempty, closed, invariant set $\mathscr{A}$ if and only if it admits a smooth Lyapunov function V with respect to $\mathscr{A}$.

Remark 2.4

Note that we made the blanket assumption that the system is complete. This property is needed in the technical proof of the converse Lyapunov theorem given in Reference 8. However, this assumption turns out to be superfluous in the case in which the closed invariant set $\mathscr{A}$ is compact. This can be shown by the following argument:

First observe that, for any given smooth (in fact, just continuous) map $f(z, \mu)$, there exist two smooth scalar functions $a(x)$ and $b(\mu)$ such that $a(x) \geqslant 1$, $b(\mu) \geqslant 1$, and $|f(x, \mu)| \leqslant a(x) b(\mu)$ for all $x \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^l$. The existence of $a(x)$ and $b(\mu)$ follows from the following argument.

For each integer i, let $C_i = \{x \in \mathbb{R}^n : |x| \leqslant i\}$ and let $D_i = \{\mu \in \mathbb{R}^l : |\mu| \leqslant i\}$. Let
$$c_i = \max\left\{1, \sup_{(x,\mu) \in C_i \times D_i} |f(x, \mu)|\right\}$$
Then it holds that
$$|f(x, \mu)| \leqslant c_i c_j, \ \forall (x, \mu) \in C_i \times D_j \tag{4}$$
Let $\sigma_i(x)$ be a nonnegative smooth function such that $\sigma_i(x) = 1$ if $x \in C_i \backslash C_{i-1}$, and $\sigma_i(x) = 0$ if $x \notin C_{i+1}$ or $x \in C_{i-2}$, where $C_i = \emptyset$ if $i \leqslant 0$. Let
$$\alpha(x) = \sum_{i=0}^{\infty} c_i \sigma_i(x)$$

Then $a(x) \geqslant c_i$ if $x \in C_i \backslash C_{i-1}$. Similarly, there exists a nonnegative smooth function $b(\mu)$ such that $b(\mu) \geqslant c_j$ if $\mu \in D_j \backslash D_{j-1}$. For any $(x, \mu) \in \mathbb{R}^n \times \mathbb{R}^l$, there exist i and j such that $x \in C_i \backslash C_{i-1}$ and $\mu \in D_j \backslash D_{j-1}$. From (4), it follows that

$$|f(x, \mu)| \leqslant c_i c_j \leqslant a(x) b(\mu)$$

Hence, $a(x)$ and $b(\mu)$ are the desired functions.

Now, to system (1), we associate the following system:

$$\dot{x} = \frac{1}{a(x)} f(x, \mu) \tag{5}$$

Because the right-hand side is bounded in norm on x, for each fixed μ (by $b(\mu)$), the system (5) is complete, i.e., $x_\mu(t, x_0)$ is defined for all $t \in \mathbb{R}$ for every fixed parameter μ. Furthermore, it can be shown (see Reference 10), that if (1) is RUGAS then so is (5). It then follows from Theorem 1 that there exists a Lyapunov function V for (5). Noting that $a(x) \geqslant 1$, one can show that V is also a Lyapunov function for the original system (1).

When dealing with families of parametrized control systems, it is usually desirable to obtain a Lyapunov function which is independent of the parameter μ. However, as illustrated in the following example, it is in general impossible to find a Lyapunov function which is independent of μ, even for a RUGAS family with parameters in a *compact* set. The example is motivated by an example in Reference 13 (see p. 170).

Example 2.5

Consider the two-dimensional parametrized family of systems:

$$\dot{x} = A(\mu)x, \qquad x \in \mathbb{R}^2, \qquad \mu \in [0, 2\pi] \tag{6}$$

where for each $\mu \in [0, 2\pi]$,

$$A(\mu) = \begin{pmatrix} -\sin^2 \mu & 1 - \sin \mu \cos \mu \\ -1 - \sin \mu \cos \mu & -\cos^2 \mu \end{pmatrix}$$

The system is RUGAS for $\mu \in [0, 2\pi]$. This can be shown as follows. First of all, for each μ, the eigenvalues of $A(\mu)$ are $(-1 - \sqrt{3}j)/2$ and $(-1 + \sqrt{3}j)/2$, where $j = \sqrt{-1}$, and moreover, one can show that for $\mu \in [0, 2\pi]$,

$$A(\mu) = T(\mu) \begin{pmatrix} \dfrac{-1 - \sqrt{3}j}{2} & 0 \\ 0 & \dfrac{-1 + \sqrt{3}j}{2} \end{pmatrix} T^{-1}(\mu)$$

where

$$T(\mu) = \begin{pmatrix} 1 & 1 \\ \dfrac{e^{(4\pi/3)j} + \sin^2 \mu}{1 - \sin \mu \cos \mu} & \dfrac{e^{(2\pi/3)j} + \sin^2 \mu}{1 - \sin \mu \cos \mu} \end{pmatrix}$$

and

$$T^{-1}(\mu)=\frac{1-\sin\mu\cos\mu}{\sqrt{3}j}\begin{pmatrix}\dfrac{e^{(2\pi/3)j}+\sin^2\mu}{1-\sin\mu\cos\mu} & -1\\ -\dfrac{e^{(4\pi/3)j}+\sin^2\mu}{1-\sin\mu\cos\mu} & 1\end{pmatrix}$$

It follows immediately that

$$e^{A(\mu)t}=T(\mu)\begin{pmatrix}e^{-(1+\sqrt{3}j)t/2} & 0\\ 0 & e^{-(1-\sqrt{3}j)t/2}\end{pmatrix}T^{-1}(\mu),\text{ for all } t$$

It is not hard to see that there exist L_1 and L_2 such that $\| T(\mu)\| \leqslant L_1$ and $\| T^{-1}(\mu)\| \leqslant L_2$ for all μ, therefore,

$$\| e^{A(\mu)t}\| \leqslant \| T(\mu)\| \, \| T^{-1}(\mu)\| \, e^{-t/2} \leqslant Le^{-t/2},\text{ for all } t>0$$

where $L=L_1L_2$. From here one sees clearly that system (6) is RUGAS for $\mu \in [0, 2\pi]$. However, we have the following conclusion:

There is *no* Lyapunov function for system (6) which is independent of μ.

Proof. Assume that there would be a Lyapunov function V for (6) which does not depend on μ. Then there would exist some $\mathcal{K}_\infty$-functions α_i for $i=1$, 2 and a continuous, positive definite function α_3 such that

$$\alpha_1(|x|) \leqslant V(x) \leqslant \alpha_2(|x|),\ \forall x$$

and

$$\frac{\partial V}{\partial x}A(\mu)x \leqslant -\alpha_3(|x|),\text{ for all } x \text{ and for all } \mu \in [0, 2\pi] \tag{7}$$

Now we let u: $\mathbb{R} \to [0, 2\pi]$ be defined by

$$u(t)=t\ (\text{mod } 2\pi) \tag{8}$$

Then V would also be a Lyapunov function for the time-varying system

$$\dot{x}=A(u(t))x(t) \tag{9}$$

because (7) would still be true when μ is replaced by $u(t)$. This would imply that system (9) is uniformly globally asymptotically stable. This is, however, impossible since with $u(t)$ defined by (8),

$$x_1(t)=-\cos t,\ x_2(t)=\sin t$$

is a solution of (9). By contradiction, we have shown that there is no Lyapunov function for (6) which is independent of μ. □

3. INPUT/STATE STABILIZABILITY OF PARAMETRIZED SYSTEMS

In this section, we consider the question of input/state stabilizability for parametrized systems, with an emphasis on compact subsets of parameters.

Consider the following *parametrized family of control systems*:

$$\dot{x} = f(x, u, \mu), \ \mu \in \mathbb{R}^l \tag{10}$$

where f is a smooth map from $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l$ to $\mathbb{R}^n$. We use $x_\mu(t, x_0, u)$ to denote solutions when the parameter μ is used.

Let Ω be a subset of $\mathbb{R}^l$. We say that a system is *smoothly robustly stabilizable with respect to a closed set $\mathscr{A}$ and all $\mu \in \Omega$* if there exist a smooth function $k(x)$ and a $\mathscr{KL}$-function β, both of which are independent of μ, such that every solution of

$$\dot{x} = f(x, k(x), \mu)$$

is defined for all $t \in \mathbb{R}$, and it holds that

$$|x_\mu(t, x_0)|_{\mathscr{A}} \leqslant \beta(|x_0|_{\mathscr{A}}, t), \ \forall t \geqslant 0, \ \forall \mu \in \Omega$$

When $\Omega = \mathbb{R}^l$, this is the same as the concept introduced in Definition 2.1 (cf. Proposition 2.2), for the system obtained by using the feedback $u = k(x)$.

A system is *robustly input/state stable (RISS) with respect to a closed set $\mathscr{A}$ and all $\mu \in \Omega$* if there exist a $\mathscr{KL}$-function β and a $\mathscr{K}$-function γ, such that for all $\mu \in \Omega$, all essentially bounded controls u, and all initial states x_0, the solution $x_\mu(t, x_0, \mu)$ is defined for all $t \geqslant 0$ and the following estimate holds

$$|x_\mu(t, x_0, u)|_{\mathscr{A}} \leqslant \beta(|x_0|_{\mathscr{A}}, t) + \gamma(\|u\|)$$

for all $t \geqslant 0$.

For systems with no parameters, we simply say 'globally input/state stable' (ISS).

Remark 3.1

Because of causality, the ISS property is equivalent to the one that would result if we would only require the estimate to be

$$|x(t, x_0, u)|_{\mathscr{A}} \leqslant \beta(|x_0|_{\mathscr{A}}, t) + \gamma(\|v_t\|) \tag{11}$$

for any *locally* essentially bounded function u, where v_t is the truncation of u at time t:

$$v_t(\tau) \stackrel{\text{def}}{=} \begin{cases} u(\tau), & \text{if } 0 \leqslant \tau \leqslant t \\ 0, & \text{if } \tau > t \end{cases}$$

A system is *smoothly robustly input/state stabilizable with respect to $\mathscr{A}$ and all $\mu \in \Omega$* if there exist a smooth map $k: \mathbb{R}^n \to \mathbb{R}^m$ and an $m \times m$ matrix Γ of smooth functions defined on $\mathbb{R}^n$, invertible everywhere, such that the system

$$\dot{x} = f(x, k(x) + \Gamma(x)v, \mu)$$

is RISS with respect to $\mathscr{A}$ for $\mu \in \Omega$.

Definition 3.2

A smooth function V: $\mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$ is said to be an *ISS-Lyapunov function* for the system (10) and for the set Ω with respect to a closed set $\mathscr{A}$ if there exist some $\mathscr{K}_\infty$-functions α_1, α_2 and χ, and a continuous positive definite function α_3 such that, for all $\mu \in \Omega$, it holds that

$$\alpha_1(|\xi|_{\mathscr{A}}) \leqslant V(\xi, \mu) \leqslant \alpha_2(|\xi|_{\mathscr{A}}), \ \forall \xi \in \mathbb{R}^n \tag{12}$$

and

$$\frac{\partial V(\xi,\mu)}{\partial \xi}\cdot f(\xi,u,\mu) \leqslant -\alpha_3(|\xi|_{\mathscr{A}}) \tag{13}$$

whenever

$$|\xi|_{\mathscr{A}} \geqslant \chi(|u|) \tag{14}$$

The following result provides a sufficient condition for input/state stability in terms of the existence of ISS-Lyapunov functions. We say that control system (10) is *forward complete* if for each $\mu \in \Omega$, each locally essentially bounded control u, and each x_0, the corresponding trajectory $x(t,x_0,u)$ exists for all $t \geqslant 0$.

Proposition 3.3

Assume that system (10) is forward complete and that it admits an ISS-Lyapunov function for $\mu \in \Omega$ with respect to a closed set $\mathscr{A}$. Then, the system (10) is RISS with respect to $\mathscr{A}$ for $\mu \in \Omega$.

Proof. For system (10), consider the following augmented system

$$\dot{x} = f(x,u,\mu), \qquad \dot{\mu} = 0 \tag{15}$$

Let $\hat{\mathscr{A}}$ denote the set $\mathscr{A} \times \Omega$. Clearly $|z|_{\hat{\mathscr{A}}} = |x|_{\mathscr{A}}$, where $z = (x,\mu)'$. Note that V is an ISS-Lyapunov function for (10) with respect to $\mathscr{A}$ if and only if V is an ISS-Lyapunov function for (15) with respect to $\hat{\mathscr{A}}$. Applying Proposition A.2 to (15), one sees that the existence of the ISS-Lyapunov function implies that the solutions of (15) satisfy

$$|z(t)|_{\hat{\mathscr{A}}} \leqslant \beta(|z(0)|_{\hat{\mathscr{A}}}) + \gamma(\|u\|)$$

for some $\mathscr{KL}$-function β and some $\mathscr{K}$ function γ. It then follows that system (10) is ISS. □

Remark 3.4

Note that when the set $\mathscr{A}$ is compact, the existence of an ISS-Lyapunov function guarantees that the system is forward complete. Thus, in the case when $\mathscr{A}$ is compact, one can drop the forward completeness assumption.

Assume now that $u = k(x)$ robustly stabilizes system (10) for all $\mu \in \mathbb{R}^l$ with respect to a closed set $\mathscr{A}$. Then by Theorem 1, one knows that there exists a smooth Lyapunov function $V_\mu(\xi) \stackrel{\text{def}}{=} V(\xi,\mu)$ such that

$$\alpha_1(|\xi|_{\mathscr{A}}) \leqslant V_\mu(\xi) \leqslant \alpha_2(|\xi|_{\mathscr{A}}) \quad \text{and} \quad \frac{\partial V_\mu(\xi)}{\partial \xi} f(\xi,k(\xi),\mu) \leqslant -\alpha_3(|\xi|_{\mathscr{A}})$$

for some $\mathscr{K}_\infty$-functions α_1, α_2 and some continuous positive definite function α_3.

It then can be shown (see References 8 and 9) that there exist some matrix $\Gamma_0(\xi,\mu)$ and a $\mathscr{K}_\infty$-function χ such that

$$\frac{\partial V_\mu}{\partial \xi} f(\xi,k(\xi)+\Gamma_0(\xi,\mu)v,\mu) \leqslant -\frac{\alpha_3(|\xi|_{\mathscr{A}})}{2}$$

for all μ whenever $|\xi|_{\mathscr{A}} \geqslant \chi(|v|)$. Moreover, for this Γ_0, the system

$$\dot{x} = f(x, k(x) + \Gamma_0(x)v, \mu)$$

is forward complete. In fact, it is shown in References 8 and 9 that Γ_0 can be chosen as $\varphi_0 I_{m\times m}$ for some smooth function φ_0 satisfying $0 < \varphi_0(\xi, \mu) \leqslant 1$ for all ξ and μ. Now for any compact subset Ω of $\mathbb{R}^l$, let

$$\varphi_1(\xi) \overset{\text{def}}{=} \min_{\mu\in\Omega} \varphi_0(\xi, \mu)$$

Then $\varphi_1(\xi)$ is locally Lipschitz since $\varphi_0(\cdot,\cdot)$ is a smooth function. Let φ_2 be a smooth function such that

$$\tfrac{1}{4}\varphi_1(\xi) \leqslant \varphi_2(\xi) \leqslant \tfrac{3}{4}\varphi_1(\xi)$$

for all ξ (such a function always exists, see Reference 8). Let

$$\Gamma \overset{\text{def}}{=} \varphi_2 I_{m\times m}$$

Clearly $\Gamma(\xi)$ is invertible everywhere, and

$$\frac{\partial V_\mu(\xi)}{\partial \xi} f(\xi, k(\xi) + \Gamma(\xi)v, \mu) \leqslant -\frac{\alpha_3(|\xi|_{\mathscr{A}})}{2}, \qquad \text{for } \mu\in\Omega$$

whenever $|\xi|_{\mathscr{A}} \geqslant \chi(|v|)$. In other words, V is an ISS-Lyapunov function for the system

$$\dot{x} = f(x, k(x) + \Gamma(x)v, \mu) \tag{16}$$

for $\mu\in\Omega$ with respect to $\mathscr{A}$. Applying Proposition 3.3, we obtain the following conclusion which generalizes the results in Reference 12 to set stability for parametrized systems.

Theorem 2

Assume that the system (10) is smoothly robustly uniformly asymptotically stabilizable with respect to a closed set $\mathscr{A}$ for all values of μ. Then for any compact subset Ω of $\mathbb{R}^l$, the system (10) is smoothly robustly input/state stabilizable with respect to $\mathscr{A}$ for all $\mu\in\Omega$.

Remark 3.5

In practice, it may be too stringent to require that a system be stabilizable for all values of μ, as the given system may only be stabilizable for those values of μ lying in a fixed compact set Ω. Thus an interesting question is: If one knows that a system is stabilizable for $\mu\in\Omega$, does it follow that the system is input/state stabilizable for $\mu\in\Omega$ with the same set Ω? In the above discussion, our approach required us to assume that the system is stable *for all* μ. However, one may often be able to proceed as follows. Assume that there is a smooth map σ: $\mathbb{R}^r \to \mathbb{R}^l$ so that the image of σ is Ω, and so that there is a compact subset $\Lambda \subseteq \mathbb{R}^r$ such that $\sigma(\Lambda) = \Omega$. Now replace $f(x, u, \mu)$ by $f(x, u, \sigma(\nu))$, and apply the above results to the system

$$\dot{x} = f(x, u, \sigma(\nu))$$

for $\nu\in\Lambda$. One obtains input/state stabilizability for all values of ν, or equivalently, the values of μ in Ω. As an illustration, assume that one knows that the system $\dot{x} = f(x, u, \mu)$ is stabilizable for $\mu\in[0,1]$. Let $\sigma(\nu) = \sin^2\nu$. Then the system $\dot{x} = f(x, u, \sin^2\nu)$ is stabilizable for all $\nu\in\mathbb{R}$.

Applying Theorem 2 to $f(x, u, \sin^2 \nu)$ with the set $[0, \pi/2]$ for the parameter ν, one concludes that the system is input/state stabilizable for all values of ν in $[0, \pi/2]$, or equivalently, for all values of μ in $[0, 1]$.

Remark 3.6

From Remarks 2.4 and 3.4, it follows that when $\mathscr{A}$ is compact, the completeness assumption is not needed in Theorem 2.

4. ADDING AN INTEGRATOR

In this section, we study the stabilizability of the following type of parametrized cascade system:

$$\dot{x} = f(x, z, \mu) \tag{17}$$

$$\dot{z} = u \tag{18}$$

where $x(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^m$, and f is a smooth map. Assume that $f(0, 0, \mu) = 0$ for all μ. To study the stabilizability problem for the above system, let us first consider the following type of parametrized cascade system:

$$\dot{x} = f(x, z, \mu) \tag{19}$$

$$\dot{z} = g(x, z, \mu) + u \tag{20}$$

where g: $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^m$ is a smooth mapping with $g(0, 0, \mu) = 0$. Let $g_0(x, \mu) \overset{\text{def}}{=} g(x, 0, \mu)$. Then g can be written as $g_0(x, \mu) + g_1(x, z, \mu)z$ for some $m \times m$ matrix of smooth functions g_1. Rewrite (20) as

$$\dot{z} = g_0(x, \mu) + g_1(x, z, \mu)z + u \tag{21}$$

For the above cascade parametrized system, we have the following conclusion.

Lemma 4.1

Assume that the system (19) is UGAS with respect to the origin when $z = 0$ for all μ, and admits an ISS-Lyapunov function V with α_1, α_2, α_3 and χ as in Definition 3.2. Let Ω be a compact subset of $\mathbb{R}^l$. Assume further that the function χ is smooth, and it holds that

$$\max_{\mu \in \Omega} |g_0(x, \mu)| = O(\sqrt{\alpha_3(|x|)}), \qquad \text{as } |x| \to 0 \tag{22}$$

Then the system (19)–(20) is smoothly robustly stabilizable with respect to the equilibrium $(x, z) = (0, 0)$ for $\mu \in \Omega$.

Proof. Let V be the ISS-Lyapunov function for (19) and α_1, α_2, α_3 and χ be as in Definition 3.2, and define

$$V_1(x, z, \mu) = V(x, \mu) + \frac{|z|^2}{2} \tag{23}$$

Then along the trajectories of (19)–(20),

$$\dot{V}_1 = \frac{\partial V}{\partial x} f(x, z, \mu) + z'u + z'g_0(x, \mu) + z'g_1(x, z, \mu)z \tag{24}$$

Take a smooth function $\varphi_1(x, z)$ satisfying

$$\varphi_1(x, z) \geqslant \max_{\mu \in \Omega} \| g_1(x, z, \mu) \|$$

and let $k_1(x, z) \stackrel{\text{def}}{=} -\varphi_1(x, z)z - z$, $u \stackrel{\text{def}}{=} u_1 + k_1(x, z)$. Then (24) implies that

$$\dot{V}_1 \leqslant \frac{\partial V}{\partial x} f(x, z, \mu) - |z|^2 + z'u_1 + z'g_0(x, \mu) \tag{25}$$

Let φ_2 be a smooth function satisfying

$$\varphi_2(x) \geqslant 1 + \frac{1}{\sqrt{\alpha_3(|x|)}} \max_{\mu \in \Omega} |g_0(x, \mu)| \tag{26}$$

Notice that such a function always exists because of condition (22). Now let

$$k_2(x, z) \stackrel{\text{def}}{=} -\varphi_2^2(x)z, \quad \text{and} \quad u_1 \stackrel{\text{def}}{=} k_2(x, z) + u_2$$

Note that when $|x| \geqslant \chi(|z|)$, $\partial V/\partial x\, f(x, z, \mu) \leqslant -\alpha_3(|x|)$, thus, (25) implies that when $|x| \geqslant \chi(|z|)$,

$$\begin{aligned}
\dot{V}_1 &\leqslant -\alpha_3(|x|) - |z|^2 + z'u_2 - |z|^2\varphi_2^2(x) + z'g_0(x, \mu) \\
&\leqslant -\frac{3\alpha_3(|x|)}{4} - |z|^2 + z'u_2 - \left(\frac{\sqrt{\alpha_3(|x|)}}{2} - \frac{|g_0(x, \mu)||z|}{\sqrt{\alpha_3(|x|)}}\right)^2 \\
&\leqslant -\frac{3\alpha_3(|x|)}{4} - |z|^2 + z'u_2
\end{aligned} \tag{27}$$

Now let us consider the case when $|x| \leqslant \chi(|z|)$. Using the smoothness of the functions χ, $\partial V/\partial x$, $f(x, z, \mu)$ and $g_0(x, \mu)$, one can rewrite the functions as follows:

$$\chi(r) = r\chi_0(r), \ \frac{\partial V}{\partial x}(x, \mu) = x'W(x, \mu)$$

and

$$f(x, z, \mu) = f(x, 0, \mu) + F_1(x, z, \mu)z, \ g_0(x, \mu) = G_0(x, \mu)x$$

where χ_0 is a smooth function, and $W(x, \mu) \in \mathbb{R}^{n \times n}$, $F_1(x, z, \mu) \in \mathbb{R}^{n \times m}$, and $G_0(x, \mu) \in \mathbb{R}^{m \times n}$ are matrices of smooth functions. With these notations, it follows from (25) that along the trajectories of (19)–(20),

$$\begin{aligned}
\dot{V}_1 &\leqslant \frac{\partial V}{\partial x} f(x, z, \mu) - |z|^2 + z'u_2 - |z|^2\varphi_2^2(x) + z'g_0(x, \mu) \\
&\leqslant \frac{\partial V}{\partial x} f(x, 0, \mu) + x'W(x, \mu)F_1(x, z, \mu)z - |z|^2 + z'u_2 + z'G_0(x, \mu)x \\
&\leqslant -\alpha_3(|x|) + |z|^2\chi_0(|z|)\,\|W(x, \mu)F_1(x, z, \mu)\| - |z|^2 + z'u_2 + |z|^2\,\|G_0(x, \mu)\|\,\chi_0(|z|)
\end{aligned} \tag{28}$$

Finally, let φ_3 be a smooth function satisfying

$$\varphi_3(x, z) \geqslant \max_{\mu \in \Omega}(\chi_0(|z|)\,\|W(x, \mu)F_1(x, z, \mu)\| + \chi_0(|z|)\,\|G_0(x, \mu)\|)$$

and let

$$u_2 = k_3(x, z) \stackrel{\text{def}}{=} -\varphi_3(x, z)z \tag{29}$$

Thus, for $u = k(x,z) \stackrel{\text{def}}{=} k_1(x,z) + k_2(x,z) + k_3(x,z)$, it follows from (28) that when $|x| \leqslant \chi(|z|)$, one has

$$\dot{V}_1 \leqslant -\alpha_3(|x|) - |z|^2 \tag{30}$$

Notice that with u_2 defined by (29), the last term in (27) is nonpositive. Combining (27) and (30), one knows that V_1 is a Lyapunov function of the system (19)–(20). We conclude that the system is smoothly robustly stabilizable. □

The proof of the lemma is a bit tedious, because, unlike the situation in the case of single systems, where the terms in the right-hand side of (20) can be cancelled in a straightforward manner by choosing a simple control law, here one needs to make some estimates which allow the design of a control law which is independent of the parameters, and which can overcome the effect of those terms.

Now we return to discuss the stabilizability of system (17)–(18). Let $f_0(x,z) \stackrel{\text{def}}{=} f(x,z,0)$ and let $f_1 \stackrel{\text{def}}{=} f - f_0$. Rewrite system (17)–(18) as

$$\dot{x} = f_0(x,z) + f_1(x,z,\mu) \tag{31}$$

$$\dot{z} = u \tag{32}$$

Assume that there exists a smooth function $k_0(x)$ for which $k_0(0) = 0$ such that $z = k_0(x)$ stabilizes system (31) with respect to the origin for all μ. By Theorem 1, there exists a Lyapunov function V for the closed-loop system

$$\dot{x} = f_0(x, k_0(x)) + f_1(x, k_0(x), \mu) \tag{33}$$

Let Ω be a compact subset of $\mathbb{R}^l$. One can show, as in the proof of Theorem 2, that there exists a smooth function φ satisfying $0 < \varphi(x) \leqslant 1$ such that V is an ISS-Lyapunov function for the system

$$\dot{x} = f_0(x, k_0(x) + \varphi(x)w) + f_1(x, k_0(x) + \varphi(x)w, \mu)$$

that is, there exist $\mathcal{K}_\infty$-functions α_1, α_2, χ, and a positive definite function α_3, such that

$$\alpha_1(|x|) \leqslant V(x,\mu) \leqslant \alpha_2(|x|)$$

and

$$\frac{\partial V}{\partial x}(f_0(x, k_0(x) + \varphi(x)w) + f_1(x, k_0(x) + \varphi(x)w, \mu)) \leqslant -\alpha_3(|x|)$$

whenever

$$|x| \geqslant \chi(|w|) \tag{34}$$

Now take $v \stackrel{\text{def}}{=} z - k_0(x)$ and $w \stackrel{\text{def}}{=} \psi(x)v$ where $\psi(x) \stackrel{\text{def}}{=} 1/\varphi(x)$. Then w satisfies the equation

$$\dot{w} = \frac{\partial \psi}{\partial x} f(x,z,\mu)\varphi(x)w + \psi(x)u - \psi(x)\frac{\partial k_0}{\partial x}(f_0(x,z) + f_1(x,z,\mu))$$

where $f = f_0 + f_1$. Write $f_1(x, k_0(x) + v, \mu)$ as $f_1(x, k_0(x), \mu) + F_1(x, v, \mu)v$ for some matrix F_1 of smooth functions. It then holds that

$$\dot{w} = \psi(x)u + g_0(x,\mu) + g_1(x,w,\mu)w - g_2(x,z)$$

where

$$g_0(x,\mu) \stackrel{\text{def}}{=} -\psi(x)\frac{\partial k_0}{\partial x}f_1(x,k_0(x),\mu)$$

$$g_1(x,w,\mu) \stackrel{\text{def}}{=} \frac{\partial \psi}{\partial x}f(x,z,\mu)\varphi(x) - \frac{\partial k_0}{\partial x}F_1(x,v,\mu)$$

and

$$g_2(x,z) \stackrel{\text{def}}{=} \psi(x)\frac{\partial k_0}{\partial x}f_0(x,z)$$

Let $u \stackrel{\text{def}}{=} \varphi(x)(u_1 - g_2(x,z))$. We have

$$\dot{x} = f_0(x, k_0(x) + \varphi(x)w) + f_1(x, k_0(x) + \varphi(x)w, \mu) \tag{35}$$

$$\dot{w} = u_1 + g_0(x,\mu) + g_1(x,w,\mu)w \tag{36}$$

Applying Lemma 4.1 to the system (35)–(36), we know that if the function χ in (34) can be chosen smooth, then there exists a smooth feedback law $u_1 = k_1(x,w)$ stabilizing the system with respect to the equilibrium 0. Noticing the relations between w, v and z and the fact that $0 < \varphi \leqslant 1$, one sees that the control law

$$u = k(x,z) \stackrel{\text{def}}{=} \varphi(x)(k_1(x,w) - g_2(x,z))$$

stabilizes system (31)–(32). Thus we have shown the following conclusion.

Proposition 4.2

Assume that the system (31) is stabilizable with respect to the origin by a smooth feedback law $z = k_0(x)$ for which $k_0(0) = 0$ for all μ. Assume further that, for a compact subset Ω of $\mathbb{R}^l$, the following holds:

1. the function χ in (34) can be chosen smooth;
2. as $|x|$ tends to 0,

$$\max_{\mu\in\Omega}\frac{\partial k_0(x)}{\partial x}f_1(x,k_0(x),\mu) = O(\sqrt{\alpha_3(|x|)}) \tag{37}$$

Then the cascade system (31)–(32) is smoothly robustly stabilizable with respect to the origin for all $\mu \in \Omega$.

Remark 4.3

A special case of Proposition 4.2 is that when system (31) is smoothly exponentially stabilizable, that is, there exists a smooth feedback law $z = k_0(x)$, such that the system (33) is UGAS, and there exist a smooth function V and *quadratic* functions α_i, $i = 1, 2, 3$ such that

$$\begin{aligned} \alpha_1(|x|) \leqslant V(x,\mu) \leqslant \alpha_2(|x|) \\ \frac{\partial V}{\partial x}f(x,k_0(x),\mu) \leqslant -\alpha_3(|x|) \end{aligned} \tag{38}$$

Assume $\alpha_3(r) \geqslant cr^2$ for some $c > 0$. Still denoting $z - k_0(x)$ by v, and write

$$f(x, k_0(x) + v, \mu) = f(x, k_0(x), \mu) + F_1(x, v, \mu)v$$

and write

$$\frac{\partial V}{\partial x} = x' W(x, \mu)$$

for some matrices F_1 and W of smooth functions. Then one has:

$$\begin{aligned}
\frac{\partial V}{\partial x} f(x, k_0(x) + v, \mu) &= \frac{\partial V}{\partial x} f(x, k_0(x), \mu) + x' W(x, \mu) F_1(x, v, \mu) v \\
&\leqslant -\frac{c|x|^2}{2} - \left(\frac{c|x|^2}{2} - x' W(x, \mu) F_1(x, v, \mu) v\right) \\
&= -\frac{c|x|^2}{2} - \frac{c}{2} |x - W(x, \mu) F_1(x, v, \mu) v|^2 + \frac{c}{2} | W(x, v) F_1(x, v, \mu) v|^2 \\
&\leqslant -\frac{c|x|^2}{2} + \frac{c}{2} \| W(x, \mu) F_1(x, v, \mu) \|^2 |v|^2
\end{aligned}$$

For the compact subset Ω of $\mathbb{R}^l$, let $\varphi(x)$ be a smooth function satisfying

$$\varphi(x) \leqslant \frac{1}{1 + \max\limits_{\mu \in \Omega,\ |v| \leqslant |x|/\sqrt{2}} \| W(x, \mu) F_1(x, v, \mu) \|}$$

Then for such a choice φ, one has

$$\frac{\partial V}{\partial x} f(x, k_0(x) + \varphi(x) w,\ \mu) \leqslant -\frac{c|x|^2}{2} + \frac{c}{2} \| W(x, \mu) F_1(x, \varphi(x) w, \mu) \|^2\ \varphi^2(x) |w|^2 \quad (39)$$

Notice then that it always holds that

$$\| W(x, \mu) F_1(x, \varphi(x) w, \mu) \| \varphi(x) \leqslant 1$$

Thus, whenever $|x| \geqslant \sqrt{2}|w|$,

$$\begin{aligned}
&\frac{\partial V}{\partial x} f(x, k_0(x) + \varphi(x) w, \mu) \\
&\leqslant -\frac{c|x|^2}{4} - \left(\frac{c|x|^2}{4} - \frac{c}{2} \| W(x, \mu) F_1(x, \varphi(x) w, \mu) \|^2 \varphi^2(x) |w|^2\right) \\
&\leqslant -\frac{c|x|^2}{4}
\end{aligned}$$

From here one sees that the function χ can be chosen as $\chi(r) = \sqrt{2}r$, which is smooth everywhere. Thus we showed that condition 1 holds. Condition 2 also holds, because $\sqrt{\alpha_3(|x|)} = O(|x|)$, and also $f_1(x, k_0(x), \mu) = O(|x|)$ since $f_1(0, k_0(0), \mu) = 0$ and μ belongs to a compact set. Thus both conditions 1 and 2 hold when α_3 is a quadratic function. Thus we recover the following result given in Reference 6:

> If system (31) is smoothly exponentially stabilizable for all μ, then for any compact set Ω, the cascade system (31)–(32) is smoothly stabilizable.

The methods used in Reference 6 could be adapted to study the general case, i.e., the case when (31) is merely smoothly, not necessarily exponentially, stabilizable. However, instead of condition 1 in Proposition 4.2, the generalization would require imposing restrictions on $\partial V/\partial x$, *in addition* to the condition (37) that was assumed in Proposition 4.2.

In many cases, condition 1 assumed in Proposition 4.2 can be relaxed. For instance, when the parameter does not appear in the control channel of (31), that is, when f_1 is independent of z, and the Lyapunov function for the closed-loop system (33) is independent of μ, one does not need to check if χ is smooth. The design can then proceed as follows.

Assume that $z = k_0(x)$ smoothly stabilizes the system

$$\dot{x} = f_0(x, z) + f_1(x, \mu) \tag{40}$$

with a Lyapunov function $V(x)$ satisfying

$$\alpha_1(|x|) \leqslant V(x) \leqslant \alpha_2(|x|)$$

and

$$\frac{\partial V}{\partial x}(f_0(x, k_0(x)) + f_1(x, \mu)) \leqslant -\alpha_3(|x|)$$

Again let $v \stackrel{\text{def}}{=} z - k_0(x)$, then

$$\dot{v} = u + g_0(x, \mu) + g_1(x, z) \tag{41}$$

where

$$g_0(x, \mu) \stackrel{\text{def}}{=} -\frac{\partial k_0}{\partial x} f_1(x, \mu), \quad \text{and} \quad g_1(x, z) \stackrel{\text{def}}{=} -\frac{\partial k_0}{\partial x} f_0(x, z)$$

Let $w = (x', v')'$ and use $F(w, u, \mu)$ to denote the map $(f_0' + f_1',\ u' + g_0' + g_1')'$. Let

$$V_1(x, v) \stackrel{\text{def}}{=} V(x) + \tfrac{1}{2}|v|^2$$

Then along the trajectories of (40)–(41),

$$\begin{aligned}\dot{V}_1 &= \frac{\partial V}{\partial x}(f_0(x, z) + f_1(x, \mu)) + v'u + v'g_0(x, \mu) + v'g_1(x, z)\\ &= \frac{\partial V}{\partial x}(f_0(x, k_0(x)) + f_1(x, \mu))\\ &\quad + \frac{\partial V}{\partial x} F_1(x, v)v + v'u + v'g_0(x, \mu) + v'g_1(x, z)\end{aligned}$$

where F_1 is a matrix of smooth functions such that

$$f_1(x, k(x) + v) = f_1(x, k(x)) + F_1(x, v)v$$

Let

$$k_1(x, v) = -g_1(x, z) - \left(\frac{\partial V}{\partial x} F_1(x, v)\right)'$$

and let $u = k_1(x, v) + u_1$. Then one has

$$\dot{V}_1 \leqslant -\alpha_3(|x|) + v'u_1 + v'g_0(x, \mu) \tag{42}$$

Now assume that for a compact set Ω, condition (37) holds. Again let $\varphi_2(x)$ be given as in (26), and let

$$u_1 = k_2(x, v) \stackrel{\text{def}}{=} -\varphi_2^2(x)v - v$$

Then for

$$u = k(x, v) \stackrel{\text{def}}{=} k_1(x, v) + k_2(x, v)$$

it follows from (42) that

$$\dot{V}_1 \leqslant -\frac{3\alpha_3(|x|)}{4} - |v|^2$$

from which it follows that V_1 is a Lyapunov function for the system

$$\begin{aligned} \dot{x} &= f_0(x, z) + f_1(x, \mu) \\ \dot{z} &= k(x, z - k_0(x)) \end{aligned}$$

Thus we established the following conclusion.

Proposition 4.4

Assume that the system (40) is smoothly stabilizable with respect to the origin for all μ, with a Lyapunov function which is independent of μ. Then the cascade system (40) and (18) is stabilizable for μ in a compact set Ω if condition (37) holds.

Remark 4.5

In the proof of Propositions 4.2 and 4.4 we needed to assume that the system was stabilizable for all values of μ. But as illustrated in Remark 3.5 one can often weaken this assumption.

Example 4.6

Consider the system

$$\dot{x} = \mu x^3 + xz, \ \mu \in [-1, 1] \tag{43}$$

$$\dot{z} = u \tag{44}$$

The closed-loop system for (43) under the feedback law $z = k_0(x) \stackrel{\text{def}}{=} -2x^2$ is

$$\dot{x} = -(2-\mu)x^3, \qquad \mu \in [-1, 1] \tag{45}$$

which is a RUGAS system with respect to the equilibrium $x = 0$ for $\mu \in [-1, 1]$, with the Lyapunov function $V(x) = x^2/2$ satisfying

$$\frac{\mathrm{d}V}{\mathrm{d}x}[-(2-\mu)x^3] = -(2-\mu)x^4 \leqslant -x^4, \qquad \forall \mu \in [-1, 1]$$

Notice that $z = -2x^2$ does not stabilize the system (43) for all values of $\mu \in \mathbb{R}$, but as indicated in Remark 3.5, one can consider the stabilization problem for system (43)–(44) for $\mu \in [-1, 1]$ by studying the system

$$\begin{aligned} \dot{x} &= x^3 \sin \nu + xz, \qquad \nu \in \mathbb{R} \\ \dot{z} &= u \end{aligned} \tag{46}$$

Let $f(x, z, \nu) \overset{\text{def}}{=} x^3 \sin \nu + xz$. Clearly $z = -2x^2$ stabilizes the system (46) for all ν, with the Lyapunov function $V(x) = \frac{1}{2}x^2$ satisfying

$$\frac{\mathrm{d}V}{\mathrm{d}x} f(x, k(x), \nu) \leqslant -x^4, \ \forall \nu \in \mathbb{R}$$

Notice that the closed-loop system is not exponentially stable, so one needs to check if condition (37) holds to see if Proposition 4.4 applies. In this example, $\alpha_3(r) = r^4$,

$$f_1(x, z, \nu) = f(x, z, \nu) - f(x, z, 0) = x^3 \sin \nu$$

and

$$\max_{\nu} \left| \frac{\mathrm{d}k}{\mathrm{d}x} f_1(x, k(x), \mu) \right| = 4x^4 = o(\sqrt{\alpha_3(|x|)}), \quad \text{as } |x| \to 0$$

By Proposition 4.4, the system (43)–(44) is stabilizable. In what follows, we will construct the desired feedback law by following the steps in the proof of the Proposition. Letting $v = z - (-2x^2)$, one has

$$\dot{v} = u + 4x^4 \sin \nu + 4x^2 z$$

Let

$$V_1(x, z) \overset{\text{def}}{=} \frac{x^2 + v^2}{2}$$

then along the trajectories of the system,

$$\dot{V}_1 = -(2 - \sin \nu)x^4 + x^2 v + uv + 4x^4 v \sin \nu + 4x^2 zv \tag{47}$$

Take $k_1(x, z) \overset{\text{def}}{=} -x^2 - 4x^2 z$ and let $u = u_1 + k_1(x, z)$. Then (47) becomes:

$$\dot{V}_1 \leqslant -x^4 + u_1 v + 4x^4 v \sin \nu \tag{48}$$

Now let

$$\varphi_2(x) \overset{\text{def}}{=} 1 + \frac{1}{\sqrt{\alpha_3(|x|)}} \max_{\nu} \left| \frac{\partial k_0}{\partial x} f_1(x, k(x), \mu) \right| = 1 + 4x^2$$

and take $u_1 = k_2(x, v) \overset{\text{def}}{=} -v\varphi_2^2(x) - v$. Then it follows from (48) that

$$\begin{aligned} \dot{V}_1 &\leqslant -x^4 - v^2 - 16x^4 v^2 + 4x^4 v \sin \nu \\ &\leqslant -x^4 - v^2 - 4x^4 \left(2v - \frac{\sin \nu}{4}\right)^2 + \frac{x^4}{4} \leqslant -\frac{3x^4}{4} - v^2 \end{aligned}$$

Therefore, the desired feedback law is

$$u = k(x, z) = k_1(x, z) + k_2(x, v) = -x^2 - 4x^2 z - v(1 + 4x^2)^2 - v$$

and the closed-loop system is

$$\begin{aligned} \dot{x} &= \mu x^3 + xz, \quad \mu \in [-1, 1] \\ \dot{z} &= -x^2 - 4x^2 z - (z + 2x^2)(1 + 4x^2)^2 - (z + 2x^2) \end{aligned}$$

which is RUGAS for $\mu \in [-1, 1]$.

APPENDIX

In this section, we recall some notions and results for set stability for nonlinear systems without parameters. For proofs of results presented here, please consult References 8 and 10.

We first recall some standard concepts from stability theory.

A function γ: $\mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a *$\mathcal{K}$-function* if it is continuous, strictly increasing and $\gamma(0) = 0$; it is a *$\mathcal{K}_\infty$-function* if it is a $\mathcal{K}$-function and also $\gamma(s) \to \infty$ as $s \to \infty$. Finally, γ is a *positive definite* function if $\gamma(s) > 0$ for all $s > 0$, and $\gamma(0) = 0$.

A function β: $\mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a *$\mathcal{KL}$-function* if for each fixed $t \geqslant 0$ the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function, and for each fixed $s \geqslant 0$ it is decreasing to zero as $t \to \infty$.

Consider the following system:

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{49}$$

where f is assumed to be smooth (i.e., infinitely differentiable). We will assume that the system is complete, and denote by $x(t, x_0)$ (and sometimes simply by $x(t)$ if there is no ambiguity from the context) the solution at time t of (49) with $x(0) = x_0$.

The following characterization of the UGAS property will be extremely useful.

Proposition A.1

The system (49) is UGAS with respect to a closed, invariant set $\mathcal{A} \subseteq \mathbb{R}^n$ if and only if there exists a $\mathcal{KL}$-function β such that, given any initial state x_0, the solution $x(t, x_0)$ satisfies

$$|x(t, x_0)|_{\mathcal{A}} \leqslant \beta(|x_0|_{\mathcal{A}}, t), \quad \text{any } t \geqslant 0 \tag{50}$$

Lyapunov functions are introduced for parametrized systems (and hence in particular for systems with no parameters) in Definition 2.3.

Theorem 3

The system (49) is UGAS with respect to $\mathcal{A}$ if and only if there exists a smooth Lyapunov function V with respect to the set $\mathcal{A}$.

Consider the following nonlinear system:

$$\dot{x} = f(x, u) \tag{51}$$

with smooth f: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$.

Recall Definition 3.2 of ISS-Lyapunov functions, considered in the special case of systems with no parameters. Then, References 8 and 9 show:

Proposition A.2

If the system (51) is forward complete and admits an ISS-Lyapunov function with respect to $\mathcal{A}$, then it is ISS with respect to $\mathcal{A}$.

Sketch of the proof. Let V be an ISS-Lyapunov function for (51) with respect to $\mathcal{A}$ with the functions $\alpha_i(x)$ ($i = 1, 2, 3$) and χ as in Definition 3.2. Pick any $x_0 \in \mathbb{R}^n$ and any bounded measurable function u. It is not hard to show that the set $S = \{\xi \in \mathbb{R}^n \mid V(\xi) \leqslant \alpha_2(\chi(\|u\|))\}$ is forward invariant. With $\zeta = \alpha_1^{-1} \circ \alpha_2 \circ \chi$, one has, for the trajectory $x(t, x_0, u)$, that $|x(t)|_{\mathcal{A}} \leqslant \zeta(\|u\|)$ if $x(t) \in S$. On the other hand,

$$\mathrm{d}V(x(t))/\mathrm{d}t = \nabla V(x(t)) \cdot f(x(t), u(t)) \leqslant -\alpha(V(x(t)))$$

when $x(t) \notin S$, where $\alpha = \alpha_3 \circ \alpha_2^{-1}$. Finally, let β_α be a $\mathcal{KL}$-function so that $y(t) \leqslant \beta_\alpha(y_0, t)$ for each solution of the differential inequality $\dot{y}(t) \leqslant -\alpha(y(t))$, $y(0) = y_0 \geqslant 0$. Then $\beta(s, t) = \alpha_1^{-1}(\beta_\alpha(\alpha_2(s), t))$ and ζ are as needed for the definition. □

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