

CONTROLLABILITY AND LINEARIZED REGULATION

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ABSTRACT

A nonlinear controllable plant, under mild technical conditions, admits a precompensator with the following property: along control trajectories joining pairs of states, the composite system (precompensator plus plant) is, up to first order, isomorphic to a parallel connection of integrators.

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1. Introduction.

This note deals with "pseudolinearization" properties of nonlinear systems. Recent research by and Baumann and Rugh ([14], [3]), and by Champetier et al. ([6], [15]) has emphasized the idea of studying families of linearizations of nonlinear systems around different operating points, and in particular the problem of obtaining compensators with the property that all closed-loop linearizations have the same dynamic behavior. In a similar spirit, we study here linearizations along *trajectories* of nonlinear systems.

The present line of work should be contrasted to that of "feedback linearization" of nonlinear systems, where one asks for a global (or almost global) reduction of the given system to linear time-invariant form (see e.g. Brockett [4], Jackubczyk and Respondek [10], Hunt et.al. [9]). The conditions for feedback linearizability are very stringent, and even when applicable, as in the "computed torque" method in robotics, the method may result in control laws that are not implementable due to the existence of actuator constraints that were ignored in the linearization process. The pseudolinearization work, on the other hand, is closely tied to the standard approach in engineering practice, where an open-loop trajectory is preplanned and a servo is built in order to regulate along it. The regulated system then corrects for disturbances and measurement errors. The price one pays for this kind of 'hierarchical' design is that only small perturbations from preplanned trajectories can be tolerated by the control system.

Since the main result is somewhat technical to state, we explain it first in intuitive and oversimplified terms. A more precise discussion of what we mean by "plant", "system", and so forth, is given in the next section. Assume that a plant is given, described by a vector differential equation

$$\dot{x}(t) = f(x(t), u(t)), \quad (1.1)$$

where $x(t)$ is the state, and $u(t)$ is the control, at time t . Assume given also a pair (\bar{x}, \bar{u}) consisting of an open loop control $\bar{u}(\cdot)$ and a reference trajectory $\bar{x}(\cdot)$ which satisfy (1.1). Such a pair may arise from a numerical optimal control procedure for command generation, or from a simulation of the plant response to a test signal \bar{u} . We think of \bar{x} as a desired trajectory. Let $(x(\cdot), u(\cdot))$ be any other such pair, corresponding to an actual control being applied, and an actual trajectory of the plant. This pair may differ from the reference pair (\bar{x}, \bar{u}) , for instance due to an initial state x_0 different from the desired \bar{x}_0 . The goal is to drive the error $x(t) - \bar{x}(t)$ to zero. Consider the differences $\delta(t) := x(t) - \bar{x}(t)$ and $v(t) := u(t) - \bar{u}(t)$. Dropping higher order terms, these quantities satisfy an equation

$$\dot{\delta}(t) = A(t)\delta(t) + B(t)v(t), \quad (1.2)$$

where $A(t) = f_x(\bar{x}(t), \bar{u}(t))$ and $B(t) = f_u(\bar{x}(t), \bar{u}(t))$ are Jacobian matrices evaluated along the reference trajectory. To account for small disturbances and errors in $\bar{x}(0)$, the usual procedure is to design a linear time-varying servo

$$v(t) = K(t)\delta(t)$$

such that $\delta(\cdot)$ is forced to decrease. The control applied on-line is then

$$u(t) = \bar{u}(t) + K(t)(x(t) - \bar{x}(t)).$$

One typically computes $K(\cdot)$ via linear quadratic optimization techniques. Note that the data for this optimization problem depends on the linearized system (A, B) , and hence on the desired (\bar{x}, \bar{u}) .

An alternative to using linear quadratic optimization in order to obtain K is to study pole shifting problems for the linear time varying systems (1.2). In the process of developing his canonical form,

Brunovsky [5] showed how to reduce (1.2) under feedback and coordinate transformations, into a *time-invariant* system (see also the fundamental paper Morse and Silverman [13]), *provided* that a rank controllability condition holds and that the Kronecker structure of $(A(\cdot), B(\cdot))$ remains constant in time. In terms of the original nonlinear system (1.1), this means that certain integers related to partial Jacobians of $f(\cdot, \cdot)$ must remain constant along the given trajectory. In any case, once a time invariant system is obtained, one can perform a pole-shifting design in the new coordinates. However, the assumption of constancy of Kronecker indices is fairly restrictive. Moreover, even for such trajectories, there is the question that two different trajectories will give rise to distinct invariants, and the time-invariant systems obtained by feedback will be different. One way to avoid the first problem, that of nonconstancy, is to add a precompensator in such a manner as to force the indices into generic form. This is precisely what is done for fixed linear time-varying systems in Kamen et.al. [11], and in the analogous case of families of systems in Sontag [16]. For pseudolinearization at *equilibrium points*, precompensation is also used in Reboulet and Champetier [15], but the form of the precompensator needed in that case is considerably simpler.

One of the main purposes of this paper is to describe a technique that allows such a K to be computed in closed form *independently* of the given reference (\bar{x}, \bar{u}) and the corresponding linear time varying system (A, B) . Roughly, the feedback K that we obtain will be expressed symbolically in terms of variables (π, ω) ; during actual operation of the closed loop system, at time t one substitutes $\bar{x}(t)$ for π and $\bar{u}(t)$ for ω . More accurately, K will include integral terms, i.e. dynamic feedback, and for technical reasons there needs to be a variable corresponding to the solution of a fundamental matrix differential equation. The form of the precompensator, as well as all changes of coordinates, are computed symbolically. One way of formalizing all this is as follows:

Given: A plant Π and a class U of (open-loop) controls.

To design: A precompensator Ξ_p to be connected as in Figure 1.

Such that: For each fixed $u(\cdot) \in U$, the time-varying system Ξ (Figure 1, large box) that results when this $u(\cdot)$ is applied, as a system with external control $v(\cdot)$ and states (z, x) , is to first order and linear coordinate change equal to a predetermined linear system.

Once the above is achieved, time-invariant linear control theory may be applied to obtain a servo which will provide satisfactory closed-loop operation in the presence of small errors in the state of Π , the external control $v(\cdot)$, and the state of Ξ_p . Note that we are requiring that *all* linearizations be isomorphic to a fixed linear system, independently of the particular $u(\cdot)$. Hence, the term *equilinearization* is probably appropriate for our objective. Moreover, we will even require that the coordinate change be in a sense precomputable independently of the particular $u(\cdot)$, so that real-time calculations are minimized.

The next problem in formalizing our objective is that of deciding upon suitable classes U of controls. In order for our method to work, we must assume some sort of finite dimensional parametrization of the possible controls in U . For instance, one may deal with the set of all controls that are, as functions of time, polynomials of degree at most three. More generally, we shall postulate the existence of an *open-loop control generator* system Ω , whose outputs are fed as controls to the plant. The idea of using autonomous systems as signal generators is of course not new: many approaches to the tracking problem rely on precisely such time-domain models for the signals to be tracked (steps, ramps, etc.). For instance, if we desire to study the response to ramps, i.e. polynomials of degree 1, it is only necessary to consider a control generator with dynamics $\dot{\omega}_1 = 0$, $\dot{\omega}_2 = \omega_1$, and output $u(t) = \omega_2(t)$. Different initial conditions $\omega_1(0)$, $\omega_2(0)$ will give rise to all possible linear $u(t)$.

Our Main Theorem assumes that such a control generator is given, and provides a solution to a

slightly modified version of the problem stated above, which we now describe. Let $Q = Q(\omega)$ be the output map of the control generator, and take Γ to be the autonomous system that has state variables partitioned as $\gamma = (\phi, \omega, \pi)$, where ω satisfies the equations for the control generator, π is the plant state trajectory with corresponding control $Q(\omega)$, and ϕ is a fundamental solution along the corresponding state and input trajectories --see section 2 for details. We view Q as a function of all of γ , and ask for a precompensator Ξ_p and a memory-free map R , to be connected as in figure 2. The system that should be easy to control is again denoted by Ξ , with external control v , and states (x, z) . The result says, roughly, that there exist Ξ_p , R , and matrices A, B , such that, for each $\gamma(\cdot) = (\phi, \omega, \pi)$ there is a linear change of variables

$$\Lambda(t) y = \begin{pmatrix} z \\ x - \bar{x} \end{pmatrix}$$

($\det(\Lambda(t)) \neq 0$ for all t .) with the following property: if $x(0)$ and $z(0)$ are arbitrary initial conditions and v is an external control, then

$$\dot{y} = Ay + Bv + o(y, v, t) .$$

More precisely, this happens for those initial conditions $\gamma(0)$ which correspond to controls which are "nonsingular" with respect to $\xi(0)$, in a sense to be clarified later and, except for special cases like that in which the system dynamics are described by polynomials or rational functions, one must restrict to compact subsets of the state space. The matrices A, B can be chosen such that the resulting system is a simple series/parallel connection of integrators. Further, the matrix $\Lambda(t)$ can also be precomputed independently of $\gamma(0)$, in the sense that there is a matrix $L(\cdot)$ such that $\Lambda(t) = L(\gamma(t))$ for all t .

A companion paper to this one, [19], deals with the issue of designing the open-loop control generators themselves. The main theorem in that reference shows, roughly, that if the plant (1.1) is controllable and if certain relatively weak assumptions are satisfied -- f must be analytic, and the state-space is for instance \mathfrak{R}^n , or a contractible open subset,-- then there exists a finite dimensional system Ω with outputs $u(\cdot)$ such that, for any two states x_1 and x_2 in a compact subset of the plant, there is some initial condition of the control generator system Ω that results in an open loop control which transfers x_1 to x_2 . Moreover, this open loop control is "nonsingular" as required in the previous paragraph. The results in the present paper are totally independent of those in [19]. However, we include below precise statements of the latter, since the existence result for control generators provides a strong mathematical motivation for the use of such generators. In fact, only length constraints precluded the integration of the material in [19] into the present paper.

In summary, combining the Main Theorem with the main result in [19], one has that, provided the plant satisfies certain reasonable assumptions, it is possible to affect any desired state transfer using a suitable open loop signal generator, and to regulate for small deviations from the corresponding trajectories using linear control design techniques. The system Γ that results can be described as an interconnection of a linear system, an internal model of the plant, and a system linear in the state; Q is a polynomial map, and Ξ_p is linear. If the original system is defined by rational equations, all functions appearing are again rational.

The systems considered are described by analytic differential equations. In the case of systems defined by rational functions, the constructions can be performed explicitly via resultant theory. We have carried them out for various examples using the Macsyma symbolic manipulation system. On the other hand, the theorem in [19] about the existence of signal generators is more of an abstract existential result, and should be probably seen as providing more of a "principle" for nonlinear control than as a practical design method in itself. This principle would be used in practice in order to motivate the search for signal generators, much as inverse Lyapunov theorems are used in the context of stability. Note however that

the Main Theorem can be typically applied directly, with the precompensator design based on the *assumption* that all controls to be applied are in a certain class (e.g., steps, ramps); these open-loop controls may be calculated for specific state transfers via numerical methods. The example worked out later illustrates this.

The plan for the paper is as follows. The next section introduces definitions and the statement of the Main Theorem. After that, we deal with various issues related to the controllability of variational systems, in particular a study of those trajectories of a given system along which the linearization (as a time-varying linear system) is controllable. Such *nonsingular trajectories* play a central role in the construction of precompensators. We then construct the precompensator needed for the Main Theorem. In the final two sections we present a worked example and some conclusions and suggestions for further work.

2. Definitions and Statement of the Main Theorem.

2.1. Systems.

An (*analytic*) system Ξ , with state space S_{Ξ} = open subset of \mathfrak{R}^n and control-value space U_{Ξ} = open subset of \mathfrak{R}^m , is described by an equation

$$\dot{x}(t) = f(x(t), u(t)) , \quad (2.1)$$

where the *dynamics* map $f: S_{\Xi} \times U_{\Xi} \rightarrow \mathfrak{R}^n$ is real-analytic. A *polynomial* system is one for which each component of f is a polynomial, and a *rational* system is one for which each component of f is a rational function. A rational function is one that can be written as a quotient of two polynomials in $n+m$ variables, *with the denominator having no zeroes in $S_{\Xi} \times U_{\Xi}$* . The system is *autonomous* if f is independent of u ; in that case the set U_{Ξ} is irrelevant. Autonomous systems will appear in the modeling of signal generators. In describing systems by their evolution equations (2.1), we shall often omit the argument t .

An *admissible control* (or *input*) u of length $T = T_u$ is a measurable essentially bounded map $u: [0, T] \rightarrow U_{\Xi}$. Essentially bounded means that there is a compact subset $K = K_u$ of U_{Ξ} such that $u(t) \in K$ for almost all t . For any admissible u and initial condition $x(0) = \xi$, the unique absolutely continuous solution $x(\cdot)$ of (2.1) at time $t \leq T$, if defined, is denoted by $x(t) = \psi(t, \xi, u)$. A pair (x, u) of functions on an interval $[0, T]$, with u an admissible control and x satisfying (2.1), i.e., $x(t) = \psi(t, x(0), u)$ for all $t \in [0, T]$, is an *admissible trajectory* on $[0, T]$. If u has length T and ξ is such that there exists an admissible trajectory (x, u) on $[0, T]$ with $x(0) = \xi$, we say that u *can be applied to ξ* . If there is an admissible trajectory on $[0, T]$ with initial $x(0) = \xi_1$ and final $x(T) = \xi_2$, we say that ξ_1 *can be controlled to ξ_2 in time T* , or that ξ_2 *can be reached from ξ_1* , and that u *steers ξ_1 to ξ_2* . If there is some $T > 0$ such that that ξ_1 can be controlled to ξ_2 in time T , we just say ξ_1 can be controlled to ξ_2 .

The system Ξ is *complete* if for every $\xi \in S_{\Xi}$, every $T > 0$, and every admissible control u , the solution $\psi(t, \xi, u)$ is well-defined for all $t \leq T$, i.e. every control can be applied to every state. Completeness will not be essential, and will not be assumed in most of the material to follow, but it will allow an elegant statement of part of the theorem from [19] quoted below.

If (x, u) is an admissible trajectory on $[0, T]$, then $z(t) := \psi(t, z(0), v)$ is defined for all $t \in [0, T]$ if $z(0)$ is sufficiently close to $x(0)$ and if v is sufficiently close to u in the essential supremum norm. More precisely, let L_{∞}^m be the Banach space (with sup norm) consisting of all essentially bounded measurable $u: [0, T] \rightarrow \mathfrak{R}^m$, for the given $T > 0$. Then the set of triples (t, ξ, u) for which the solution $\psi(\tau, \xi, u)$ is defined for all $\tau \leq t$ is an open subset of $\mathfrak{R} \times S_{\Xi} \times L_{\infty}^m$. Further, for each fixed t , $\psi(t, \xi, u)$ is infinitely differentiable as a function of $(\xi, u) \in S_{\Xi} \times L_{\infty}^m$, and the mapping $(\xi, u) \rightarrow \psi(\cdot, \xi, u)$, with values in $C^0([0, T], \mathfrak{R}^n)$ (with sup norm) is continuous. These are well-known classical results; see for instance Lee and Marcus [12] and references there, as well as [19], where we also establish continuity and differentiability with respect to other norms on controls.

2.2. Variational Systems.

Let (x, u) on $[0, T]$ provide an admissible trajectory for the system Ξ . The (total) *variational system* associated to Ξ along (x, u) is the linearization of Ξ along this trajectory, that is, the linear time varying system $D_{x,u}\Xi$ defined by the evolution equations

$$\begin{aligned} \dot{\lambda}(t) &= f_x(x(t), u(t))\lambda(t) + f_u(x(t), u(t))v(t) , \\ t &\in [0, T] , \end{aligned} \quad (2.2)$$

where f_x, f_u denote Jacobians of f with respect to the first n variables and the last m variables respectively, and where $\lambda(t) \in \mathfrak{R}^n$ and $v(t) \in \mathfrak{R}^m$ for all t . (Strictly speaking, time-varying systems are not "systems" with our definition.)

We shall say that Ξ is *linearly controllable* along (x,u) if (2.2) is (completely) controllable in $[0,T]$, i.e. for each λ_1 and λ_2 in \mathfrak{R}^n there is a measurable essentially bounded v such that, solving (2.2) with this v and with initial condition $\lambda(0)=\lambda_1$ results in $\lambda(T)=\lambda_2$.

An interpretation of this property can be given via the map $(\xi,u) \rightarrow \psi(T,\xi,u)$ defined on an open subset of $S_{\Xi} \times L_{\infty}^m$. The Frechet differential of this map at a given (ξ,u) , evaluated at a tangent element $(\lambda_0, v) \in \mathfrak{R}^n \times L_{\infty}^m$ is precisely the solution $\lambda(T)$ at time T of (2.2) with $\lambda(0)=\lambda_0$. In particular, consider a fixed pair (T,ξ) , and the partial map $\alpha(u) := \psi(T,\xi,u)$. Then its differential at u , evaluated at v , is obtained by solving (2.2) with this v and with $\lambda(0)=0$. Thus α is nonsingular at u precisely if (2.2) is reachable from zero, i.e. if for each λ_2 there is a v such that $\lambda(0)=0$ results in $\lambda(T)=\lambda_2$. Since reachability from zero is equivalent to complete controllability for time-varying (continuous-time) systems, we conclude that Ξ is linearly controllable along (x,u) if and only if α has full rank at u .

We shall need also a more general notion of variational system. This notion will apply to systems which have inputs of two kinds: a "reference signal" $\gamma(t)$ and a "control" $v(t)$. For such systems, we shall be interested in linearizations with respect to perturbations in the states and control v , along trajectories corresponding to each possible reference signal γ . Formally, consider a system for which the control value set U_{Ξ} has the form $U_1 \times U_2$, where U_1 and U_2 are open subsets of \mathfrak{R}^k and \mathfrak{R}^l respectively. Partition admissible controls in the form (γ, v) to reflect the structure of U_{Ξ} . (We shall often write $(x, \bar{\gamma}, v)$ instead of $(x, (\gamma, v))$ for admissible trajectories.) Write the system equations as

$$\dot{x} = f(x, \gamma, v) . \quad (2.3)$$

The *partial variational system with respect to x and v along the admissible trajectory $(\bar{x}, \bar{\gamma}, \bar{v})$* is the time varying linear system

$$\begin{aligned} \dot{\lambda}(t) = & \\ & f_x(\bar{x}(t), \bar{\gamma}(t), \bar{v}(t))\lambda(t) + \\ & f_v(\bar{x}(t), \bar{\gamma}(t), \bar{v}(t))v(t) , \\ & t \in [0, T] \end{aligned} \quad (2.4)$$

(note that $v(t) \in \mathfrak{R}^l$ for each t). Fix such an admissible trajectory $(\bar{x}, \bar{\gamma}, \bar{v})$ on the interval $[0, T]$. Expanding the differentiable map f , we conclude the existence of a function $\phi(t, a, b)$, which is defined for $0 \leq t \leq T$, for a in a neighborhood of $0 \in \mathfrak{R}^n$, and for b in a neighborhood of $0 \in \mathfrak{R}^l$, such that $\phi(t, a, b) = o(a, b)$ for each t , ϕ is analytic in (a, b) for each t , and the following property holds: For each admissible trajectory $(x, \bar{\gamma}, v)$ on $[0, T]$ with (x, v) sufficiently close to (\bar{x}, \bar{v}) , if $\delta(t) := x(t) - \bar{x}(t)$ and $v(t) := v(t) - \bar{v}(t)$ then

$$\begin{aligned} \delta(t) = & \\ & f_x(\bar{x}(t), \bar{\gamma}(t), \bar{v}(t))\delta(t) + \\ & f_v(\bar{x}(t), \bar{\gamma}(t), \bar{v}(t))v(t) + \\ & \phi(t, \delta(t), v(t)), \quad t \in [0, T]. \end{aligned} \quad (2.5)$$

Further, if $\bar{\gamma}$ is continuous as a function of t - in our application, $\bar{\gamma}$ will be even analytic in t , - then ϕ is *uniformly* $o(a, b)$, i.e. $\phi(t, a, b) / \|(a, b)\| \rightarrow 0$ as $(a, b) \rightarrow 0$ uniformly on $t \in [0, T]$. Thus the time-varying system (2.4) provides a good approximation to the perturbed system (2.5), and a feedback design based on the former can be expected to perform well for the latter, as long as δ and v are kept small.

2.3. Pseudointegrators.

Let the system Ξ be as in (2.3), and assume that the dimension of the state space S_{Ξ} is a multiple of l , say lb . Fix an admissible trajectory $(\bar{x}, \bar{\gamma}, \bar{v})$ of Ξ . We shall say that Ξ is a *pseudointegrator along* $(\bar{x}, \bar{\gamma}, \bar{v})$ iff there exists a differentiable matrix function $\Lambda(t)$ defined on $[0, T]$, with $\det(\Lambda(t)) \neq 0$ for all t , such that, with the notations $A(t) := f_{\bar{x}}(\bar{x}(t), \bar{\gamma}(t), \bar{v}(t))$ and $B(t) := f_{\bar{v}}(\bar{x}(t), \bar{\gamma}(t), \bar{v}(t))$, the following equations hold:

$$\Lambda(t)A = A(t)\Lambda(t) - \dot{\Lambda}(t), \quad \Lambda(t)B = B(t),$$

where A and B are the following (*constant*) matrices:

$$\mathbf{A} = \begin{array}{cccccccc} 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & \\ \mathbf{I} & 0 & 0 & \cdot & \cdot & 0 & 0 & \\ 0 & \mathbf{I} & 0 & \cdot & \cdot & 0 & 0 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \cdot & \cdot & \mathbf{I} & 0 & \end{array} \quad \mathbf{B} = \begin{array}{c} \mathbf{I} \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array}$$

(each block is of size l by l , and there are b block rows). (2.6)

This property can be interpreted as follows. Assume that, as above, $\delta(t)$ and $v(t)$ denote small perturbations of \bar{x} , \bar{v} respectively. Define $\varepsilon(t)$ by the coordinate change $\delta(t) := \Lambda(t)\varepsilon(t)$; then ε satisfies an equation like

$$\dot{\varepsilon}(t) = A\varepsilon(t) + Bv(t) + \eta(t, \varepsilon(t), v(t)), \quad (2.7)$$

where $\eta(t, a, b)$ is again $o(a, b)$ for each t (uniformly if $\bar{\gamma}$ is continuous). In the ε coordinates, the partial variational system is up to first order a *constant* linear system.

Note that a pseudointegrator must be an "index invariant" system in the sense of Morse and Silverman [13]. (Index invariant systems are more general, since they cannot be brought to constant form under coordinate changes alone, but feedback must also be used.) A particular consequence of our results will be that one can build a precompensator that makes all linearizations along suitably nondegenerate trajectories index invariant, and with the *same* indexes appearing along all such trajectories.

Systems (2.7) have very nice local controllability properties, and any feedback control designed in terms of $\varepsilon(t)$ gives a corresponding feedback law for the original system via the similarity $\Lambda(t)$ (which gives rise to a Lyapunov transformation, since its determinant is bounded away from 0 on $[0, T]$).

We introduce one last definition. The system Ξ is a pseudointegrator *uniformly* with respect to a class C of signals γ iff there exists an lb by lb matrix \mathbf{L} of analytic functions defined on an open subset of \mathfrak{R}^k such that, for all γ in C , the definition of pseudointegrator is satisfied with $\Lambda(t) = \mathbf{L}(\gamma(t))$. Thus the coordinate change can be "precomputed" independently of γ . We shall sometimes abuse notation and use the notation " Λ " also for \mathbf{L} , where no confusion arises.

2.4. Reference Signals and Plant.

From now on we fix an arbitrary system as in (2.1). We shall denote this system by Π , and refer to it as the *plant*. The notations n , m , S_{Π} , U_{Π} , f will be reserved for the data associated to this particular system. Also fixed will be an autonomous system Ω , with state space $S_{\Omega} \subseteq \mathfrak{R}^r$ and dynamics denoted by

P , as well as an ("output") analytic map $Q: S_\Omega \rightarrow U_\Pi$. The notation $\Omega \downarrow \Pi$ will stand for the system obtained by feeding the output of Ω as a control to Π , i.e. the autonomous system

$$\begin{aligned} \dot{\omega} &= P(\omega) \\ \dot{\pi} &= f(\pi, Q(\omega)) \end{aligned} \quad (2.8)$$

on the state space $S_\Omega \times S_\Pi$. Our main objective is to study the regulation of the plant Π along trajectories that result from the application of controls produced by the "open-loop control generator" Ω . (The result from Sontag [19] quoted below establishes that a natural property of abstract controllability of Π is enough to guarantee the existence of suitable "open-loop control generators" Ω ; that fact provides a motivation for the present model.) Among such trajectories we shall be interested in particular in those along which the plant is linearly controllable. Since trajectories are parametrized by the initial states of Ω and Π , we express the desired controllability property in terms of $S_\Omega \times S_\Pi$, as follows.

Consider a pair $(w_0, p_0) \in S_\Omega \times S_\Pi$. At least for small enough times $T > 0$, the solution $(\omega(t), \pi(t))$ of (2.8) with $\omega(0) = w_0$ and $\pi(0) = p_0$ is defined on the interval $[0, T]$. We shall say that (w_0, p_0) is *nondegenerate* iff for some such T , Π is linearly controllable along the ensuing admissible trajectory $(\pi, Q(\omega))$ of Π .^{*} The set of such pairs will be denoted by $ND(\Omega \downarrow \Pi)$. The *trajectory* (ω, π) will be called nondegenerate if (w_0, p_0) is. (It will be remarked later that $(\omega(t), \pi(t))$ is again in $ND(\Omega \downarrow \Pi)$, for each $t \leq T$.)

It is easy to see that $ND(\Omega \downarrow \Pi)$ is an open set; in fact, we prove later the following stronger result, which implies in particular that -if nonempty- $ND(\Omega \downarrow \Pi)$ is an open dense subset:

Lemma 2.1: There is an analytic function $\Delta: S_\Omega \times S_\Pi \rightarrow \mathfrak{R}$ with the following property: (w_0, p_0) is nondegenerate if and only if $\Delta(w_0, p_0) \neq 0$. When $\Omega \downarrow \Pi$ is a polynomial (resp., rational) system, Δ can be chosen to be a polynomial (resp., rational) function.

For polynomial or rational systems, the function Δ could in principle be obtained algorithmically using symbolic manipulation systems, but in practice only low dimensional systems can be dealt with, because of computational complexity considerations.

To the given combination (2.8) of control generator and plant, we associate one last system, the *reference generator* Γ . This is an autonomous system, whose state space is the set $S_\Gamma := GL(n) \times S_\Omega \times S_\Pi$, where $GL(n)$ is the set of $n \times n$ invertible matrices viewed as an open subset of \mathfrak{R}^{n^2} . It is described by the set of equations

$$\begin{aligned} \dot{\Phi} &= a(\pi, \omega) \Phi \\ \dot{\omega} &= P(\omega) \\ \dot{\pi} &= f(\pi, Q(\omega)) \end{aligned} \quad (2.9)$$

where $a(p, w)$ is the Jacobian matrix $f_x(p, Q(w))$ for each $p \in S_\Pi$ and $w \in S_\Omega$. The state $(\Phi(t), \omega(t), \pi(t))$ of Γ at time t will be denoted by $\gamma(t)$.

Note that, if (π, ω) is any given admissible trajectory for (2.8) on $[0, T]$, and if $\Phi(0)$ is an arbitrary element of $GL(n)$, then there is also a well defined state trajectory γ for Γ on $[0, T]$, having the given second and third coordinates (ω, π) and with the given $\Phi(0)$. In other words, the first equation in (2.9) can be solved on all of $[0, T]$; this is because for fixed (ω, π) it is linear in Φ . In fact, Φ is a fundamental solution for the variational system of Π along $(\pi, Q(\omega))$. We shall say that $\gamma(0)$, and the corresponding trajectory γ

^{*}Here $Q(\omega)$ denotes the function defined by $Q(\omega)(t) := Q(\omega(t))$.

on $[0, T]$, are nondegenerate if $(\omega(0), \pi(0))$ is. A *nondegenerate reference signal* will mean, from now on, a nondegenerate trajectory γ of Γ . We may see $Q(\omega)$ as a function of γ which depends only of the ω -coordinate of γ .

Note that the system Γ consists of the equation for ω (which in the proof of the result in [19] happens to be linear), which feeds into the last equation (an internal model of the plant) and both of these feed into the first equation, which is itself a system linear in the state with input (π, ω) .

2.5. Statement of the Main Theorem.

Consider the following situation. Let $\Omega \downarrow \Pi$ and Γ be as above, and assume given in addition an integer k , an m by nk matrix R of analytic functions defined on an open subset W of S_Γ , and constant matrices $C \in \mathfrak{R}^{nk \times nk}$ and $D \in \mathfrak{R}^{nk \times n}$. These data give rise to a "regulated system" Ξ defined as follows. The system Ξ is of the type described in equation (2.3), having two kinds of controls, $\gamma(\cdot)$ and $v(\cdot)$, where $U_1 = W$ and $U_2 = \mathfrak{R}^n$. Its state $(z(t), x(t)) \in \mathfrak{R}^{nk} \times S_\Pi$ satisfies the simultaneous equations:

$$\dot{z} = Cz + Dv, \quad (2.10)$$

$$\dot{x} = f(x, u), \quad (2.11)$$

$$u = Q(\omega) + R(\gamma)z. \quad (2.12)$$

Here $\omega(t)$ is the S_Ω -coordinate of $\gamma(t) \in W \subseteq GL(n) \times S_\Omega \times S_\Pi$.

The system Ξ consists of a cascade of two systems: a linear time-invariant *precompensator* Ξ_p (equation (2.10)) with input $v(t)$, and the original plant Π . The control $u(t)$ to Π which is applied at any given time t is an analytic function of the reference signal $\gamma(t)$ and of the state of the precompensator (in fact, affine in the latter). Alternatively, we may think of the control law (2.12) as the output of Ξ_p . Figure 2 illustrates the resulting system diagram; the larger box corresponds to the regulated system Ξ . The input v is an external control, and γ , generated by Γ , acts as a reference signal.

The objective is to build precompensators in such a way that, for any nondegenerate reference signal γ , the system Ξ behaves as a (fixed) constant linear system for small z , $x(0) =$ small perturbation of $\pi(0)$, and small external controls v . More precisely, let (ω, π) be an admissible trajectory of (2.8) on $[0, T]$, and pick any Φ_0 . There results an admissible trajectory $\gamma = (\Phi, \omega, \pi)$ of the reference generator system (2.9) on the interval $[0, T]$. We consider this signal γ as a reference input $\bar{\gamma}$ for (2.10-2.12). Together with the choice $\bar{v}(t) \equiv 0$, $\bar{z}(t) \equiv 0$, $\bar{x} = \pi$, we have an admissible trajectory for (2.10-2.12) on the time interval $[0, T]$. We may then consider the variational system of Ξ along this admissible trajectory $((0, \pi), (\gamma, 0))$. The main result of this paper is as follows.

Main Theorem. Let $\Omega \downarrow \Pi$ be as above, and assume that O is any given compact subset of $ND(\Omega \downarrow \Pi)$. Denote $G := GL(n) \times O$. Then there exist:

- an integer k ,
- an m by nk matrix R of analytic functions defined on an open subset W of S_Γ which contains G , and
- constant matrices $C \in \mathfrak{R}^{nk \times nk}$ and $D \in \mathfrak{R}^{nk \times n}$,

with the following property: uniformly on reference signals γ which satisfy that $\gamma(t) \in G$ for all t , the system (2.10-2.12) is a pseudointegrator along $((0, \pi), (\gamma, 0))$. Furthermore, if $\Omega \downarrow \Pi$ is a rational system then the conclusions are true even if O is all of $ND(\Omega \downarrow \Pi)$ and the entries of R can be chosen to be rational functions.

In applying the definition of pseudointegrator to the above system, note that the integers 'b' and 'l' in section 2.3 become, respectively, k+1 and n.

Although mathematically independent of the material in this paper, it is worth quoting precisely the result in [19], which lends justification to the use of signals generators. For technical reasons, we need there the following property of linear growth in controls for the plant Π :

$$(*) \quad U_{\Pi} = \mathfrak{R}^m \text{ and there is a continuous function } \beta: S_{\Pi} \rightarrow \mathfrak{R} \text{ such that} \\ \|f_u(\xi, \mu)\| \leq \beta(\xi) \text{ for all } \xi \in S_{\Pi} \text{ and all } \mu \in U_{\Pi}.$$

(By $\|f_u\|$ we denote the norm of the Jacobian of f with respect to u , for any fixed operator norm.) As discussed in [19], this assumption can be relaxed in various ways. The system Π is *controllable* iff for each π_1 and π_2 in S_{Π} , π_1 can be controlled to π_2 . An *equilibrium point* for Π is a pair (π, μ) , $\pi \in S_{\Pi}$, $\mu \in U_{\Pi}$, such that $f(\pi, \mu) = 0$. Then we have:

Theorem[19]. Assume that the controllable plant Π satisfies (*) and *either* that Π has some equilibrium point *or* that Π is complete and S_{Π} is simply connected. Let \mathbf{C} be any compact subset of S_{Π} . There exists then an autonomous polynomial system Ω , a polynomial map Q , and a compact subset O of $ND(\Omega \downarrow \Pi)$ such that the following property holds:

$$\text{for each } \pi_1 \text{ and } \pi_2 \text{ in } \mathbf{C} \text{ there are a } T > 0 \text{ and an admissible trajectory } (\omega, \pi) \text{ of the system (2.8)} \\ \text{with } \pi(0) = \pi_1 \text{ and } \pi(T) = \pi_2 \text{ and such that } (\omega(t), \pi(t)) \in O \text{ for all } t \in [0, T].n$$

From the two theorems together it follows then that for a controllable plant, -provided the weak extra requirements are satisfied,- for each compact subset \mathbf{C} of S_{Π} there are a signal generator and a precompensator such that the following holds: for any given pair of states in \mathbf{C} , there is an initial condition of the signal generator which gives rise to an open-loop input that controls one of these states to the other, and along the ensuing trajectory the composite system (precompensator + plant) is up to first order and coordinate change a simple parallel/series connection of integrators.

Important notational convention. Most vectors appearing in this paper are column vectors. In order to save space when displaying them, we shall often use the alternative notation

$$(a_1 : \dots : a_r) \tag{2.13}$$

(note the ":" instead of

$$\begin{array}{c} | a_1 | \\ | \cdot | \\ | \cdot | \\ | a_r | \end{array} .$$

If the a_i are scalars, this is the same as the transpose of the row vector (a_1, \dots, a_r) , but we will mostly deal with cases in which the a_i are themselves column vectors, in which case (2.13) would correspond to, in more usual but cumbersome notation, (a_1', \dots, a_r') (primes indicate transpose).

3. Linearized Controllability.

In this section we include some general facts about linearized controllability, which are needed for the proofs of the main results.

3.1. Controllability of time-varying linear systems.

First we shall develop some elementary linear system theory. A good general reference for the latter is Chen [7], where proofs of the results in this subsection can be found. We state the needed facts for further reference, in a language more suitable for our applications. Given the time-varying linear system

$$\dot{x}(t) = C(t)x(t) + D(t)u(t) , \quad (3.1)$$

with $t \in [0, T]$, (C is an l by l matrix, and D is l by m), this is (completely) controllable on $[0, T]$ iff there exists no $p_0 \neq 0$ such that

$$\langle p_0, \Phi(0, t)D(t) \rangle = 0 \quad \text{for almost all } t \in [0, T] ,$$

where the fundamental matrix Φ satisfies

$$\begin{aligned} \partial \Phi(t, \tau) / \partial t &= C(t)\Phi(t, \tau) \quad \text{for all } t, \tau, \\ \Phi(0, 0) &= I . \end{aligned}$$

Equivalently, with $p(t) := \Phi'(0, t)p_0$, controllability is equivalent to the impossibility of simultaneous vanishing almost everywhere of all the switching functions

$$\phi_\mu(t) := \langle p(t), D_\mu(t) \rangle , \quad \mu = 1, \dots, m,$$

($D_\mu := \mu$ -th column of D), for a nonzero solution of the adjoint equation

$$\dot{p}(t) = -C'(t)p(t) , \quad p(0) = p_0 .$$

When all the entries of C and D are smooth functions of time on an open interval containing $[0, T]$, a sufficient condition for controllability is that

$$\dim \text{span} \{d_{\kappa\mu}, \kappa \geq 0, \mu = 1, \dots, m\} = l , \quad (3.2)$$

where

$$d_{\kappa\mu} := d^\kappa / dt^\kappa |_{t=0} [\Phi(0, t)D_\mu(t)] . \quad (3.3)$$

When (3.1) is analytic, meaning that all entries of C and D are analytic functions of time, condition (3.2) is necessary as well as sufficient.

3.2. Vector Fields.

Let W be an open subset of an Euclidean space \mathfrak{R}^s . Using the natural global coordinate chart for W given by the embedding in \mathfrak{R}^s , we may (and will) identify functions $f: W \rightarrow \mathfrak{R}^s$ with vector fields on W . These will be represented as s -dimensional column vectors of functions on W . An *analytic* vector field is one that is real analytic when seen as a function $W \rightarrow \mathfrak{R}^s$. A *polynomial*, (resp., *rational*), vector field is one which corresponds to a polynomial (resp., rational) map f . Let f, g be vector fields on W . Their Lie bracket $[f, g]$ is then the vector field on W defined by the formula

$$[f, g](x) := g_x(x)f(x) - f_x(x)g(x) , \quad (3.4)$$

where $(\cdot)_x$ indicates Jacobian. If f and g are polynomial or rational then $[f, g]$ is also polynomial or rational respectively. For each vector field F , we use the notation ad_F for the following induced linear operator on vector fields:

$$\text{ad}_F(g) := [F, g] .$$

Consider now the following situation. Given are analytic vector fields

$$A, b_1, \dots, b_m$$

defined on an open subset L of \mathfrak{R}^l . We let e^{sA} denote the flow of A , i.e., $e^{sA}\xi_0$ is the solution $\xi(s)$ at time s (if it exists) of

$$\dot{\xi}(t) = A(\xi(t)), \quad \xi(0) = \xi_0 \quad (3.5)$$

(and is undefined if the solution does not exist on $[0, s]$). We use the notation $(\cdot)_*$ for differentials. Then, the Baker-Campbell-Hausdorff formula gives that, for each $\mu=1, \dots, m$ and $\kappa \geq 0$,

$$d^\kappa/dt^\kappa|_{t=0} [(e^{-tA})_* b_\mu(e^{tA}(\xi))] = \text{ad}_A^\kappa(b_\mu)(\xi) .$$

We consider (3.5) as an autonomous system on \mathfrak{R}^l . Given an admissible $\xi(t)$ on $[0, T]$, the variational system along $\xi(\cdot)$ (there are no explicit controls here) has equations

$$\dot{\lambda}(t) = C(t)\lambda(t) ,$$

where we are denoting

$$C(t) := A_x(\xi(t)) .$$

Let Φ be the fundamental matrix associated to this C . It is easy to see that in this situation,

$$\Phi(t, \tau) = (e^{(t-\tau)A})_* .$$

Still for the given vector fields, and the given admissible trajectory ξ on $[0, T]$ as in (3.5), consider now the time-varying linear system (3.1) with the above C ($n \times n$) and where D is the matrix whose μ -th column, $\mu=1, \dots, m$, is

$$D_\mu(t) := b_\mu(\xi(t)) = b_\mu(e^{tA}(\xi_0)) ,$$

where ξ_0 is $\xi(0)$. Formula (3.3) becomes then

$$d_{\kappa\mu} = \text{ad}_A^\kappa(b_\mu)(\xi_0) . \quad (3.6)$$

Since A and the b_μ 's are analytic, the obtained system (3.1) also is, and hence the latter is controllable if and only if these vectors $d_{\kappa\mu}$ span a space of dimension n . This will be used in the next subsection.

3.3. Nondegenerate pairs.

Fix now a system $\Omega \downarrow \Pi$ as in (2.8) consisting of the plant Π and open-loop control generator Ω . Nondegenerate pairs (ω, π) allow the parametrization of state trajectories and controls along which the variational system is controllable. The above results provide an easy characterization of these pairs, as follows. Let l be $n+r$, take the open set L to be $S_\Omega \times S_\Pi$, and introduce

$$A(\omega, \pi) := \begin{pmatrix} P(\omega) \\ f(\pi, Q(\omega)) \end{pmatrix} , \quad (3.7)$$

$$b_\mu(\omega, \pi) := \begin{pmatrix} 0 \\ g_\mu(\omega, \pi) \end{pmatrix} , \quad (3.8)$$

where for $\mu = 1, \dots, m$, g_μ is the μ -th column of the Jacobian of f with respect to $u \in \mathbb{R}^m$ evaluated at $(\pi, Q(\omega))$:

$$f_u(\pi, Q(\omega)) .$$

Assume now given an admissible trajectory (ω, π) on $[0, T]$. The matrix functions $C(t)$, $D(t)$, $\Phi(t, \tau)$ are defined from this data as in the previous section. We may also consider the variational system of the plant Π along the admissible trajectory $(\pi, Q(\omega))$ (where $Q(\omega)$ appears now as an external control). This is a time-varying system

$$\dot{\delta}(t) = C^2(t)\delta(t) + D^2(t)v(t),$$

where

$$\begin{aligned} C^2(t) &:= f_x(\pi(t), Q(\omega(t))), \\ D^2(t) &:= f_u(\pi(t), Q(\omega(t))), \end{aligned}$$

(C^2 is an n by n matrix, and D^2 is n by m .) Let $\Phi^2(t, \tau)$ be the fundamental matrix for C^2 . Then, Π is linearly controllable along $(\pi, Q(\omega))$ if and only if the vectors

$$d_{\kappa\mu}^2 := d^{\kappa}/dt^{\kappa}|_{t=0} [\Phi^2(0, t)D_{\mu}^2(t)], \quad \kappa \geq 0, \mu = 1, \dots, m,$$

span a space of dimension n . (D_{μ}^2 denotes the μ -th column of D^2 .) Note that $C(t)$ and $D(t)$ have the following partitioned structure:

$$C(t) = \begin{pmatrix} * & 0 \\ * & C^2(t) \end{pmatrix}$$

(the $(1, 1)$ -block is of size r by r), and

$$D(t) = \begin{pmatrix} 0 \\ D^2(t) \end{pmatrix}$$

(the 0 block is size r by m). From the form of C it then follows that there is also a partitioned structure

$$\Phi(t, \tau) = \begin{pmatrix} * & 0 \\ * & \Phi^2(t, \tau) \end{pmatrix}.$$

Thus, it follows that for each t in $[0, T]$,

$$\Phi(0, t)D_{\mu}(t) = \begin{pmatrix} 0 \\ \Phi^2(0, t)D_{\mu}^2(t) \end{pmatrix},$$

and so taking derivatives that

$$d_{\kappa\mu}^2 = \begin{pmatrix} 0 \\ d_{\kappa\mu}^2 \end{pmatrix}$$

for all $\kappa \geq 0$ and $\mu = 1, \dots, m$. We have proved the following result:

Proposition 3.1: $(\omega_0, \pi_0) \in ND(\Omega \downarrow \Pi)$ iff $\dim \text{span} \{ad_A^{\kappa} b_{\mu}(\omega_0, \pi_0), \kappa \geq 0, \mu = 1, \dots, m\} = n$.

We may restate this as follows. For each integer k , let

$$\Delta_k(\omega, \pi) := \sum \rho_j(\omega, \pi)^2,$$

(sum over $j \in J$) where $\{\rho_j, j \in J\}$ are all the possible $n \times n$ minors of the k -th reachability matrix

$$R_k := \{b_1, \dots, b_m, ad_A b_1, \dots, ad_A^{k-1} b_m\}.$$

Note that Δ_k is analytic [resp., polynomial, rational] if P , Q , and f all are, and it is in the k -th reachability ideal I_k generated over analytic functions by the $n \times n$ minors of R_k . From the block forms in the definitions of A and the b_{μ} 's, it follows that R_k has the structure

$$\begin{pmatrix} 0 \\ S_k \end{pmatrix}, \tag{3.9}$$

where the 0 block has size $r \times km$. Thus the minors ρ_j are the maximal possible nonzero minors, and the span in proposition 3.1 has dimension exactly n iff one of these is nonzero, for some k . Thus:

Corollary 3.2: $(\omega_o, \pi_o) \in ND(\Omega \downarrow \Pi)$ iff $\Delta_k(\omega_o, \pi_o) \neq 0$ for some k, n

Let O be any compact subset of $ND(\Omega \downarrow \Pi)$. For each $(\omega_o, \pi_o) \in O$, there is then an integer k_o such that $\Delta_{k_o}(\omega_o, \pi_o) \neq 0$. Since Δ_k is continuous, there is a neighborhood V_o of (ω_o, π_o) in $ND(\Omega \downarrow \Pi)$ such that $\Delta_{k_o}(\omega, \pi) \neq 0$ (and hence also $\Delta_{\kappa}(\omega, \pi) \neq 0$ for all $\kappa \geq k_o$) for all (ω, π) in V_o . Taking a finite subcover of O by such neighborhoods V_o , and choosing the largest k_o , we conclude:

Corollary 3.3: For each compact subset O of $ND(\Omega \downarrow \Pi)$ there is an integer k such that $\Delta_k(\omega, \pi) \neq 0$ for all (ω, π) in O, n

We now wish to show that a suitable infinite combination of the squares of the functions Δ_k will be analytic, so as to establish lemma 2.1. This construction must be done over C , since in general a uniformly convergent series of real-analytic functions will not be necessarily again analytic.

By definition of real-analyticity, there exists for the above vector field A an open subset V of C^l , $l = r+n$, such that

$$V \cap \mathfrak{R}^l = S_{\Omega} \times S_{\Pi},$$

and an extension of A to a vector function analytic on V , which we shall denote with the same letter A . Similarly with b_1, \dots, b_m . Intersecting if necessary, we assume that a common V as above has been chosen. The elements in the Lie algebra generated by A and the b_{μ} 's all admit analytic continuations to the same open subset V , since they are defined by differentiations and algebraic operations on the entries of the original vector fields. (More generally, all vectors whose entries are in the differential algebra generated by the entries of A and the b_{μ} 's admit analytic continuations to the same V .) It follows finally that there is an extension of each Δ_k to an analytic function on V , which we denote by Γ_k . (Note that, contrary to Δ_k , the vanishing of Γ_k does *not* imply the vanishing of Γ_{κ} for $\kappa < k$.)

Consider the set V as a (separable) Hausdorff manifold. By an elementary topological reasoning, (see for instance Brickell and Clark [2], lemma 3.4.3,) there exists a countable covering $\{V_{\kappa}, \kappa \geq 0\}$ of V by compact subsets of V such that, for each κ , V_{κ} is contained in the interior $\text{int}V_{\kappa+1}$ of $V_{\kappa+1}$. We let

$$O_{\kappa} := V_{\kappa} \cap \mathfrak{R}^l$$

for each $\kappa \geq 0$. Since V_{κ} is compact and \mathfrak{R}^l is closed in C^l , this is a compact subset of $V \cap \mathfrak{R}^l = S_{\Omega} \times S_{\Pi}$. For each k , the continuous function $|\Gamma_k|$ is thus bounded above on V_{κ} by some positive constant C_{κ} . Replacing Γ_k by the quotient Γ_k/C_{κ} , we may assume that $|\Gamma_k| \leq 1$ on V_{κ} , for each k . (And Γ_k is still analytic on V .) Note that, since the V_{κ} form an ascending chain, for all k it holds that

$$|\Gamma_{\kappa}(v)| \leq 1 \text{ for all } v \in V_{\kappa} \text{ and all } \kappa \geq k.$$

Consider finally the function defined by the series

$$\Gamma(v) := \sum 2^{-\kappa} \Gamma_{\kappa}(v)$$

(sum from $\kappa=0$ to ∞). Given any $v \in V$, there is a k such that $v \in V_{\kappa}$. Thus, the terms in the above series are majorized in norm by $2^{-\kappa}$ for all $\kappa \geq k$. It follows that the series is absolutely convergent and that Γ is well-defined. To prove that Γ is analytic, it is sufficient to remark that the series converges uniformly on compacts (see for instance Dieudonné [8], chapter 9, section 12). So let K be any compact subset of V .

Since the open sets $\text{int}V_\kappa$ also cover V , there is a finite subcover of K by such sets; since these form a chain, there is in fact an integer k such that

$$K \subseteq V_k .$$

For any $v \in K$, then,

$$\begin{aligned} & |\Gamma(v) - \sum_{\kappa=0}^{k-1} 2^{-\kappa} \Gamma_\kappa(v)| \leq \\ & \sum_{\kappa=k}^{\infty} 2^{-\kappa} |\Gamma_\kappa(v)| \leq 2^{-k} . \end{aligned}$$

(first sum from $\kappa=0$ to $k-1$ and second from $\kappa=k$ to ∞). Uniform convergence follows. Finally, let Δ be the restriction of Γ to $S_\Omega \times S_\Pi$. Thus Δ is real-analytic, and it can be expressed as

$$\Delta = \sum_{\kappa=0}^{\infty} a_\kappa \Delta_\kappa ,$$

(sum from $\kappa \geq 0$) for positive a_κ 's. So $\Delta(\omega, \pi)$ is nonzero if and only if some $\Delta_\kappa(\omega, \pi)$ is, and the first part of lemma 2.1 is established.

Assume now that $\Omega \downarrow \Pi$ is rational. Then, the functions Δ_κ are all elements of the ring $\text{Rat}(S_\Omega \times S_\Pi)$ consisting of all rational functions on \mathfrak{R}^{r+n} which have no poles on $S_\Omega \times S_\Pi$. This ring is a fraction ring of the polynomial ring on $r+n$ variables, and hence is Noetherian since the polynomial ring is. (See e.g. [B], II.2.4, corollary 2, for Noetherian fraction rings.) Thus there exists an integer k such that, for every κ , Δ_κ is in the ideal generated by

$$\{\Delta_1, \dots, \Delta_k\} . \tag{3.10}$$

If (ω_0, π_0) is in $ND(\Omega \downarrow \Pi)$, then by corollary 3.2 there is some κ such that $\Delta_\kappa(\omega_0, \pi_0)$ is nonzero, and hence, by choice of k , one of the generators in (3.10) is nonzero there. This means that $\Delta_\kappa(\omega_0, \pi_0) \neq 0$, given the construction of the Δ_κ 's. If $\Omega \downarrow \Pi$ is a polynomial, -rather than rational,- system the argument is the same. In the first case Δ_κ is rational, in the second polynomial. Choosing $\Delta := \Delta_\kappa$ satisfies the last statement in lemma 2.1. Furthermore, this argument shows also that corollary 3.3 remains true even if $O = ND(\Omega \downarrow \Pi)$, provided that $ND(\Omega \downarrow \Pi)$ be rational.

Finally, we remark that if (ω_0, π_0) is in $ND(\Omega \downarrow \Pi)$, and if (ω, π) is an admissible trajectory on $[0, T]$ with $(\omega(0), \pi(0)) = (\omega_0, \pi_0)$, then for each $\tau \in [0, T]$ it holds that again $(\omega(\tau), \pi(\tau)) \in ND(\Omega \downarrow \Pi)$. This is because otherwise, for the corresponding variational system the vectors

$$\{\Phi(\tau, t)D(t), t \in [\tau, T]\} ,$$

and hence also the vectors

$$\{\Phi(0, t)D(t), t \in [\tau, T]\} ,$$

would span a space of dimension less than $r+n$. By analyticity, this would imply that also the vectors

$$\{\Phi(0, t)D(t), t \in [0, T]\}$$

span a lower dimensional space, contradicting the fact that (ω_0, π_0) is in $ND(\Omega \downarrow \Pi)$.

4. Proof of the Main Theorem.

The proof of the main theorem involves a number of intermediate constructions. We shall assume first that k has been chosen, and will then determine a value such that the theorem holds. Similarly, we shall assume given a set of analytic functions on (an open subset of S_T which contains) G ,

$$\{\rho_{\kappa\mu\nu}, \mu=1, \dots, m, \nu=1, \dots, n, \kappa=1, \dots, k\},$$

also to be determined later. With this data, R is the block matrix

$$R := (R_1, \dots, R_k),$$

and each R_κ is the m by n matrix with the following elements in position (μ, ν) , $\mu=1, \dots, m$, $\nu=1, \dots, n$:

$$(R_\kappa)_{\mu\nu} := \rho_{\kappa\mu\nu}.$$

The desired matrices C and D have the same block forms as A and B respectively have in equation (2.6), except that each block now has size n by n , and there are k block rows. We partition the coordinates of the precompensator as $z = (Z_1 : \dots : Z_k)$, with each $Z_\kappa = (z_{1\kappa}, \dots, z_{n\kappa})$.

Note that, with the above choice of R , the μ -th coordinate of u in (2.12) satisfies

$$u_\mu = Q_\mu(\omega) + \sum_{\kappa=1}^k \sum_{\nu=1}^n \rho_{\kappa\mu\nu}(\gamma) z_{\nu\kappa}.$$

(first sum from $\kappa=1$ to k and second sum $\nu=1$ to n). Thus,

$$\begin{aligned} \partial f(x, Q(\omega) + R(\gamma)z) / \partial z_{\nu\kappa} = \\ \sum_{\mu=1}^m \rho_{\kappa\mu\nu}(\gamma) f_{u_\mu}(x, Q(\omega) + R(\gamma)z). \end{aligned} \quad (4.1)$$

(sum $\mu=1$ to m) (Equality of n -vectors.)

Assume now that, as in the statement of the Main Theorem, $\gamma = (\Phi, \omega, \pi)$ is a nondegenerate reference signal on $[0, T]$.

Let Ξ be the system given by (2.10-2.12). We wish to apply the definition of pseudointegrator, as given in section 2.3, to this system, except that we denote the dynamics map "f" in (2.3) instead by \hat{f} , so as to avoid confusion with the "f" map that describes the plant Π . Note that

$$\hat{f}((z, x), \gamma, v) = \begin{pmatrix} Cz + Dv \\ f(x, Q(\omega) + R(\gamma)z) \end{pmatrix}.$$

We now consider the definition of pseudointegrator along (with the notations in section 2.3) the admissible trajectory $(\bar{x}, \bar{\gamma}, \bar{v})$, where \bar{x} denotes the pair $\begin{pmatrix} 0 \\ \pi \end{pmatrix}$, $\bar{\gamma}$ is the given reference signal, and \bar{v} is identically zero.

The Jacobian matrix A becomes in this case:

$$A(t) = \begin{pmatrix} C & 0 \\ J(t) & a(t) \end{pmatrix},$$

where C is, recall, an $n\kappa \times n\kappa$ constant matrix, the '0' block is $n\kappa \times n$, $a(t)$ is $n \times n$, and $J(t)$ is $n \times n\kappa$. Explicitly, these are as follows:

$$a(t) = f_x(\pi(t), Q(\omega(t)) + R(\gamma(t))0) = f_x(\pi(t), Q(\omega(t))),$$

consistently with the previous definition of a , in (2.9),

$$J(t) = [S_1(t), \dots, S_k(t)],$$

and for all $\kappa=1, \dots, k$

$$S_{\kappa}(t) = \sum_{\mu=1}^m [\text{from}=\text{""}, \text{to}=\text{""}] f_{u_{\mu}}(\pi(t), Q(\omega(t))) \cdot [\rho_{\kappa\mu 1}(\gamma(t)), \dots, \rho_{\kappa\mu n}(\gamma(t))] \quad (4.2)$$

(sum $\mu=1$ to m) Each term in this sum is the product of an $n \times 1$ by a $1 \times n$ matrix, the latter being the μ -th row of R_{κ} . For simplicity, we shall introduce the following notation:

$$g_{\mu}(w, p) := f_{u_{\mu}}(p, w), \quad p \in S_{\Pi}, \quad w \in S_{\Omega}.$$

(To be thought of as a vector field on $S_{\Omega} \times S_{\Pi}$.) Also, note that

$$B(t) = \begin{pmatrix} D \\ 0 \end{pmatrix},$$

where the 0 block is size $n \times n$. We search for an $n(k+1) \times n(k+1)$ matrix function $\Lambda(t)$ defined on $[0, T]$, nonsingular for all t , such that

$$\Lambda(t)A = A(t)\Lambda(t) - \dot{\Lambda}(t), \quad \Lambda(t)B = B(t), \quad (4.3)$$

where A, B are as in equation (2.6) (and b, l there are $h+1, n$ here.) This is the matrix that will give the change of coordinates that makes the system time-invariant. We shall construct the matrix Λ in the following special form:

$$\Lambda = \begin{array}{c|c} & \begin{array}{c} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array} \\ \hline I & \end{array},$$

$$\Lambda_{\kappa} \cdot \dots \cdot \Lambda_1 \quad | \quad \Lambda_0$$

where each Λ_{κ} , $\kappa = 0, \dots, k$, is an $n \times n$ matrix of analytic functions of t , with Λ_0 invertible for each t , and with $\Lambda_{\kappa} \equiv 0$. (The "0" blocks are each n by n , and the identity I is $nk \times nk$.) Since $\det \Lambda = \det \Lambda_0$, Λ will be indeed invertible for all t . Furthermore, since $\Lambda_{\kappa} \equiv 0$, the second equation $\Lambda(t)B = B(t)$ in (4.3) is satisfied for all t . Now we see what conditions are needed on the entries of R in order that the first equation in (4.3) be satisfied for suitable Λ . Calculating,

$$\Lambda(t)A = \begin{pmatrix} C & 0 \\ J_3 & 0 \end{pmatrix},$$

$$A(t)\Lambda(t) - \dot{\Lambda}(t) = \begin{pmatrix} C & 0 \\ J_1 & J_2 \end{pmatrix}$$

(using the same partitioned structure as Λ above), where, omitting t arguments for simplicity:

$$J_2 = a\Lambda_0 - \dot{\Lambda}_0,$$

$$J_3 = [\Lambda_{k-1}, \Lambda_{k-2}, \dots, \Lambda_0],$$

and

$$J_1 = J + a[\Lambda_k, \dots, \Lambda_1] - [\dot{\Lambda}_k, \dots, \dot{\Lambda}_1].$$

Thus, the following equations must hold for the sought Λ_{κ} :

$$\dot{\Lambda}_0 = a\Lambda_0$$

$$\dot{\Lambda}_{\kappa} = a\Lambda_{\kappa} + S_{\kappa-\kappa+1} - \Lambda_{\kappa-1},$$

$$\kappa = 1, \dots, k,$$

$$\Lambda_k \equiv 0.$$

(4.5)

The first equation is equivalent to saying that Λ_0 must be a solution of the fundamental equation for the plant. So we shall choose Λ_0 to be Φ , the first coordinate of the given reference signal γ . We also let Λ_k be identically 0, and by induction on descending $\kappa = k-1, \dots, 1$, let

$$\Lambda_\kappa := a\Lambda_{\kappa+1} - \dot{\Lambda}_{\kappa+1} + S_{k-\kappa}, \quad (4.6)$$

or equivalently, with $(Df)(t) := \dot{f}(t)$ (componentwise differentiation for vector functions),

$$\Lambda_\kappa = (a-D)^{k-\kappa-1}S_1 + (a-D)^{k-\kappa-2}S_2 + \dots + S_{k-\kappa}. \quad (4.7)$$

Thus the ρ 's must be chosen so that precisely equation (4.5) holds when $\kappa=1$, the equations for the other κ 's being satisfied by construction. That is, we need that

$$(a-D)\Lambda_1 + S_k = \Phi,$$

or equivalently, substituting Λ_1 from (4.7) into this equation:

$$\Phi = \sum_{\kappa=1}^k (a-D)^{k-\kappa} S_\kappa.$$

(sum $\kappa=1$ to k). Recalling the definition of the S_κ , equation (4.2), and letting Φ_v be the v -th column of Φ , for each $v=1, \dots, n$, we need that the following equation hold for each such v :

$$\Phi_v = \sum_{\kappa=1}^k \sum_{\mu=1}^m (a-D)^{k-\kappa} [\rho_{\kappa\mu v}(\gamma) g_\mu(\pi)]. \quad (4.8)$$

(sum $\kappa=1$ to k). These equations are decoupled for the different v , so we fix any $v=1, \dots, n$ and show how to obtain the $\rho_{\kappa\mu v}$ for this v . Let

$$\phi: S_\Gamma \rightarrow \mathfrak{R}^n \quad (4.9)$$

be the projection

$$\phi(\Phi, \omega, \pi) := v\text{-th column of } \Phi. \quad (4.10)$$

Still for the particular v chosen, denote

$$\sigma_{\kappa\mu} := (-1)^\kappa \rho_{k-\kappa, \mu, v}, \quad \kappa = 0, \dots, k-1.$$

It will be sufficient to establish the following property:

(P) There are analytic functions $\{\sigma_{\kappa\mu}, \kappa=1, \dots, k-1, \mu=1, \dots, m\}$ defined on G such that, for each reference signal $\gamma(t)$ for which $\gamma(t) \in G$ for all t ,

$$\phi(\gamma(t)) = \sum_{\kappa=0}^k \sum_{\mu=1}^m (D-a(t))^\kappa [\sigma_{\kappa\mu}(\gamma(t)) g_\mu(\pi(t))].$$

(sum $\kappa=0$ to $k-1$ and then $\mu=1$ to m). (And, if the system is rational, the functions $\sigma_{\kappa\mu}$ also are.) The fact that ϕ is the particular function in (4.10) is not needed in order to establish property (*). We shall see that this holds for any analytic ϕ (rational, in the rational case).

Introduce the vector field F on S_Γ that defines the dynamics of Γ , i.e.

$$F(\gamma) = F(\Phi, \omega, \pi) := (a(\pi, \omega)\Phi : P(\omega) : f(\pi, \omega)). \quad (4.11)$$

(Recall the convened notation for block column vectors, and that that we represent the element $a(\pi, \omega)\Phi$ of $GL(n)$ as a vector in \mathfrak{R}^{n^2} . Thus $F(\gamma)$ is a vector of size n^2+r+n .) Let H be any vector field on S_Γ of the block form (with respect to the partition (Φ, ω, π) of coordinates in S_Γ):

$$H(\gamma) = H(\Phi, \omega, \pi) = (* : 0 : h) . \quad (4.12)$$

Then, the Lie bracket $[F, H]$ again has the form in the right hand side of (4.12), and if $\bar{\gamma}$ is any reference signal, the following equality holds (use definition of Lie bracket and chain rule):

$$\text{ad}_F(H)(\bar{\gamma}(t)) = [F, H](\bar{\gamma}(t)) = (* : 0 : (D-a(t))h(t)) ,$$

where again $D = d/dt$ and $a(t) = f_x(\bar{\pi}(t), \bar{\omega}(t))$. In particular, assume that, for each $\kappa = 0, \dots, k-1$ and each $\mu = 1, \dots, m$, we denote

$$G_\mu := (0 : 0 : g_\mu) \quad (4.13)$$

(blocks of sizes n^2, r , and n respectively), and

$$H_{\kappa\mu}(\gamma) := \sigma_{\kappa\mu}(\gamma)G_\mu(\pi, \omega)$$

with the σ 's still to be determined. Thus we have the following situation. For any choice of $\sigma_{\kappa\mu}$'s,

$$\sum_{\kappa=0}^{k-1} \text{ad}_F^\kappa(H_{\kappa\mu}) = (* : 0 : \theta) , \quad (4.14)$$

(sum $\kappa=0$ to $k-1$), for some θ , seen as an equation for vector fields on S_T . And, property (P) is satisfied iff there exist $\sigma_{\kappa\mu}$'s for which θ equals the given ϕ . Let now A be as in (3.7) and each b_μ as in (3.8). We then have the partitioned structure

$$\text{ad}_F^k(G_\mu) = \begin{pmatrix} * \\ \text{ad}_F^k b_\mu \end{pmatrix} , \quad (4.15)$$

where the $**$ block has size n^2 .

Choose an integer k as in corollary 3.3, for the given compact O in the statement of the Main Theorem (or globally, in the rational case). With the notations in subsection 3.3, the desired integer " k " in the above constructions will be taken to be equal to this k . Let $a_j := \rho_j/\Delta_k$ for each $j \in J$. These are all analytic functions defined on an open set containing O (rational if $\Omega \downarrow \Pi$ is a rational system), and they satisfy

$$\sum_{j \in J} a_j \rho_j = 1 .$$

(sum over $j \in J$). Consider the extended reachability matrix

$$R^* := (G_1, \dots, \text{ad}_F^{k-1} G_m) .$$

From equations (4.15) and (3.9) we conclude that

$$R^* = (* : 0 : S) ,$$

where S is an $n \times km$ matrix.

We may think of the ρ_j , $j \in J$, as all the $n \times n$ minors of S , since all other minors of R are necessarily zero. If ρ_j is the determinant of the matrix S_j of S obtained when picking columns in the ordered set $j = \{i_1, \dots, i_n\}$, where $i_1 < \dots < i_n$, we let E_j be the km by n matrix obtained as follows. Let D_j be the cofactor matrix of S_j ; then the i_1 -th row of E_j is by definition the 1 -th row of D_j , and all other rows i , i not in the set j , are set to zero. Then,

$$SE_j = \rho_j I ,$$

where I is an $n \times n$ identity matrix. Finally, introduce the matrix $E := \sum_{j \in J} a_j E_j$, again a $km \times n$ matrix. We conclude that

$$R^* E = (* : 0 : I) .$$

(Sizes of blocks are $n^2 \times n$, $r \times n$, and $n \times n$ respectively). Now, given any function ϕ as in (4.9), we let B be the vector $E\phi$. It follows that R^*B has the form in the right-hand side of (4.14), with $\theta =$ the desired ϕ . Thus we have an equation

$$\sum_{\kappa=0}^{k-1} \sum_{\mu=1}^m \beta_{\kappa\mu} \text{ad}_F^\kappa(G_\mu) = (* : 0 : \phi), \quad (4.16)$$

where the $\beta_{\kappa\mu}$'s are the entries of the vector B . Note that the $\beta_{\kappa\mu}$'s are analytic in O , and are rational if $\Omega \downarrow \Pi$ is rational.

We are left then with the problem of finding the functions $\sigma_{\kappa\mu}$, assuming that the $\beta_{\kappa\mu}$ as above have been constructed. But this follows from the following lemma:

Lemma 4.1: Assume that F, G_1, \dots, G_m, H are vector fields on an open set in \mathfrak{R}^n , and that there are an integer k and functions $\{\beta_{\kappa\mu}, \kappa=0, \dots, k-1, \mu=1, \dots, m\}$ such that

$$H = \sum_{\kappa=0}^{k-1} \sum_{\mu=1}^m \beta_{\kappa\mu} \text{ad}_F^\kappa(G_\mu). \quad (4.17)$$

Then, there exist functions $\{\sigma_{\kappa\mu}, \kappa=0, \dots, k-1, \mu=1, \dots, m\}$ such that

$$H = \sum_{\kappa=0}^{k-1} \sum_{\mu=1}^m \text{ad}_F^\kappa(\sigma_{\kappa\mu} G_\mu).$$

Further, the functions $\{\sigma_{\kappa\mu}\}$ can be chosen in the differential algebra generated by the $\beta_{\kappa\mu}$ and the entries of F .

Proof: We prove this by induction on k . For $k=1$, the result is trivial. So assume it is proved for k . Take now an expression

$$H = \sum_{\kappa=0}^k \sum_{\mu=1}^m \beta_{\kappa\mu} \text{ad}_F^\kappa(G_\mu). \quad (4.18)$$

Repeated use of the derivation formula

$$[F, \alpha G] = L_F(\alpha)G + \alpha[F, G]$$

(where $L_F(\alpha)$ is the Lie derivative $\text{grad}(\alpha) \cdot F$) results for each $\mu=1, \dots, m$ in

$$\text{ad}_F^k(\beta_{\kappa\mu} G_\mu) = \sum_{\kappa=0}^{k-1} \alpha_{\kappa\mu} \text{ad}_F^\kappa(G_\mu) + \beta_{\kappa\mu} \text{ad}_F^k(G_\mu)$$

for suitable functions $\alpha_{\kappa\mu}$. Solving for the last term, and substituting in the right hand side of (4.18) there results the equation

$$H = H_1 + \sum_{\mu=1}^m \text{ad}_F^k(\beta_{\kappa\mu} G_\mu),$$

where

$$H_1 = \sum_{\kappa=0}^{k-1} \sum_{\mu=1}^m (\beta_{\kappa\mu} - \alpha_{\kappa\mu}) \text{ad}_F^\kappa(G_\mu).$$

By induction, H_1 , and hence also H , can be reduced to the desired form.

It only remains to establish the 'uniform' character of Λ . That is, we wish to find a matrix \mathbf{L} of functions of γ such that $\Lambda(t) = \mathbf{L}(\gamma(t))$ for all signals $\gamma(\cdot)$ as in the theorem. The block structure of \mathbf{L} is as that of Λ displayed in equation (4.4), and we denote by $\mathbf{L}_0, \dots, \mathbf{L}_k$ the corresponding submatrices of \mathbf{L} . We again define $\mathbf{L}_0 := \Phi$ (first block of coordinates of γ), seen now as a symbolic coordinate, and $\mathbf{L}_k = 0$. By induction on descending $\kappa = k-1, \dots, 1$, we define the $n \times n$ matrix \mathbf{L}_κ so that (4.6) holds along references, i.e.

$$\mathbf{L}_\kappa := f_x(\pi, Q(\omega))\mathbf{L}_{\kappa+1} - \mathbf{L}'_{\kappa+1}F + \mathbf{S}_{\kappa-\kappa}, \tag{4.19}$$

where the \mathbf{S}_κ are defined essentially as before: same equation as (4.2), but symbolically on coordinates, and where $\mathbf{L}'_\kappa F$ is the $n \times n$ matrix whose (i,j) -th entry is

$$L_F((\mathbf{L}_\kappa)_{ij}),$$

the Lie derivative by F of the (i,j) -th entry of \mathbf{L}_κ . This completes the proof of the theorem.

Finally, note that Λ^{-1} (needed in the computation of closed-loop control laws, see example later,) can be also precomputed, since \mathbf{L}^{-1} admits a simple expression $W.T, T =$

$$\begin{array}{c|c} \mathbf{I} & \begin{array}{c} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array} \\ \hline \mathbf{N}_\kappa \cdot \cdot \cdot \mathbf{N}_1 & \mathbf{N}_0 \end{array},$$

$W =$ a block matrix with Φ^{-1} in the diagonal, and $N_0 = I, N_\kappa = 0,$ and $N_\kappa = -\mathbf{L}_\kappa$ for other κ .

5. A worked example.

Here we work out an example illustrating the Main Theorem. For simplicity, and because of the length of the paper, we shall only consider an (academic) example in dimension 1. We shall reports on more experimental results, for more realistic problems, in the future; see for instance [20]. The example to consider has state space \mathfrak{X} and control set also \mathfrak{X} . The equations are

$$\dot{x} = 1 + u.\sin^2 x . \quad (5.1)$$

We picked this example because it is almost as pathological as possible in dimension 1; note that the only way to cancel the nonlinearity at (or near) $x = 0, \pi$, etc., is by using an infinite gain. We shall design a closed-loop controller that will regulate along all possible reference trajectories corresponding to step inputs.

Since we are interested in constant inputs, we take $r=1$ and $S_\Omega := \mathfrak{X}$ with equations

$$\dot{\omega} = 0$$

for the autonomous system Ω . Use coordinates (w,p) for pairs of states in $S_\Omega \times S_\Pi$. The relevant vector fields are then

$$g = g_1 = \begin{pmatrix} 0 \\ \sin^2 p \end{pmatrix}, \quad f = \begin{pmatrix} 0 \\ 1+w\sin^2 p \end{pmatrix} .$$

We then compute $[f,g]$ and $[f,[f,g]]$, and obtain the equation

$$(w+2)g + (1/2)[f,[f,g]] = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Note, incidentally, that the existence of such an equation shows that $ND(\Omega \downarrow \Pi)$ is the whole state space $S_\Omega \times S_\Pi$ of the input-generator + plant. (These, as well as all the computations to follow, were performed using the Macsyma symbolic manipulation system.) We now introduce the vector fields F and $G = G_1$ from equations (4.11-4.13); these are, explicitly,

$$F(\gamma) = (w.\sin 2p.\Phi : 0 : 1+w\sin^2 p), \quad G := (0 : 0 : \sin^2 p) .$$

Note that the fundamental solution Φ satisfies the equation

$$\dot{\Phi} = \Phi.w.\sin 2p . \quad (5.2)$$

We now apply lemma 4.1, with (from above) $k = 3$ and

$$\beta_{01} = \Phi(w+2), \quad \beta_{11} = 0, \quad \beta_{21} = \Phi/2 .$$

We obtain the σ 's, and from these the ρ 's, which dropping the indexes that are always equal to 1 ($n=1, m=1$.) are then:

$$\begin{aligned} \rho_1 &= \Phi/2, \quad \rho_2 = \Phi.w.\sin 2p, \\ \rho_3 &= \Phi.(-4w^2\sin^4 p + (3w^2-2w)\sin^2 p + 2w + 2). \end{aligned} \quad (5.3)$$

These were obtained from the formulas

$$\begin{aligned} \rho_1 &= \beta_2, \quad \rho_2 = 2L_F\beta_2 - \beta_1, \\ \rho_3 &= \beta_0 - L_F\beta_1 + L_F^2\beta_2, \end{aligned}$$

as in lemma (5.2). Finally, the coordinate change matrix \mathbf{L} has

$$\begin{aligned} \mathbf{L}_1 &= \Phi.\cos p \sin p.(w.\sin^2 p - 1), \\ \mathbf{L}_2 &= (\Phi/2)\sin^2 p . \end{aligned} \quad (5.4)$$

The inverse of \mathbf{L} is computed trivially from this.

The composite system Ξ in Figure 2 is now described by the equations

$$\begin{aligned}\dot{z}_1 &= v, \quad \dot{z}_2 = z_1, \quad \dot{z}_3 = z_1, \\ \dot{x} &= 1 + (\rho_1 z_1 + \rho_2 z_2 + \rho_3 z_3) \sin^2 x,\end{aligned}$$

where the ρ_i are as in equation (5.3). With the change of coordinates

$$\mathbf{L}(\gamma(t)) \begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} z \\ x-p \end{pmatrix},$$

that is, with

$$y = (x - p - \mathbf{L}_2(\gamma)z_2 - \mathbf{L}_1(\gamma)z_3)/\phi, \quad (5.5)$$

one gets the equations

$$\dot{z}_1 = v, \quad \dot{z}_2 = z_1, \quad \dot{z}_3 = z_1,$$

and

$$\begin{aligned}\dot{y} &= -(((8\Phi \sin^4 p - 6\Phi \sin^2 p)w^2 + (4\Phi \sin^2 p - 4\Phi)w - 4\Phi)z_3 \\ &\quad - 2\Phi w z_2 \sin 2p - \Phi z_1 - 2w) \sin(((2\Phi w \cos p \sin^3 p - 2\Phi \cos p \sin p)z_3 + \Phi \sin^2 p z_2 + 2\Phi y + 2p)/2)^2 \\ &\quad + ((10\Phi \sin^4 p - 12\Phi \sin^6 p)w^2 + 4\Phi \sin^2 p - 2\Phi)z_3 \\ &\quad + 6\Phi w z_2 \cos p \sin^3 p + \Phi z_1 \sin^2 p + 4\Phi w y \cos p \sin p + 2w \sin^2 p)/(2\Phi).\end{aligned}$$

A Taylor expansion of this last term with respect to (z_1, z_2, z_3, y) gives indeed that

$$\dot{y} = z_3 + \alpha(\gamma, z_1, z_2, z_3, y),$$

where $\alpha(\gamma(t), z_1, z_2, z_3, y)$ is $o(z_1, z_2, z_3, y)$ for each t .

Now close the loop by assigning all poles at, for instance, -2 . This results in a feedback law in terms of the (z, x) coordinates of the precompensator, $v = \mathbf{KL}^{-1}(z: x-p)$, where \mathbf{K} is the gain matrix

$$\mathbf{K} = [k_1, k_2, k_3, k_4] = [-8, 24, 32, 16]$$

that assigns poles at -2 for a system in control canonical form. The resulting closed-loop controller is as follows:

$$\begin{aligned}z_1 &= a_1(t)z_1 + a_2(t)z_2 + a_3(t)z_3 + a_4(t)e(t), \\ \dot{z}_2 &= z_1, \quad \dot{z}_3 = z_2, \\ \dot{p} &= 1 + w \cdot \sin^2 p, \\ \dot{\Phi} &= \Phi \cdot w \cdot \sin 2p, \\ u &= w + b_1(t)z_1 + b_2(t)z_2 + b_3(t)z_3,\end{aligned}$$

where

$$\begin{aligned}a_1 &= k_1, \quad a_2 = k_2 - (k_4/2) \sin^2 p, \\ a_3 &= k_3 - k_4(1 - w \cdot \sin^2 p) \sin p \cos p, \quad a_4 = k_4/\Phi,\end{aligned}$$

and

$$\begin{aligned}b_1 &= \Phi(t)/2, \quad b_2 = \Phi(t) \cdot w(t) \cdot \sin 2p(t), \\ b_3 &= \Phi(t) \cdot (-4w^2(t) \sin^4 p(t) + (3w^2(t) - 2w(t)) \sin^2 p(t) + 2w(t) + 2).\end{aligned}$$

Note that $e(t) = x(t) - p(t)$, where $x(t)$ is the measured state. (With the alternative coordinates as in (5.5), the state $(z: y)$ is seen to indeed satisfy up to first order a linear constant differential equation with all eigenvalues at -2 .)

The controller can be somewhat simplified by changing to coordinates $z' := \Phi z$, so that Φ disappears completely. But the actual computational complexity is not decreased, since the equations for the precompensator variables have now an extra term.

Now we simulate the behavior of the system for a typical constant input, say $w \equiv 10$ on the interval $[0,10]$. The ensuing trajectory for the model plant, with $p(0) = 0$, results in $p(10) = 33.53$. Thus there are 22 points in this trajectory where the term $\sin^2 x$ multiplying the control vanishes. Intuitively, the system should be hard to control especially about these points. Assume now that there is an error in the initial state (or a disturbance occurs at time $t=0+$), and that the true initial state $x(0)$ of the plant happens to be $x(0) = 0.1$. Then, after $t = 10$ seconds, the plant *without* our servo would be at state $x = 34.05$. (Of course, for this simple example, in dimension 1 and with a constant control, the 'wrong' trajectory is just a phase shift from the reference trajectory.) If instead our regulator is used, the closed loop system is such that the state $x(10)$ is 33.54. The enclosed graph (Figure 3) shows a comparison between the unregulated error (solid line) and the error $x(t)-p(t)$ for our closed-loop system (dotted line).

6. Final remarks.

There is a need for much more theoretical analysis as well as experimentation before it will be clear whether our precompensator design is useful in real problems. We expect the method to be applied in conjunction with open-loop command generators acting at a higher level of design. These other methods could involve, for instance, optimal control techniques. If it is known, for example, that all controls are relay ("bang-bang"), then one may design compensators for constant reference controls. Otherwise one may design on the basis of an assumption of piecewise polynomials of a certain degree, say 3. The relative performance when the control happens not to be in one such class is probably very hard to study theoretically, and a large amount of simulation may be needed in each particular situation.

A note on the complexity of the compensator. The dimension of the obtained compensators is clearly huge: nk , where k is obtained from the Lie algebraic manipulations, and n is the dimension of the original system. This number is at least n^2 , so it would seem that the method presented is impractical. However, it is important to realize that *dimension* of a system is most emphatically *not* a measure of *complexity* of implementation. In the necessary numerical solutions of the corresponding systems of ode's, the limiting factor is more the amount of computation needed for updating each coordinate. In the design given, all coordinates z_{ij} of the precompensator evolve according to a simple integration, except only for the first block of n coordinates. Thus the complexity is more like that of a system of dimension n . Of course, the design may be impractical for other reasons, but dimensionality per se doesn't seem to be as serious an issue as it would first appear.

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