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A concept of local observability

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A notion of local observability, which is natural in the context of nonlinear input/output regulation, is introduced. A simple characterization is provided, a comparison is made with other local nonlinear observability definitions, and its behavior under constant-rate sampling is analyzed.

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1. Introduction

Observability is a fundamental system-theoretic property which reflects the possibility of estimating internal states on the basis of input/output data. In the context of nonlinear systems, various possible ways of formalizing this property give rise to different, nonequivalent, notions. These notions differ mainly in the choice of inputs used in testing observability (see for instance a discussion in the introduction to [10]) or in the type of local behavior desired (see [5]).

For continuous-time systems the fundamental global observability results are found in [13,14], where the necessary techniques are developed in the context of the minimal realization problem. Here we shall be interested more in *local* observability: intuitively, distinguishing states only from their neighbors. In that area, by far the most important work is that of Hermann and Krener in [5] (see [3] for related results), who introduced and compared various notions of local observability and obtained Lie-theoretic characterizations of some of these. The best results relate to 'local weak observability', which we shall review below. In what follows, we refer to this later concept simply as *HK-observability*.

When dealing with problems of input/output stabilization [11], where observation and control alternate and a closed-loop design is Lyapunov stable, a notion of local observability somewhat different to those in [5] appears naturally. We shall call this notion *L-* (for *Lyapunov*) *observability*. We show below that the new notion can be characterized in a fashion analogous to that of HK-observability, coinciding with it in the analytic case but admitting a somewhat nicer characterization in the general smooth case. Technically, the results elaborate on the material in [5].

Recent work of the author has concentrated on the study of the preservation of system-theoretic properties under sampling; see for instance [12,9]. We present a result along these lines regarding L-observability; in fact, this was one of the main motivations for studying L-observability in the first place. Note that the term 'sampling' is used here in the sense of digital control (constant-rate sampling) as opposed to the very different case of arbitrary (non-equispaced) piecewise constant controls as in [1,2], or [4]. Nonetheless, the techniques used in proving some of the statements are closely related to the later reference.

2. Lyapunov observability

The systems Σ to be considered are described by equations

$$\dot{x}(t) = f(x(t), \mu(t)) \tag{2.1a}$$

$$v(t) = h(x(t)), \tag{2.1b}$$

where states $x(t)$ belong to a smooth (i.e. C^∞) Hausdorff second countable connected n-dimensional manifold $\mathbf{X}$, controls μ take values in a subset $\mathbf{U}$ of a manifold $\mathscr{U}$, and measurements $y(t)$ take values in $\mathscr{R}^p$, for some p. The map $h : \mathbf{X} \to \mathscr{R}^p$ is smooth, and $F_u = f(\cdot, u)$ is a smooth vector field for each $u \in \mathbf{U}$. An *analytic* system is one for which h and all F_u are real-analytic. In Section 3, we shall need that the dynamics be smooth in $\mu(t)$, more precisely:

(2.2) f extends to a smooth map $\mathbf{X} \times \mathscr{U} \to T\mathbf{X}$, int(U) is connected, and $\mathbf{U} \subseteq$ clos(int(U)).

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A *control (function)* is a map

$\mu : [0, T] \to \mathbf{U}$

which is measurable and essentially bounded; $T = |\mu|$ is the *length* of μ.

The following notations will be used. Let $\sigma = (s_1, \ldots, s_r)$ be a sequence of positive real numbers, with $T := \Sigma s_i$, and let $u := (u_1, \ldots, u_r)$ be a sequence of elements in $\mathbf{U}$. Then u_σ is the control function μ of length T defined as follows:

$$\mu(t) := u_i \quad \text{if } t \in [s_0 + \cdots + s_i, s_0 + \cdots + s_{i+1}),$$

$i = 0, \ldots, r$ (denoting $s_0 := 0$). The state $x(\tau)$ at time $\tau \in [0, T]$ corresponding to solving (2.1a) with $x(0) = \xi$ and control μ, is denoted by $\phi[\tau, \xi, \mu]$; the corresponding output measurement $h(\phi[\tau, \xi, \mu])$ is denoted by $\eta[\tau, \xi, \mu]$. When $\tau = T$ (= length of μ) these are denoted just by $\phi[\xi, \mu]$ and $\eta[\xi, \mu]$ respectively.

For a system Σ as in (2.1), let $\mathscr{L}$ be the Lie algebra of vector fields generated by all the $\{F_u, u \in \mathbf{U}\}$, and let $\mathscr{L}_0$ be the ideal of $\mathscr{L}$ generated by all the vector fields of the form $F_u - F_v$ with u, v in $\mathbf{U}$. The *accessibility rank condition* at $x \in \mathbf{X}$ is the condition

$$\dim \mathscr{L}(x) = n. \qquad (\mathrm{ARC})_x$$

The *strong accessibility rank condition* at x is the condition

$$\dim \mathscr{L}_0(x) = n. \qquad (\mathrm{SARC})_x$$

(These are standard properties considered in nonlinear control; details about the later can be found in [15].)

The *system* Σ satisfies either of the above conditions if it holds for all $x \in \mathbf{X}$. For vector fields F and functions g we denote the Lie derivative $Fg = \langle \mathrm{d}g, F \rangle$; in local coordinates, this is $\mathrm{d}g \cdot F$, where $\mathrm{d}g$ is the gradient of g. For a map $\theta : \mathbf{X} \to \mathscr{R}^r$, $\mathrm{d}\theta$ denotes its differential (Jacobian in local coordinates). Hermann and Krener introduced $\mathrm{d}\mathscr{H}$, the smallest $\mathscr{R}$-vector space of one-forms that contains $\mathrm{d}h_1, \ldots, \mathrm{d}h_p$ ($h_i :=$ i-th component of h) and is closed under Lie differentiation by elements of $\mathscr{L}$. In analogy to the discrete-time situation in [10], we may call $\mathrm{d}\mathscr{H}$ the *observation space* of Σ. Its generators are the elements of the form

$$\mathrm{d}(F_{u_1} \cdots F_{u_r} h_i), \qquad (2.3)$$

for all possible sequences $(u_1, \ldots, u_r)$ in $\mathbf{U}^r$ and $i = 1, \ldots, p$.

The *observability rank condition* at $x \in \mathbf{X}$ introduced in [5] is

$$\dim \mathrm{d}\mathscr{H}(x) = n. \qquad (\mathrm{ORC})_x$$

Let W be a subset of $\mathbf{X}$. The states ξ, ξ' in W are *indistinguishable inside* W (denoted $\xi \mathrm{I}_W \xi'$) if the following property holds: given any $T \geq 0$, and any control μ of length T for which

$$\phi[\tau, \xi, \mu] \in W \quad \text{and} \quad \phi[\tau, \xi', \mu] \in W$$

for all $0 \leq \tau \leq T$, necessarily $\eta[\xi, \mu] = \eta[\xi', \mu]$. (That is, ξ and ξ' cannot be distinguished by i/o experiments if trajectories are to stay in W.) Denote also $I_W := \{\xi' \text{ s.t. } \xi \mathrm{I}_W \xi'\}$. The following notion of ('local weak') observability at $x \in \mathbf{X}$ was introduced in [5]:

$(\mathrm{HK\text{-}obs})_x$ There exists a neighborhood W of x such that, for every open neighborhood $V \subseteq W$ of x, $I_V(x) = \{x\}$.

The intuition behind this definition is that states close to x should be 'instantaneously' distinguishable from x. As discussed in the introduction, we wish to study a somewhat different notion of local observability at x, more closely related to stabilization problems:

$(\mathrm{L\text{-}obs})_x$ For every neighborhood W of x, there exists a neighborhood $V \subseteq W$ of x such that $I_W(x) \cap V = \{x\}$.

Note the difference in the quantifier order with the previous definition. Intuitively, this says that states close to x should be distinguishable from x without 'large' excursions. System-theoretically, such a property is of interest if we are trying to control Σ to the state x using only output measurements, and such that the closed-loop system is Lyapunov stable; precise details about such questions are provided in [11]. Mathematically, one may expect that this requirement will be more natural than (HK-obs) when deailing with nonanalytic systems, since no 'instantaneous' properties are involved.

The relevant results of Hermann and Krener relating (HK-obs) and (ORC) can be summarized as follows ([5], Theorems 3.1, 3.11, and 3.12):

(2.4) For any x, $(\mathrm{ORC})_x \Rightarrow (\mathrm{HK\text{-}obs})_x$.

(2.5) (HK-obs)$_x$ holds *for all* x $\Rightarrow$ (ORC)$_x$ is true *on an open dense subset* of **X**.

Furthermore, if Σ satisfies the ARC and Σ is *analytic* then also:

(2.6) (ORC)$_x$ holds for all x $\Leftrightarrow$ (HK-obs)$_x$ holds for all x.

For (L-obs), there is a characterization analogous to (2.6) but valid in the general smooth case:

2.7. Theorem. *If Σ satisfies the ARC, then the following statements are equivalent*:

(a) (L-obs)$_x$ *holds at all* x;

(b) (ORC)$_x$ *holds on an open dense subset of* **X**.

Note that (b) is in principle easier to check computationally than the satisfiability of (ORC)$_x$ *for all* x. For instance, most natural examples of smooth nonanalytic systems (e.g. in the approximation of switching discontinuities by a rapidly changing function) are *piecewise analytic*; in such a situation it is enough in order to check (b) to find a *single* point in each domain of analyticity where (ORC)$_x$ holds. For analytic systems, (b) is equivalent to (ORC) holding everywhere, but not so in the general case. The proof of Theorem 2.7 will follow from a few lemmas.

2.8. Lemma. (HK-obs)$_x$ $\Rightarrow$ (L-obs)$_x$.

Proof. Fix first a neighborhood W of x as in the definition of (HK-obs)$_x$. Now let W' be any neighborhood of x. Let V be any open neighborhood of x contained in $W \cap W'$. Thus, $I_{W'} \cap V$ is contained in I_V. By (HK-obs)$_x$, $I_V(x) = \{x\}$, so also $I_{W'}(x) \cap V = \{x\}$, as wanted. □

2.9. Lemma. *If Σ satisfies the ARC, then the following statements are equivalent*:

(i) (L-obs)$_x$ *holds everywhere*;

(ii) (L-obs)$_x$ *holds on a dense subset of* **X**.

Proof. Assume that the condition holds on the dense subset D of **X**; we must prove it holds everywhere. Pick then any $x \in \mathbf{X}$ and any neighborhood W of x. By the ARC and Chow's theorem (see [6]) there exists an open subset W' of W such that every $z \in W'$ is reachable from x by trajectories staying inside W. Pick any z in $D \cap W'$. Since $z \in D$, there is a neighborhood $V' \subseteq W$ of z such that $I_W(z) \cap V' = \{z\}$. And since $z \in W'$, there is a control μ of length T such that $\phi[\tau, x, \mu] \in W$ for all $0 \le \tau \le T$ and $\phi[x, \mu] = z$. By continuity of the map $(\xi, \tau) \to \phi[\tau, \xi, \mu]$, there exists a neighborhood $V \subseteq W$ of x with the following properties:

(i) $\phi[\cdot, \mu]$ maps V into V', and

(ii) $\phi[\tau, \xi, \mu] \in W$ for each $\xi \in V$ and all $0 \le \tau \le T$.

We claim that $I_W(x) \cap V = \{x\}$, so that V is as desired. Indeed, pick any ξ in V. Applying μ to ξ results in a trajectory included in W and leading up to a state ξ' in V'. But ξ' can be distinguished from $z = \phi[x, \mu]$ without leaving W, say by a control μ'. Thus the concatenation of μ and μ' distinguishes ξ from x without leaving W. □

2.10. Lemma. *If* (L-obs)$_x$ *holds on a dense subset of* **X**, *then also* (ORC)$_x$ *holds on a dense* (*hence, open dense*) *set.*

Proof. Assume that (L-obs) holds on D. The argument used in the proof of [5], Theorem 3.11, can be adapted to the present case almost without changes. We use the terminology from that reference. Assume that there exists an open subset $B \subseteq \mathbf{X}$ where the ORC does not hold; without loss of generality we may assume that $\dim \mathscr{H}(x) \equiv k$ (constant) $< n$ there. There is then an open subset $W \subseteq B$ where the 'strong indistinguishability relation' SI is regular, with equivalence classes being submanifolds of dimension $n - k$. Pick now any x in $D \cap W$ and any neighborhood $V \subseteq W$ of x. Since SI(x) is connected, there is a z in V which is strongly indistinguishable from x. If μ is any control such that $\phi[\cdot, x, \mu]$ and $\phi[\cdot, z, \mu]$ both remain in W, the existence of a quotient dynamics on **X**/SI implies that $\eta[x, \mu] = \eta[z, \mu]$. Thus $z \in I_W(x) \cap V$, contradicting (L-obs) at x. □

We can now complete the proof of Theorem 2.7. That (a) implies (b) follows from Lemma 2.10. Conversely, assume that (b) holds. By (2.4), (HK-obs)$_x$ holds on a dense set, so by Lemma 2.8 this is also true of (L-obs)$_x$. It then follows from Lemma 2.9 that (L-obs)$_x$ holds at all x, as desired. □

3. Sampled observability

If δ is a positive real and $u = (u_1, \ldots, u_r) \in \mathbf{U}^r$, consider the control $\mu = u_\sigma$ obtained using $\sigma := \delta^r$

$=(\delta,\ldots,\delta)$ (r times); we call such a control a *δ-sampled control*, and denote it also by u_δ^r. Note that its length is $T=r\delta$. If $z=\phi[x, u_\delta^r]$ for some sequence $u=(u_1,\ldots,u_r)$, then z is k-step *δ-sampled reachable* from x; the set of such states is denoted by $A_\delta^r(x)$. See [12,9] for various results on sampled controllability; here we want to study a suitable sampled (local) observability notion.

For the rest of this paper, hypothesis (2.2) *is assumed to hold for all systems considered.*

Let W be a subset of $\mathbf{X}$ and pick any $\delta>0$. By analogy with the previous section, we shall say that states ξ, $\xi'\in W$ are *δ-indistinguishable inside* W ($\xi \mathrm{I}_W^\delta \xi'$) if the following property holds: given any $r>0$ and any δ-sampled control $\mu=u_\delta^r$ of length $T=r\delta$ for which

$$\phi[\tau,\xi,\mu]\in W \quad \text{and} \quad \phi[\tau,\xi',\mu]\in W$$

for all $0\le\tau\le T$, necessarily $\eta[\xi,\mu]=\eta[\xi',\mu]$. We write $I_W^\delta(\xi):=\{\xi' \text{ s.t. } \xi \mathrm{I}_W^\delta \xi'\}$. We then have a notion of Lyapunov sampled observability at $x\in\mathbf{X}$:

(L-s.obs)$_x$ For every neighborhood W of x, there exists a real $\Delta>0$ such that, for each $0<\delta<\Delta$ there is a neighborhood $V\subseteq W$ of x such that $I_W^\delta(x)\cap V=\{x\}$.

Other definitions of local sampled observability are of course possible, even of a 'Lyapunov' type; the above one seems to be the easiest to work with. Also, a notion of 'weak' observability can be defined as for discrete-time systems in [8]: in that case both positive time and (ideal) negative time trajectories are allowed in testing indistinguishability. Another possibility is to make sampling periods δ to be uniformly bounded on compacts, in the style of the sampled controllability results mentioned earlier.

It will be technically convenient to introduce a rank condition associated to sampled observability. If $\mu^1,\ldots,\mu^k$ are (non-necessarily sampled) controls, let

$$\eta^{\mu^1\cdots\mu^k}(\xi):=\begin{pmatrix}\eta[\xi,\mu^1]\\ \eta[\xi,\mu^k]\end{pmatrix}. \tag{3.1}$$

Consider now the following *sampled observability rank condition* at x:

(s.ORC)$_x$ There exist integers $k>0$ and $n_1,\ldots,n_k>0$, a real $\Delta>0$, and sequences $u^i\in\mathbf{U}^{n_i}$, such that for each $0<\delta<\Delta$ the following condition holds: let $\mu^i:=u^i_{\sigma_i}$, $\sigma_i:=\delta^{n_i}$; then

$$\operatorname{rank} d\eta^{\mu^1\cdots\mu^k}(x)=n. \tag{3.2}$$

(See [7] for a related notion in the context of discrete-time systems with no controls.)

3.3. Theorem. *Assume that Σ satisfies the SARC. Then each of the following statements is equivalent to* (a) *and* (b) *in Theorem* 2.7:

(c) (L-s.obs)$_x$ *holds at all x;*

(d) (s.ORC)$_x$ *holds on an open dense subset of* $\mathbf{X}$.

Since $I_W(x)\subseteq I_W^\delta(x)$ for any W, δ, x, it is clear that (c) implies (a). That (b) implies (d) follows from:

3.4. Lemma. (ORC)$_x$ $\Rightarrow$ (s.ORC)$_x$.

This result will be established in Section 4. In a way totally analogous to that of Section 2, the Theorem will then follow from these two lemmas:

3.5. Lemma. (s.ORC)$_x$ $\Rightarrow$ (L-s.obs)$_x$.

3.6. Lemma. *If Σ satisfies the SARC then the following are equivalent:*

(i) (L-s.obs)$_x$ *holds everywhere;*

(ii) (L-s.obs)$_x$ *holds on a dense set.*

Proof of Lemma 3.5. This is basically the inverse function theorem. Pick any x in $\mathbf{X}$, and let W be any neighborhood of x. Let the sequences u^i be as in the definition of the (s.ORC)$_x$. The lengths of the corresponding controls μ^i are uniformly bounded by δN, where $N:=\max\{n_i\}$; further they take values on a finite (hence compact) subset of $\mathbf{U}$. It follows from continuity on controls and initial conditions that, for some Δ small enough, and for some neighborhood $V'\subseteq W$ of x, the trajectories $\phi[\cdot,\xi,\mu^i]$ remain in W whenever $\xi\in V'$, $0\le\tau\le n_i\delta$, and $0\le\delta\le\Delta$. Without loss of generality, Δ is small enough that

$$\operatorname{rank} d\eta^{\mu^1\cdots\mu^k}(x)=n \quad \text{for } 0\le\delta\le\Delta.$$

Pick any such δ. The map $\eta^{\mu^1\cdots\mu^k}$ is thus, by the inverse function theorem, locally 1–1 at x. Let V be a neighborhood of x contained in V' where this map is 1–1. It follows that trajectories starting at

$\xi \in V$ corresponding to the controls μ^i remain in W, and for each such ξ there is an i such that $\eta[x, \mu^i] \neq \eta[\xi, \mu^i]$. Thus $I_W^\delta(x) \cap V = \{x\}$, as desired. □

Before establishing Lemma 3.6, we need the auxiliary result of Lemma 3.9 given below, on sampled controllability. This result plays for Lemma 3.6 a role analogous to that of Chow's theorem for the proof of Lemma 2.9 in the previous section. We shall state a somewhat stronger form than needed (cf. Remark 3.10). If r is a positive integer, and $T > 0$ is real, denote

$$\mathscr{R}_r(T) := \{\sigma = (s_1, \ldots, s_r) \in \mathscr{R} \text{ s.t. } \text{all } s_i > 0 \text{ and } \Sigma s_i = T\}, \tag{3.7}$$

seen as a manifold of dimension $r - 1$. Further, for any fixed r, T as above, $u \in U^r$, and $x \in X$, let Φ be the map

$$\Phi : \mathscr{R}_r(T) \to \mathbf{X} : \sigma \to \phi[x, u_\sigma]. \tag{3.8}$$

3.9. Lemma. *Let $x \in \mathbf{X}$ and let W be an open neighborhood of x. Assume given T and r as above, and a sequence $\sigma \in \mathscr{R}_r(T)$ such that*

(a) *$\phi[\tau, x, u_\sigma] \subseteq W$ for each $\tau \leq T$, and*

(b) *Φ has full rank differential at σ.*

Let $z := \phi[x, u_\sigma]$. Assume given a neighborhood $V_z \subseteq W$ of z. Then there are a neighborhood $V_x \subseteq W$ of x and a $\Delta > 0$ such that, for each $0 < \delta < \Delta$, there is a δ-sampled control μ of length $T = T(\delta)$ such that $z = \phi[x, \mu]$ and:

(i) *$\phi[\tau, \xi, \mu] \subseteq W$ for all $\tau \leq T(\delta)$ and all $\xi \in V_x$, and*

(ii) *$\phi[V_x, \mu] \subseteq V_z$.*

Proof. (Sketch). This is a refinement of the result in Theorem 2.2 of [12], so only the idea is given here. That reference shows that, for each δ small enough, there is a δ-sampled control μ with $z = \phi[x, \mu]$ satisfying (i), and further, μ is close to u_σ in a suitable sense, *uniformly* with respect to δ. Let then V' be any neighborhood of z whose closure is included in V_z, and let V_x be the pre-image of V' under the map $\phi[\cdot, u_\sigma]$, intersected with W. Thus,

$$\phi[V_x, u_\sigma] \subseteq V' \subseteq \text{clos}(V') \subseteq V_z.$$

Picking if necessary a smaller Δ, (ii) is also satisfied. □

Proof of Lemma 3.6. Assume that $(\text{L-s.obs})_x$ holds for $x \in D$ dense. Pick any $x \in X$ and any neighborhood W of x. Since the SARC holds, there exist σ, T, r as in the previous lemma, so that (a) and (b) there hold. (This is a consequence of 'normal strong—or 'fixed time'—accessibility'; see for instance [9], Lemma 2.2.) Further, since Φ is an open map at σ, we may assume that $z = \phi[x, u_\sigma]$ is in the (dense) set D. Let $V_z := W$, and apply the previous lemma to this data. Let V_x, Δ_x be as in the conclusion of the lemma. Since z is in D, there is a Δ_z and for each $0 < \delta < \Delta_z$ there is a neighborhood V_δ of z in W such that $I_W^\delta(z) \cap V_\delta = \{z\}$. Let $\Delta := \min\{\Delta_x, \Delta_z\}$, and pick any $0 \leq \delta \leq \Delta$ and the corresponding δ-sampled μ as in Lemma 3.9. Let V be the preimage of V_δ under $\phi[\cdot, \mu]$, intersected with V_x. Thus, the δ-sampled control μ sends x into z, and sends any $\xi \in V$ into some $\xi' \in V_\delta$, with trajectories remaining in W. Concatening with a sampled control which distinguishes ξ' from z, it follows that $I_W^\delta(x) \cap V = \{x\}$, as desired. □

3.10. Remark. Lemma 3.6 admits a stronger version, which is of interest if the sampled observability rank condition holds uniformly on δ over D. More precisely, assume that we require that the neighborhoods V in the definition of sampled observability at x can be choosen independently of δ, for $0 \leq \delta \leq \Delta$. We then have an analogue of Lemma 3.6 for this stronger property. Indeed, the only change needed in the above proof is the choice $V_z :=$ this common neighborhood, instead of $V_z = W$ as above. Now all V_δ are equal to V_z, so the neighborhood V of x obtained at the end of the proof is again independent of δ. Thus the same type of uniform sampled observability holds at x. We leave as a topic for further research the clarification of the relation between this uniform notion and that developed in this paper.

4. Proof of Lemma 3.4

The proof will be a generalization of that in [4]. In that paper, the author establishes what can be interpreted as a non-controlled, non-constant-rate version of Lemma 3.4. Since the result is local, we work with coordinates in $\mathbf{X} = \mathscr{R}^n$.

Assume then that $(\text{ORC})_x$ holds. Thus there is a finite set of control values $\{v_i, i = 1, \ldots, k\}$ such that (denoting $F_i := F_{v_i}$) the row vectors

$$\{\mathrm{d} F_{i_1} \cdots F_{i_r} h_j(x), 1 \leq i_s \leq k, 1 \leq j \leq p, 0 \leq r \leq q\} \tag{4.1}$$

span an n-dimensional space, for some $q > 0$. Equivalently, there exists an integer l and a sequence of control values $u = (u_1, \ldots, u_r)$ such that, (with $F_i := F_{u_i}$) the rows

$$\left\{ \mathrm{d}F_1^{i_1} \ldots F_l^{i_l} h_j(x),\ i_s \geq 0,\ \Sigma i_s \leq q,\ i \leq j \leq p \right\} \tag{4.2}$$

span an n-dimensional space, for (the same) $q > 0$: the second statement follows from the first simply by taking $l := kq$ and considering the sequence $(u_1, \ldots, u_r)$ obtained by repeating q times the sequence $(v_1, \ldots, v_k)$. This second form will be used in what follows.

Choose an ordering for the set of all l-vectors $(i_1, \ldots, i_l)$ of nonnegative integers i_s such that $\Sigma i_s \leq q$, or equivalently, for all monomials of degree $\leq q$ in l variables $t_1, \ldots, t_l$. With this ordering, let $a(t_1, \ldots, t_l)$ be the row vector (say, with N columns) consisting of all the monomials

$$t_1^{i_1} \cdots t_l^{i_l} / (i_1! \cdots i_l!) \tag{4.3}$$

with $\Sigma i_s \leq q$. For each $j = 1, \ldots, p$, let D_j be the matrix whose rows are the n-vectors

$$\mathrm{d}F_1^{i_1} \cdots F_l^{i_l} h_j(x),$$

listed with the above order for exponents $(i_1, \ldots, i_l)$. Consider the Taylor expansion of degree q,

$$\mathrm{d}\eta^{u(t_1, \ldots, t_l)}(x) = A(t_1, \ldots, t_l) D + B(t_1, \ldots, t_l), \tag{4.4}$$

where the p by pN matrix $A(t_1, \ldots, t_l)$ is defined as

$$A(t_1, \ldots, t_l) = \begin{pmatrix} a(t_1, \ldots, t_l) & 0 & \cdots & 0 \\ 0 & a(t_1, \ldots, t_l) & & \vdots \\ \vdots & & & 0 \\ 0 & 0 & \cdots & a(t_1, \ldots, t_l) \end{pmatrix}, \tag{4.5}$$

$$D = \begin{pmatrix} D_1 \\ \vdots \\ D_p \end{pmatrix} \tag{4.6}$$

and $B(t_1, \ldots, t_l)$ is a p by n matrix all whose entries $b_{ij}(t_1, \ldots, t_l)$ satisfy that

$$b_{ij}(t_1, \ldots, t_l) / \|(t_1, \ldots, t_l)\|^q \to 0 \quad \text{as } (t_1, \ldots, t_l) \to 0. \tag{4.7}$$

Choose positive integers $\{s_{ij}\}$, $i = 1, \ldots, N$, $j = 1, \ldots, l$, such that the multivariable Vandermonde matrix

$$\begin{pmatrix} a(s_{11}, \ldots, s_{1l}) \\ \vdots \\ a(s_{N1}, \ldots, s_{Nl}) \end{pmatrix} \tag{4.8}$$

has rank N. Thus the (pN by pN) matrix

$$C := \begin{pmatrix} A(s_{11}, \ldots, s_{1l}) \\ \vdots \\ A(s_{N1}, \ldots, s_{Nl}) \end{pmatrix} \tag{4.9}$$

has rank pN.

For each $i = 1, \ldots, N$ let u^i be the sequence in U^{n_i} defined as follows:

$$n_i := s_{i1} + \cdots + s_{il}, \tag{4.10}$$

$$u^i := (u_1, \ldots, u_1, u_2, \ldots, u_2, \ldots, u_l, \ldots, u_l), \tag{4.11}$$

where u_j is repeated s_{ij} times. Finally, let

$$C(\delta) := \begin{pmatrix} A(\delta s_{11}, \ldots, \delta s_{1l}) \\ \vdots \\ A(\delta s_{N1}, \ldots, \delta s_{Nl}) \end{pmatrix} \tag{4.12}$$

and $E(\delta) :=$ the pN by n matrix with blocks of rows

$$B(\delta s_{i1}, \ldots, \delta s_{il}), \quad i = 1, \ldots, N.$$

Note that the entries of $E(\delta)$ are $o(\delta^q)$ and that $C(\delta) = C \cdot R(\delta)$, where $R(\delta)$ is a diagonal matrix containing powers of δ of degree at most q.

Let σ_L denote the smallest singular value of a matrix L (i.e., σ_L^2 is the smallest nonzero eigenvalue of $L'L$). Applying repeatedly the general fact that (for the Euclidean norm) $\|Lz\| \geq \sigma_L \|z\|$, we obtain, for $0 \leq \delta \leq 1$,

$$\|C(\delta) Dz\| \geq \sigma_C \sigma_D \delta^q \|z\| \tag{4.13}$$

for all vectors z. Let $\alpha := \sigma_C \sigma_D > 0$. Then, for all z,

$$\|(C(\delta) D + E(\delta)) z\| \geq (\alpha - \delta^{-q} \|E(\delta)\|) \delta^q \|z\|. \tag{4.14}$$

For δ small enough, $\alpha - \delta^{-q}\|E(\delta)\| > 0$, and so $C(\delta)D + E(\delta)$ has rank n. Thus there is a $\Delta > 0$ such that, whenever $0 < \delta < \Delta$,

$$\mathrm{d}\eta^{\mu^1 \cdots \mu^N}(x) = C(\delta) D + E(\delta) \tag{4.15}$$

has rank n, for $\mu^i := u^i_{\sigma_i}$, $\sigma_i := \delta^{n_i}$. This establishes the sampled ORC property, as desired. □

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