On Linear Systems and Noncommutative Rings

by

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ABSTRACT

This paper studies some problems appearing in the extension of the theory of linear dynamical systems to the case in which parameters are taken from noncommutative rings. Purely algebraic statements of some of the problems are also obtained.

Through systems defined by operator rings, the theory of linear systems over rings may be applied to other areas of automata and control theory; several such applications are outlined.

Introduction. The algebraic theory of linear constant systems with coefficients over an arbitrary field has gone through major developments during the past decade. A comprehensive account is given in Kalman, Falb and Arbib [9, Chap. 10]. Since a considerable number of the results remain valid for the case of commutative rings, it is natural to develop the theory in this direction. A systematic study of the realization problem over commutative rings was initiated by Rouchaleau [16] and Rouchaleau, Wyman and Kalman [17]. An exposition may be found in Eilenberg [4, Chap. 16], or Sontag [21].

The situation with noncommutative rings is completely different, however, essentially because the Cayley-Hamilton theorem fails to hold. Part B of the present paper grew out of an effort to isolate the *exact* condition(s) for extending the commutative theory. These conditions are further studied from a purely algebraic viewpoint in Sontag [19] which may be viewed as a complement to this work. Part C of the paper shows that the situation is not at all surprising when viewed in the context of automata theory and rational power series.

Why study systems over arbitrary rings? Various answers are possible. As pointed out in the work of Rouchaleau, it is natural to study systems which have integer parameters, because integers are more "natural" for computers than real numbers. Other reasons are the design of error-correcting codes, the study of partial difference equations, and the consideration of certain systems defined on groups, as indicated and developed in the case of codes in Johnston

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[7]. Another application, at least at the level of an analogy, is the global study of time-varying control systems (Kamen [11]). Perhaps equally important is the hope that the study and understanding of such linear systems may be eventually useful for the formulation of more general, i.e., nonlinear, theories. It is also of general scientific interest to know the ultimate limits of linear system theory.

An entirely different application is due to Kamen [10] (see also Newcomb [13]), where discrete time systems appear not as the object of study for their own sake but as a tool in the theory of certain infinite-dimensional continuous-time systems from the point of view of distributions. The model presented in Part D generalizes this approach. The examples given there should suffice to indicate how the introduction of more or less sophisticated operator rings (even non-commutative) can be useful in studying a broad class of systems which are "essentially" finite-dimensional. A detailed discussion of the commutative case will be given in Sontag [20], but the setup is introduced here in its full generality along with a general realizability result. The rings considered in Part B play a dominant role in this construction, but Part D can also be read independently if the reader is willing to restrict himself to considering commutative rings instead of arbitrary "$\mathscr{F}\mathscr{A}$ rings".

A different approach to systems which can be studied using rings of operators is given by Wyman [24].

A. Preliminaries

Although no deep algebraic results will be used, it is assumed the reader feels comfortable with the definitions and elementary properties of rings and modules. *All* rings will be assumed associative and with a unit element, and all modules unitary (i.e., 1 acts as the identity); unless otherwise stated, *module* will stand for "left module" and homomorphisms will be written on the right; for right modules they will be written on the left. If R is a ring, R^n will denote the free R-module in n generators, considered as a left module; consequently, matrices will be written to the right of the (row) vectors in R^n and the endomorphism ring End (R^n) will be identified with the matrix ring $R^{n\times n}$ by expressing transformations in the given basis.

Given an R-module M, $\langle g_1,\cdots,g_m\rangle_R$ will denote the (left) submodule generated by the elements $g_1,\cdots,g_m$ of M. Let R^{op} denote the ring which coincides with R as an abelian group but such that for all a, b the product ab is defined to be equal to ba in R. Any left R-module is naturally a right R^{op}-module and vice versa.

A subsemigroup T of the multiplicative semigroup of R is called a *left denominator set* in R if and only if for all r in R, t in T, (i) $Tr \cap Rt$ is nonempty, and (ii) if either $rt = 0$ or $tr = 0$, then $r = 0$. For such a T one can construct (as in Cohn [3, Chap. 0]) a ring containing R, the *localization* $T^{-1}R$ consisting of all quotients $t^{-1}r$. Defining $(t^{-1}r)\hat{g} := (tg)^{-1}(rg)$, a routine calculation proves:

(0.1) *If g: $R \to S$ is an injective ring homomorphism such that Tg consists of units in S, there is a unique extension of g to an inclusion $\hat{g}$: $T^{-1}R \to S$.*

For an R-module M, denote by $M((x))$ the set of (truncated) Laurent series in the indeterminate x, i.e., the set of all (formal) sums $\sum_{i \geq k} a_i x^i$ for some integer k and a_i in M. (We will omit mention of k when irrelevant.) We endow $R((x)$ with coefficientwise addition and with the (convolution) multiplication defined by

$$\left(\sum a_i x^i\right)\left(\sum b_j x^j\right) = \sum c_k x^k, \qquad \text{where } c_k := \sum_{i+j=k} a_i b_j.$$

Since the sum is clearly finite, $R((x))$ is a ring; it can also be seen as the localization of the usual power series ring $R[[x]]$ at $\{x^n, n \in \mathbf{N}\}$. Given an R-module M, define the $R((x))$-module $M((x))$ in the same way. For any $s = s(x) = \sum a_i x^i \neq 0$ in $M((x))$ define $o(s) := \min\{k : a_k \neq 0\}$, $\deg s := -o(s)$; $a_{o(s)}$ will be referred to as the "coefficient of least order (in x)" and if $s \in M[x^{-1}] \subseteq M((x))$, $a_{o(s)}$ will be also called the "leading coefficient (in x^{-1})". Both for notational convenience in Parts C and D and for historical reasons, we will usually take $x = z^{-1}$; then for example in $\mathbf{Z}((z^{-1}))$, $o(2z+1) = -1$, $\deg(2z+1) = 1$ and $a_{-1} = 2$, and the coefficient of least order of $z^{-1} + 5z^{-2}$ is 1. The units of $R((x))$ are the nonzero elements whose coefficient of least order is a unit in R. So if $p \in M[x^{-1}]$ and $q \in R[x^{-1}]$ is a unit, then $q^{-1}p$ is a well-defined element of $M((x))$ and can be clearly calculated by the so-called "long division" algorithm.

Observe that for any R-module M there exists a canonical inclusion $M[z] \to M((z^{-1}))$, giving rise to an exact sequence of $R[z]$-modules

$$0 \to M[z] \to M((z^{-1})) \xrightarrow{p} M((z^{-1}))/M[z] \to 0 \tag{0.2}$$

with p the canonical projection. $M((z^{-1}))/M[z]$ can be identified with $z^{-1}M[[z^{-1}]]$, the R-module of power series in z^{-1} with zero constant term.

For the linear systems results, this paper is based upon Kalman [8], Kalman, Arbib and Falb [9, Chap. 10], and Eilenberg [4, Chap. 16]: the first two references treat the case of R a field and the third R a commutative ring. All results which extend without restrictions to any R and are proved in an analogous way will be quoted freely from the above. The Laurent series formalism used here is borrowed from Wyman [23]. The reader should have no trouble in translating the language in the previous references into the one used here. Another useful source, especially for rationality, is Fliess [6].

We review briefly the fundamental notions. R will denote a ring, and U and Y will stand in each case for the *input* and *output modules*; U will be finitely generated (i.e., an *image* of some free module of finite rank R^m) and Y will be *included* in a free module R^p. This covers most cases of interest, e.g., *scalar* systems (when $U = Y = R$), and those in Part D. All of Section 1 holds without *any* restrictions on Y and Section 2 holds without restrictions on U. Unless otherwise specified, a fixed system of generators $e_1, \cdots, e_m$ is implicitly assumed for U (in other words, we really look at pairs (U, presentation of U)). We let Ω, Γ denote the $R[z]$-modules $U[z]$, $z^{-1}Y[[z^{-1}]]$.

A (*discrete-time, constant, R-linear dynamical*) system Σ consists of an R-module X (the *state space*) and homomorphisms (F, G, H) where $G: U \to X$, $F: X \to X$, and $H: X \to Y$. The *result* of Σ is the map $f: U((z^{-1})) \to Y((z^{-1}))$

given by $\sum u_i z^{-i} \mapsto \sum y_k z^{-k}$, where $y_k := \sum_{i \leq k-1} u_i G F^{k-i-1} H$. Then $\bar{f}$ is easily seen to be an $R((z^{-1}))$-homomorphism, so in particular $f = f_\Sigma$ given by the composition (see (0.2)) $\Omega \to U((z^{-1})) \xrightarrow{\bar{f}} Y((z^{-1})) \xrightarrow{p} \Gamma$ is an $R[z]$-homomorphism. (Any such f defines a unique $\bar{f}$, and $\bar{f}$ defines a unique f, so there are no ambiguities.) Giving a system Σ is equivalent to giving an $R[z]$-module X and $R[z]$-homomorphisms $\mathcal{G}$ and $\mathcal{H}$ where $zx := xF$, $az^n\mathcal{G} := aGF^n$ and $x\mathcal{H} := \sum xF^nHz^{-n}$. See the suggested references for details. In fact, $f = \mathcal{G}\mathcal{H}$. Denote by $g_i := e_iG$ the *accessible generators* of Σ. Then the *reachable part* of X is $X_r = \Omega\mathcal{G} = \langle g_1, \cdots, g_m \rangle_{R[z]} \subseteq X$; since X_r is F-stable, by restricting everything to X_r we get a subsystem Σ_r of Σ.

We call Σ *reachable in time k* if the restriction of $\mathcal{G}$ to the R-submodule of Ω consisting of the polynomials of degree less than or equal to $k-1$ is surjective. It is *observable in time k* provided that $xF^iH = 0$, $i = 0, \cdots, k$, implies $x = 0$. We call Σ *reachable* (*observable*) *in bounded time* if and only if it is reachable (observable) in time k for some k. Finally, we call $x \in X$ *controllable* if there exists k and $\omega \in \Omega$, deg $\omega < k$, such that $xF^k + \omega\mathcal{G} = 0$. We call Σ *controllable* when every $x \in X$ is; for each x the condition can be restated as:

(0.3) *There exist* $a_{ij} \in R$, $i = 1, \cdots, m$, $j = 1, \cdots, k-1$, *with* $xF^k = \sum_{i,j} a_{ij} g_i F^j$.

Controllable in bounded time means that some k as above can be found for Σ independently of x.

Conversely, any $R[z]$-homomorphism $f: \Omega \to \Gamma$ (for short, an *I/O map*) can be factorized as $\Omega \xrightarrow{\mathcal{G}} X_f \xrightarrow{\mathcal{H}} \Gamma$ with $X_f := \Omega/\ker f$, and $\mathcal{G}$, $\mathcal{H}$ the induced maps. Call the system Σ_f obtained from X_f, $\mathcal{G}$ and $\mathcal{H}$ the *canonical realization* of f. In general, any Σ such that $f_\Sigma = f$ is called a *realization* of f. The canonical realization is characterized up to a unique isomorphism by the properties of reachability and observability; we have (Eilenberg [4, p. 419]):

(0.4) *For any realization* Σ *of* f *there is a* (*unique*) *surjective* $R[z]$-*homomorphism* $\varphi: (X_\Sigma)_r \to X_f$ *such that* $G_\Sigma\varphi = G_f$.

If f has some *finite* realization Σ (i.e., X_Σ is finitely generated) f will be called *sequential*; we have (Eilenberg [4, p. 413]):

(0.5) *Assume that* U *is a free R-module and* f *is sequential. Then* f *has a realization* Σ *with* $X_\Sigma \simeq R^n$, *for some* n.

When X_f is finitely generated, we call f *canonically sequential.*

By definition, any canonically sequential f is sequential. The converse, although true for commutative rings, is not true in general. The rings which satisfy the converse turn out to be precisely those over which much of the standard linear system theory remains valid. These rings can be equivalently defined by a number of highly desirable system-theoretic properties, as we shall see. The property of observability in bounded time, on the other hand, is in a certain sense dual to the facts discussed above.

B. Discrete-Time R-Systems

(1) $\mathcal{F}\mathcal{A}$-Rings and Reachability.

(1.1) Definition. A ring R is called *finitely accessible* if and only if every sequential $f: \Omega \to \Gamma$ is even canonically sequential. (We say, for simplicity, that R is $\mathcal{F}\mathcal{A}$.)

The following holds without any assumption on the ring R.

(1.2) LEMMA. *Σ_r is reachable in bounded time if and only if X_r is finitely generated.*

Proof. If Σ_r is reachable in time n, then $g \in \langle g_1, \cdots, g_m, g_1F, \cdots, g_mF^n \rangle_R$ for all g in X_r. Conversely, if $x_h = \sum_{j=1}^{m} \omega_j^{(h)} g_j$, $h = 1, \cdots, n$, generate X_r, then Σ_r is reachable in time $k := \max\{\deg \omega_j^{(h)}, j = 1, \cdots, m, h = 1, \cdots, n\}$. □

(1.3) Definition. Given an R-module M with an R-endomorphism F and $g \in M$, we say that g is *recurrent* for F when there is some integer k and $a_0, \cdots, a_{k-1}$ in R such that $gF^k = \sum_{i=0}^{k-1} a_i g F^i$.

We omit mention of F if clear from the context.

Observe that this definition corresponds to controllability of g in a system for which $G = g$. An equivalent definition, via the natural $R[z]$-module structure on M, is that some monic q in $R[z]$ annihilates the element g, or that $\langle g \rangle_{R[z]}$ is finitely generated over R. As $X_r = \sum \langle g_i \rangle_{R[z]}$, each g_i being recurrent is a sufficient condition for X_r being finitely generated. Another sufficient condition is controllability:

(1.4) LEMMA. *If each g_i in Σ is controllable, then X_r is controllable in bounded time, and, in particular, finitely generated.*

Proof. For each $h = 1, \cdots, m$ we have, as in (0.3), $g_hF^{k_h} \in \langle g_1, \cdots, g_mF^{k_h-1} \rangle_R$. If $k > k_h$ for all h, then $g_hF^k \in \langle g_1, \cdots, g_mF^{k-1} \rangle_R$ for every h. Hence, by an easy induction, each $\langle g_h \rangle_{R[z]}$ and hence also X_r is in $\langle g_1, \cdots, g_mF^{k-1} \rangle_R$. So X_r is controllable in time k and finitely generated. □

(1.5) LEMMA. *Suppose in Σ each g_i is recurrent* [*controllable*]. *Then each accessible generator $\hat{g}_i$ of X_{f_Σ} is recurrent* [*controllable*].

Proof. By the definition of "accessible generator" and by (0.4), $\hat{g}_i = g_i\varphi$. Clearly $g_iF^j\varphi = \hat{g}_iF_f^j$ for all $j \geq 0$. Hence any relation $\sum_{i,j} a_j g_i F^j = 0$ is mapped into $\sum_{i,j} a_j \hat{g}_i F_f^j = 0$. □

(1.6.) THEOREM. *For any ring R the following are equivalent.* (*a*) *R is an $\mathcal{F}\mathcal{A}$ ring.* (*b*) *For every finitely generated R-module M and any F in* End (M) *any finite set $\{g_1, \cdots, g_m\}$ in M has a common recurrency.* (*c*) *Same as* (*b*) *with $m = 1$.* (*d*) *For any n and $F \in R^{n \times n}$ any g in R^n is recurrent.* (*e*) *Σ_r is reachable in bounded time for every finite R-system Σ.* (*f*) *For every finite R-system Σ reachable states are controllable.* (*g*) *Same as* (*f*) *with controllable in bounded time.* (*h*) *X_f is controllable for any sequential f.* (*i*) *Same as* (*h*) *in bounded time.*

Proof. We shall prove (a) ⇒ (d) ⇒ (c) ⇒ (b) ⇒ (e) ⇒ (g) ⇒ (f) ⇒ (i) ⇒ (h) ⇒ (a).

(a) ⇒ (d): Given F, g, define $\Sigma := (F, G, H)$ by $U := R$ (as a left module), $Y := R^n$, $X := R^n$, $G: 1 \mapsto g$, $H := 1_X$, F as given. Since Σ is observable, Σ_{f_Σ} is isomorphic to Σ_r. So $\langle g \rangle_{R[z]} = X_r \simeq X_{f_\Sigma}$. Since f_Σ is sequential, by (a), X_{f_Σ} is finitely generated. So g is recurrent.

(d) ⇒ (c): Let $\varphi: R^n \to M$ by an R-epimorphism with $g\varphi = g_1$ and $\varphi F = \hat{F}\varphi$ for some $\hat{F} \in R^{n \times n}$, $g \in R^n$, $n \in \mathbf{N}$. Then, a recurrency for g induces one for g_1.

(c) ⇒ (b): Consider R^m as a bimodule and define $\hat{M} := R^m \otimes_R M$. Then $\hat{M}$ is a finitely generated (left) R-module. Define $g := \sum_{i=1}^m e_i \otimes g_i \in \hat{M}$ and $\hat{F} := 1_{R^m} \otimes F \in \operatorname{End}(\hat{M})$. By (c), there exist k and $a_0, \cdots, a_{k-1}, a_k = 1$ in R such that

$$\begin{aligned} 0 &= \sum_{j=0}^{k} a_j g \hat{F}^j = \sum_{j=0}^{k} a_j \left(\sum_{i=1}^{m} e_i \otimes g_i \right) \hat{F}^j, \\ &= \sum_{j=0}^{k} a_j \left(\sum_{i=1}^{m} e_i \otimes g_i F^j \right), \\ &= \sum_{i=1}^{m} e_i \otimes \left(\sum_{j=0}^{k} a_j g_i F^j \right). \end{aligned}$$

Each term in parentheses on the last line must vanish, which proves (b).

(b) ⇒ (e): If each g_i is recurrent, apply (1.2).

(e) ⇒ (g): Given that Σ_r is reachable in time k, $gF^k \in \langle g_1, \cdots, g_m F^{k-1} \rangle_R$ for any $g \in X$. So Σ_r is controllable in time k.

(g) ⇒ (f): Trivial.

(f) ⇒ (i): Take a finite Σ realizing f. By (f) each g_i is controllable. Apply (1.5) and (1.4).

(i) ⇒ (h): Trivial.

(h) ⇒ (a): By (h) X_f is controllable. Applying (1.4) shows that X_f is finitely generated. Hence f is canonically sequential. □

Remark. Alternative characterizations of $\mathscr{F}\mathscr{A}$ rings can be given in terms of a Cayley-Hamilton type matrix condition as well as in terms of integral extensions of R (see Sontag [19, Theorem 1.3]).

(2) $\mathscr{F}\mathscr{O}$ Rings and Observability.

We begin by dualizing (1.6.e).

(2.1) Definition. A ring R is *finitely observable* if and only if every finitely generated and observable Σ is even observable in bounded time. (We say, for simplicity, that R is $\mathscr{F}\mathscr{O}$).

(2.2) *Observation.* For any R-modules, M, N and homomorphisms $T: M \to M$ and $H: M \to N$ the following are equivalent. (a) The descending chain $\{\bigcap_{i=0}^s \ker T^i H\}_{s \geq 0}$ has length k. (b) $\bigcap_{i=0}^{k-1} \ker T^i H \subseteq \ker T^k H$. (c) $\bigcap_{i=0}^{k-1} \ker T^i H$ is T-invariant.

(2.3) PROPOSITION. *For any ring R the following are equivalent. (a) R is $\mathscr{F}\mathscr{O}$. (b) For any finitely generated R-module M, any $p \in \mathbf{N}$, $F \in \operatorname{End}(M)$, and $H \in \operatorname{Hom}(M, R^p)$, condition (2.2.a) holds for some k. (c) For any $n \in \mathbf{N}$, $T \in R^{n \times n}$, and $h \in R^{n \times 1}$, condition (2.2.a) holds for some k.*

Proof. Since (b) trivially implies (c), we prove (c) $\Rightarrow$ (a) and (a) $\Rightarrow$ (b).

(c) $\Rightarrow$ (a): Given $\Sigma = (F, G, H)$ observable (and finite), choose n and an epimorphism $\varphi: R^n \to X$ with $\varphi F = T\varphi$ for some $T \in R^{n \times n}$. If $Y \subseteq R^p$, let h_j, $j = 1, \cdots, p$, be the composition of φH with the projection on the jth coordinate. Let $\hat{k}$ be the maximum of the lengths of the chains $\{\bigcap_{i=0}^{s} \ker T^i h_j\}_{s \geq 0}$. Suppose $g \in X$ is such that $gF^iH = 0$ for $i = 1, \cdots, \hat{k}$. Choose $x \in R^n$ with $x\varphi = g$. Then $xT^ih_j = 0$, $i = 1, \cdots, \hat{k}$, $j = 1, \cdots, p$. It follows that $gF^iH = 0$ for every $i = 1, 2, \cdots$. By observability of Σ, this implies $g = 0$. So Σ is observable in time $\hat{k}$.

(a) $\Rightarrow$ (b): Given F, H as in the hypothesis, let $M: = \bigcap_{i=0}^{\infty} \ker F^iH$ and observe that M is F-invariant and $M \subseteq \ker H$. So if $X: = R^n/M$ and $\varphi: R^n \to X$ is the canonical projection, φ induces F_Σ, H_Σ with $\varphi F_\Sigma = F\varphi$ and $\varphi H_\Sigma = H$. Define $\Sigma: = (F_\Sigma, 0, H_\Sigma)$. Clearly Σ is an observable finite system, hence observable in time k. So k then satisfies (2.2.a). □

The class of $\mathscr{F}\mathscr{O}$ rings is very large, but except for the case of rings without zero-divisors, the condition is quite independent of the $\mathscr{F}\mathscr{A}$-condition. From the usual "duality" of linear system theory, it would be expected that a statement such as "R is an $\mathscr{F}\mathscr{A}$ ring if and only if R^{op} is an $\mathscr{F}\mathscr{O}$ ring" would hold; it can be proved however that just the "only if" part holds in general (Sontag [19, Sections (3) and (4)]).

For all rings embeddable in a division ring the k in (2.2) exists and is in fact bounded by the cardinality of any set of generators for M. Such rings are an extremely simple example of "effectively $\mathscr{F}\mathscr{O}$ rings", defined below.

(2.4) Definition. A ring R is called *effectively* $\mathscr{F}\mathscr{A}$ (resp. $\mathscr{F}\mathscr{O}$) if given any finite system Σ over R a bound on reachability time (resp. observability time) exists and is effectively calculable.

It is of course implicit in the definition that R is itself effectively describable and that "given" means in terms of, say, matrices. For any Σ, $f_\Sigma = 0$ if and only if $e_jGF^iH = 0$ for $j = 1, \cdots, m$ and all $i \geq 0$. If R is $\mathscr{F}\mathscr{O}$, it is enough to check the condition for $i = 1, \cdots, k$. If R is $\mathscr{F}\mathscr{A}$, then there is a k such that e_jGF^i for $i > k$ is a combination of terms with $i \leq k$. So in any case we have

(2.5) PROPOSITION. *If R is either effectively $\mathscr{F}\mathscr{A}$ or $\mathscr{F}\mathscr{O}$, then the equality of behavior of a pair of systems is decidable.*

(2.6) *Remark.* The decidability question in (2.5) is by no means trivial. In fact, it is intimately related to an outstanding open problem in the developing theory of Lindenmayer systems, namely the so-called equivalence problem (see Lindenmayer [12], Nielsen [14], Salomaa [18]).

The relation with the latter is obtained by giving a construction which embeds Lindenmayer systems (essentially *nonlinear* objects) in discrete-time R-systems.

Let $X = \{x_1, \cdots, x_n\}$ be an alphabet, and X^* the free monoid generated by it. Define a product in the set $(X^*)^n$ by $(u_1, \cdots, u_n)\cdot(v_1, \cdots, v_n): = (u_1(v_1, \cdots, v_n), \cdots, u_n(v_1, \cdots, v_n))$, where $u_i(v_1, \cdots, v_n)$ is the word obtained by replacing each occurrence of x_j in u_i by v_j. This operation is clearly associative, and admits the identity $(x_1, \cdots, x_n)$. We denote the monoid so obtained by M.

Choosing (and fixing) an arbitrary field k, we denote the corresponding monoid algebra $k[M]$ by R.

Now let $\delta: X^* \to X^*$ be an arbitrary monoid homomorphism, which we shall write (exceptionally) on the left of its argument. Let π be in X^*. We shall interpret δ as the production rule and π as the axiom of a Lindenmayer system. Consider the R-system given by $U = X = Y: = R$, $H: =$ identity, $F: =$ map sending each r in R to $r\cdot(\delta(x_1),\cdots,\delta(x_n))$, and $G: r \mapsto r\cdot(\pi,\cdots,\pi)$. Then the "impulse response" of the system, i.e., the sequence $\{GF^iH\}$, is the sequence $\{(\delta^i(\pi),\cdots,\delta^i(\pi)),\ i = 0, 1,\cdots\}$. Hence equivalence of Lindenmayer systems corresponds to equal input/output behavior of the associated R-systems.

Finally, as a "bonus" from the construction in the last paragraph, many generalizations of Lindenmayer systems are immediate. For example, taking $k: =$ real numbers, systems with a convex combination of G's as before would correspond to only knowing a (finite) distribution of probabilities of axioms; a similar stochastic interpretation would hold with respect to F. Quotients of R, on the other hand, would correspond to certain cell configurations determining the extinction of the process.

C. Rationality

(3) Rational Power Series.

Let R be a ring, x an indeterminate and $R((x))$ as in Part A. A subset S of $R((x))$ is called *rationally closed* if and only if the inverse of any invertible element of S is again in S.

(3.1) Definition. The ring $R^*[(x)]$ of *rational Laurent series* in x over R is the smallest rationally closed subring of $R((x))$ containing $R[x]$. The subring of $R^*[(x)]$ consisting of series s with $o(s) \geq 0$ (the ring of *rational power series* in x) is denoted by $R[(x)]$; the subset of series s with $o(s) \geq 1$ (the "causal" power series) is denoted by $R_c[(x)]$.

(3.2) *Remark.* It is easy to verify that the definition of $R[(x)]$ given here coincides with the standard one as the smallest rationally closed subring of $R[[x]]$ including the polynomials. In any case, observe that $s \in R((x))$ is rational if and only if it may be expressed as products and sums of polynomials and inverses of such combinations when the coefficients of least order are units in R. In the case R is a commutative field, $R^*[(x)]$ reduces to $R(x)$, the field of rational functions.

The following is a particular case of a result due to Schützenberger (see Eilenberg [4, Chap. 7, Theorem 5.1]; the result is true without the assumption that R is commutative).

(3.3) *If* $s \in R((z^{-1}))$, *then* $s \in R_c[(z^{-1})]$ *if and only if there is a sequential* f *with* $U = Y = R$ *and* $s = 1f$.

We shall frequently identify any s as in (3.3) with the corresponding f.

If R is commutative terms can be regrouped and common denominators chosen so that any rational power series is a quotient of two polynomials. This property will give us yet another characterization of $\mathscr{F}\mathscr{A}$ rings.

(3.4) Definition. A series $s \in z^{-1}M[[z^{-1}]]$, M an R-module, is called *elementary rational* if it can be expressed in the form $q^{-1}p$ with $q \in R[z]$ monic and $p \in M[z]$.

For such an $s = q^{-1}p$, clearly deg $q <$ deg p. A standard argument shows that $s \in z^{-1}M[[z^{-1}]]$ is elementary rational if and only if its coefficients satisfy a monic recurrence relation. Equivalently, s is elementary rational if the columns of its "Hankel matrix" generate a finitely generated R-module.

An input/output map f: $\Omega \to \Gamma$ with $Y \subseteq R^p$ gives rise to a matrix of series with elements $f_{ij} := e_i f \pi_j \in z^{-1}R[[z^{-1}]]$, where π_j, $j = 1, \cdots, p$ indicates the result of applying the coordinate maps for Y to the coefficients of a series in $z^{-1}Y[[z^{-1}]]$. As in the commutative case, f is sequential if and only if each f_{ij} is rational.

By definition of the (quotient) $R[z]$-module structure on Γ, $qe_i f \in Y[z]$ if and only if $qe_i f = 0$ in Γ. For the canonical realization, $f = \mathscr{G}\mathscr{H}$ and $\mathscr{H}$ is injective, so that this last condition is equivalent to $qg_i = 0$. Hence we have

(3.5) LEMMA. *Let f: $\Omega \to \Gamma$ be an input/output map and $q \in R[z]$. Then $q(e_i f)$ is a polynomial if and only if q annihilates g_i in X_f.*

(3.6) THEOREM. *For any ring R, the following are equivalent. (a) R is an $\mathscr{F}\mathscr{A}$ ring. (b) Any input/output map f is sequential if and only if all f_{ij} are elementary rational. (c) Same as (b), with a common denominator. (d) The set T of monic polynomials in z is a left denominator set in $R[z]$ and $T^{-1}(R[z])$ can be naturally identified with $R^*[(z^{-1})]$.*

Proof. We shall prove (a) $\Leftrightarrow$ (c) and (c) $\Rightarrow$ (b) $\Rightarrow$ (d) $\Rightarrow$ (c).

(a) $\Leftrightarrow$ (c): For any polynomial q in $R[z]$, one has $qe_i f \pi_j = 0$ for $j = 1, \cdots, p$ if and only if $qe_i f = 0$. By (3.5), this is equivalent to q annihilating every g_i, so the equivalence of (a) and (b) in (1.6) gives the result.

(c) $\Rightarrow$ (b): Trivial.

(b) $\Rightarrow$ (d): If s in $R^*[(z^{-1})]$ is nonzero, let $n \geq 0$ be an integer such that $z^{-n}s$ is in $R_c[(z^{-1})]$. By (3.3), $z^{-n}s = 1f$ for some sequential f; so, by (b), it is elementary rational, say equal to $q^{-1}p$ with p in $R[z]$, q in T. Then $s = q^{-1}(z^n p)$. In particular, for every nonzero p in $R[z]$ and q in T, pq^{-1} is in $R^*[(z^{-1})]$. So it follows that there exist p_1 in $R[z]$ and q_1 in T such that $pq^{-1} = q_1^{-1}p_1$. Hence $q_1 p = p_1 q \in Tp \cap Rq$. Thus T is a left-denominator set, and the inclusion of $R[z]$ in $R^*[(z^{-1})]$ extends to an isomorphism as in (0.1).

(d) $\Rightarrow$ (c): If f is sequential, then each f_{ij} is in $R^*[(z^{-1})]$. Then (d) says that each f_{ij} is elementary rational. By Cohn [3, Exercise 0.5.2]—or by an argument similar to our proof of (1.6.c)—all f_{ij} can be written with a common denominator. □

(3.7) *Remark.* It clearly follows that T is a *right* denominator set when R^{op} is an $\mathscr{F}\mathscr{A}$ ring. Since the property of being $\mathscr{F}\mathscr{A}$ is left-right independent (Sontag [19, Section (4)]), it is perfectly possible that in a ring R *every* power series in $R^*[(z^{-1})]$ be expressible as $q^{-1}p$ while, for some, *no* representation of the form pq^{-1} exists. In fact, R may even be a (non-commutative) left principal-ideal domain, and still have this property. □

(3.8) *Remark.* There is a natural generalization of the Fatou-type problems which appear in the theory of systems over commutative integral domains (see for instance Eilenberg [4, Chap. 16, Section 12], Rouchaleau [16], Fliess [6]). Assume that R is a left Noetherian ring with no zero divisors. Let K be any division ring containing it (for example, its field of left fractions, which exists because of the Noetherian condition). The result is

$$(*) \qquad R[[z^{-1}]] \cap K[(z^{-1})] = R[(z^{-1})].$$

This is equivalent to the following. Let $\{A_i\}$ be a sequence of $m \times p$ matrices over R. In particular, they are matrices over K, and as such define a corresponding input-output map. Assume that there is a realization over K, i.e., a triple of matrices (F, G, H), of appropriate dimensions, with entries in K, such that $A_n = GF^nH$ for all n. Then $(*)$ asserts that there are matrices F', G', H' with entries in R (possibly of higher dimensions) such that $A_n = G'F'^nH'$ for all n. The utility of such a result lies in that the techniques involving determinants (over K) of Hankel matrices become available (see Fliess [6, II.1.d], and Artin [1, Chap. 4] for determinants in the noncommutative case).

The proof of $(*)$ is fairly straightforward. The inclusion of $R[(z^{-1})]$ in the left side is obvious. Take now $s = \sum a_i z^{-i}$ in $R[[z^{-1}]]$ and suppose it is rational over K. Being a division ring, K^{op} is an $\mathcal{FA}$ ring. So there are an n and a *right* recurrence

$$(**) \qquad a_h = \sum_{u=1}^{n} a_{h-u}k_u \qquad \text{for all } h \geq n$$

with k_u in K. Call M the (left) R-submodule of R^n generated by the set of all $x_i = (a_i, a_{i+1}, \cdots, a_{i+n-1})$, $i = 0, 1, \cdots$. By the Noetherian assumption, M is finitely generated. In particular, there is an m with $x_m = \sum_{i<m} r_i x_i$. Then, when $0 \leq j \leq n-1$,

$$(***) \qquad a_{m+j} = \sum_{i=0}^{m-1} r_i a_{i+j}.$$

Assuming that $(***)$ is true for all $j < t$ (where $n \leq t$), we show it is also true for $j = t$. Indeed, by $(**)$, $a_{m+t} = \sum_u a_{m+t-u}k_u = \sum_u (\sum_i r_i a_{i+t-u})k_u = \sum_i r_i (\sum_u a_{i+t-u}k_u) = \sum_i r_i a_{i+t}$, as wanted. Then $(***)$ is true for all j, and s is elementary rational over R. So $(*)$ holds.

Observe that the only property of K needed for the proof is that K^{op} is $\mathcal{FA}$.

D. Applications to Finitary Linear Systems

(4) Admissible Triples.

(4.1) Definition. Given a right B-module A, a B-endomorphism α of A is called *transcendental* (over B) if and only if no expression of the form $b_n\alpha^n + b_{n-1}\alpha^{n-1} + \cdots + b_0$, with all b_i in B, $b_n \neq 0$, acts as the zero map on A.

Observe that this implies that every $b \neq 0$ acts as a non-zero map, i.e., that A is a faithful B-module. The definition of transcendental is equivalent to requiring that the smallest subring of End (A) containing both B and α be in fact (isomorphic to) the polynomial ring $B[\alpha]$.

(4.2) Definition. Assume that A, B, α are as in (4.1). The triple (A, B, α) is called *admissible* if and only if the inclusion of $B[\alpha]$ into End (A) extends to an inclusion of $B^*[(\alpha^{-1})]$ into End (A).

In other words, admissible (A, B, α)'s correspond to the faithful $B^*[(\alpha^{-1})]$-modules. In practice, however, A, B, α are usually given, and it must be checked whether α induces such a representation. In the case of the rings introduced in Section B, the following is a useful criterion.

(4.3) LEMMA. *If A, B, α are as in (4.1) and B is an $\mathscr{F}\mathscr{A}$ ring, a necessary and sufficient condition for admissibility of (A, B, α) is that all monic polynomials $\alpha^n + b_{n-1}\alpha^{n-1} + \cdots + b_0$ be bijections.*

Proof. Necessity is clear, because monic polynomials are units in $B^*[(\alpha^{-1})]$.

Sufficiency. Denote by i the inclusion of $B[\alpha]$ in End (A). Then, by (0.1) and Theorem (3.6), i extends to an inclusion of $B^*[(\alpha^{-1})]$ into End (A). □

The condition in (4.3) is intended to be interpreted as stating the existence and uniqueness of solutions of equations (differential, difference, etc.). This is precisely the sense in which we are interested in admissible triples.

When B is an algebraically closed field, a triple (A, B, α) is admissible if and only if the spectrum of α (as an element of the B-algebra End (A)) is empty; then for many cases we have a negative result. For example, if $B = \mathbf{C}$, if A is a Banach space, and α is continuous, (A, B, α) can never be admissible (see Taylor [22, Theorems 4.7-B, C and 5.2-B]).

Note also that (4.3) still holds if B^{op}, instead of B, is $\mathscr{F}\mathscr{A}$ (apply the left-right dual of (0.1) and (3.7)). The lemma is *not* true, however, without some $\mathscr{F}\mathscr{A}$ condition, and a counterexample can be constructed using for B a ring of noncommutative polynomials.

The notion of admissibility can be made less restrictive, but it involves increasing somewhat the complexity of the treatment. The generalization is analogous to the replacement of differential by integral equations.

(4.4.) Definition. If (A, B, α) is admissible, a map $f\colon A \to A$ is called (strictly causal) *rational* if it is (via the given inclusion) in $B_c[(\alpha^{-1})]$. A map $f\colon A^m \to A^p$ is *rational* if and only if each coordinate map is.

The relation given by f in (4.4) can almost always be interpreted as a polynomial (difference, differential, difference-differential,$\cdots$) equation:

(4.5) PROPOSITION. *Assume that (A, B, α) is admissible and that B^{op} is an $\mathscr{F}\mathscr{A}$ ring. Given m, p, consider an equation $yQ = uP$ with $P \in B^{m\times p}[\alpha]$, $Q \in B^{p\times p}[\alpha]$ and such that the leading coefficient of Q is a unit in $B^{p\times p}$ and* deg *$Q >$* deg *P. Then there is a unique solution y for each $u \in A^m$ which defines a rational $f\colon A^m \to A^p$. Conversely, for any such f there are P, Q as before such that $uf = y$ if and only if $yQ = uP$; moreover, Q may be chosen as a scalar matrix. These properties characterize $\mathscr{F}\mathscr{A}$ rings.*

Proof. If Q is invertible in $B^{p\times p}$, the equation is equivalent to $y = uPQ^{-1}$, with $PQ^{-1} \in (B_c[(\alpha^{-1})])^{m\times p}$ by the degree requirement, and hence corresponding to an f as in (4.4). Conversely, if B^{op} is an $\mathscr{F}\mathscr{A}$ ring, any rational $f\colon A^m \to A^p$ can be written as Pq^{-1} with $q \in B[\alpha]$, using (3.6) and (3.7). □

(4.6) *Examples.* We give a few examples of admissible triples, each corresponding to a particular type of equation. They will be used later in the definition of different kinds of systems.

(a) *Difference equations.* Let B be an arbitrary ring, $A := B((z^{-1}))$, $\alpha := z$. The structure of a $B^*[(z^{-1})]$-module on A is the one given by right multiplication. This kind of "standard" triple will be useful in the study of arbitrary triples.

(b) *Differential equations.* Let $A := C_+^\infty$ consist of all infinitely differentiable functions $f: \mathbf{R} \to \mathbf{R}$ with support bounded to the left. Take $B := \mathbf{R}$ and $\alpha :=$ derivative operator. Given such a function, assume $f(t) = 0$ for all $t < t_0$. From well-known theorems on ordinary differential equations, given real numbers $b_0, \cdots, b_{n-1}$, there is a unique g (also in A) satisfying $g^{(n)} + b_{n-1}g^{(n-1)} + \cdots + b_0 g = f$ and $g(t) = 0$ for all $t < t_0$. This proves that (A, B, α) satisfies the condition stated in (4.3).

The restriction of *infinite differentiability* which appears in this and the other "continuous-time" examples *is not at all essential*, and can in fact be replaced by local (Lebesgue) integrability. This involves relaxing the definition of admissibility as indicated above.

(c) *Delay-differential equations using generalized functions.* Let A be the algebra of real-valued distributions with support bounded to the left with convolution product, $B := \mathbf{R}[\delta_{a_1}, \cdots, \delta_{a_r}]$ for $a_1, \cdots, a_r \in \mathbf{R}$ and $\alpha := \delta_0'$. (δ_a is the Dirac distribution centered at $a > 0$). These systems were defined and studied by Kamen [10]. Admissibility follows from his Theorem 1 and Proposition 4.

(d) *Delay-differential equations* again. Let $A := C_+^\infty$, $\sigma_a :=$ shift operator, $B := \mathbf{R}[\sigma_{a_1}, \cdots, \sigma_{a_r}]$ and $\alpha :=$ derivation. Admissibility is a consequence of the theory of delay-differential (or "hereditary") systems by setting initial conditions to zero on a conveniently chosen interval. This approach permits one to speak of the "input at time t" and obtain for systems explicit criteria for reachability, etc. (see Section (7) below). Of course, it does not allow consideration of irregular inputs as does (c). The setup can be extended to allow for more general functional-differential equations.

(e) *Delays.* Let $A := \{$all continuous $f: \mathbf{R} \to \mathbf{R}$ with support bounded to the left$\}$, $B := \mathbf{R}$ and $\alpha :=$ 1-second shift. Admissibility is easy. More generally, α can be a time-varying delay.

(f) *Certain "infinite-dimensional" difference equations.* Here let M be a faithful right B-module, where B is an arbitrary ring. Denote $A := M((z^{-1}))$, and $\alpha := z$. Then $A := M((z^{-1}))$ is naturally a $B((z^{-1}))$-module. Moreover, A is a *faithful module.* In fact, since M is a faithful B-module, $b_k \neq 0$ implies the existence of some $m_k \in M$ such that $m_k b_k \neq 0$. Hence if $b = \sum_{i \geq k} b_i z^i$ (with $b_k \neq 0$), then $m_k b \neq 0$. In particular, there is an inclusion of $B^*[(\alpha^{-1})]$ into End(A). So admissibility follows. This example and the next one will be discussed further in Section (6).

(g) *Certain "infinite-dimensional" differential equations.* Let X be a real (or complex) Banach space, and B an algebra of linear bounded operators on X. Define A as the additive group of all infinitely differentiable functions $\mathbf{R} \to X$ with support bounded to the left. Let α be the derivative operator. As in example (b), there is existence and uniqueness of solutions. So, if B is an $\mathcal{FA}$ ring, (A, B, α) is admissible.

(h) *Substitution.* Assume that (A, B, α) is admissible, with B an $\mathscr{F}\mathscr{A}$ ring and s an element of $B_c[(\alpha^{-1})]$ which is a unit in $B^*[(\alpha^{-1})]$. A monic polynomial in s^{-1} with coefficients in B is clearly an invertible element of $B^*[(\alpha^{-1})]$. It follows that (A, B, s^{-1}) is also admissible. In the particular case of example (f), s can be more arbitrary, i.e., not necessarily rational, but the same conclusion follows.

(i) *Variations on the above.* One possibility is to allow for esoteric time sets (this might be useful in some studies of periodic behaviors). Another is to give non-standard (in the sense of logic) models of the above: for example, a model of (b) where there is an "infinitely differentiable" impulse "function"; first-order statements about (b) would then be true also for the non-standard model. The use of "time-varying" operators for α (for example, $(f\alpha)(t) := e^{-t}f'(t)$, or an arbitrary Volterra operator) is also allowed, provided that α commutes with elements of B (e.g., in (b)). Approaches formalized in terms of Mikusiński's operational calculus can also be given.

(5) Finitary Systems.

We shall need the following result due to Richard [15] (it is, of course, trivial when B is commutative).

(5.1) *For any ring B the restriction of the canonical isomorphism between* $(B((x)))^{n\times n}$ *and* $B^{n\times n}((x))$ *gives a canonical isomorphism between* $(B^*[(x)])^{n\times n}$ *and* $(B^{n\times n})^*[(x)]$.

(5.2) Definition. Given an admissible triple (A, B, α), a (constant, linear, finitary) (A, B, α)-*system* is a triple of matrices (F, G, H), where $F \in B^{n\times n}$, $G \in B^{m\times n}$, and $H \in B^{n\times p}$. The *input* map $L: A^m \to A^n$ is the group homomorphism which assigns to each $\omega \in A^m$ the unique solution $x \in A^n$ of the system of equations $x\alpha = xF + \omega G$. The *result* is $LH: A^m \to A^p$.

(5.3) *Remark.* The input map is well defined. By (5.1)

$$(B^{n\times n})^*[(\alpha^{-1})] = (B^*[(\alpha^{-1})])^{n\times n} \subseteq (\mathrm{End}\,(A))^{n\times n} = \mathrm{End}\,(A^n).$$

Since $\alpha I - F$ is a unit in $(B^{n\times n})^*[(\alpha^{-1})]$, clearly $L = G(\alpha I - F)^{-1}$ is well defined. □

Everything is set up so that we can state

(5.4) THEOREM. *Consider a fixed admissible (A, B, α). Then the result of any (A, B, α)-system is rational. Conversely, every rational map $f: A^m \to A^p$ can be realized as the result of such a system.*

Proof. Observe that the faithfulness of the action of $B[(\alpha^{-1})]$ on A implies the equivalence of the given problem with that of the correspondence between power series of the form $\sum_{i\geq 0} GF^iH\alpha^{-i-1}$ (F, G, H being B-matrices) and those in $B_c[(\alpha^{-1})]^{m\times p}$. Since the above power series correspond to discrete B-systems with free state modules, (3.3) and (0.5) give the theorem. □

(5.5) *Observation.* Far from being an abstract result, the preceding allows us to *explicitly compute* realizations, provided such a method exists for B. See, for example, Rouchaleau [16] and Kamen [10].

(5.6) *Discussion.* Various types of interpretations are possible. For the most usual one, think of elements of A as defining the time-evolution of inputs, "states" and outputs. The elements of B would correspond to the input-output functions of certain components, while α^{-1} denotes the input-output behavior of a fixed system Σ_0. An (A, B, α)-system will then correspond to a set of n copies of the basic system Σ_0, interconnected (and feeding back) via adders and operators in B. The inputs act via similar connections, and outputs are obtained by forming suitable combinations of the outputs of the Σ_0's. Then (5.4) gives a characterization of the input-output behavior of the composite system.

Via the same reasoning, the results in "minimal" realizations over B can be used to find systems with equal input-output relations but requiring the fewest possible number of copies of Σ_0.

Under this interpretation, for example, (4.6.a) corresponds to the usual discrete-time systems over B, while (4.6.b) defines the linear continuous-time systems used in control theory.

Example (4.6.h) allows "hierarchical" considerations: once a system is defined and optimized as above, it can be used as a "building block" for new, more complex, systems. For "completely integrally closed" domains it is possible to give an algorithm to decide whether a given (A, B, α)-system can be simulated by using any number of components of a fixed type; this algorithm is a direct application of the theory of partial realizations.

A different "hierarchical" reasoning is also possible, by iterating on B instead of α. Indeed, assume that B is itself of the form $C[(\beta^{-1})]$, where (A, C, β) is admissible. Then, once that an input-output map on $B_c[(\alpha^{-1})]$ has been realized over B, the procedure can be iterated by realizing each of the entries of F, G, H over the ring C. The precise description of this process corresponds to the theory of *n-component* linear systems, which we sketch below for the case $n = 2$. Assume that α, β are two commuting C-endomorphisms of A, algebraically independent over C. A suitable definition of admissibility allows to embed in End (A) the ring of rational power series in two variables. Now systems correspond to equations of the type

$$X_1\alpha = X_1F_{11}+X_2F_{12}+\omega G_1$$

$$X_2\beta = X_1F_{21}+X_2F_{22}+\omega G_2$$

$$Y = X_1H_1+X_2H_2$$

with ω in A^m, X_1 in A^r, X_2 in A^s and all matrices over C. The realization theorem (5.4) generalizes immediately, since the set of rational power series over C in two variables α^{-1}, β^{-1} can be identified with $B[(\alpha^{-1})]$, where $B = C[(\beta^{-1})]$. This two-step realization procedure gives realizations of rather small dimensions. An example of the above kind of system is given by the most general kind of delay-differential systems, where α denotes differentiation and β is the delay. Another example is given by two-dimensional filters, i.e., systems with "time set" $\mathbf{Z} \oplus \mathbf{Z}$, where α and β correspond to the two possible shifts. □

(6) Some Examples: "Cellular" Systems or Partial Difference Equations.

In this section we shall assume for simplicity that all systems considered are *scalar* (i.e., $m = p = 1$). The interpretations can be easily extended to the general case.

(6.1) *Example.* Let R be a ring and C a group. Denote by M_1 the set of all functions $C \to R$, with the pointwise R-module structure. Let $B: = R[C]$ be the group ring of C over R. There is a natural action of B on M_1, given as follows. If $f: C \to R$ and $B = \sum r_g g$, define $f \cdot b: = h$, where $h(g) = \sum_{x \in C} (xf) \cdot r_{x^{-1}g}$. Applying $b \in B$ to the characteristic function of the identity of C, it is clear that M_1 is a faithful B-module. The construction in example (4.6.f) is then applicable, giving an admissible triple (A_1, B, α).

We may interpret (6.1) as follows. Assume (F, G, H) is an (A_1, B, α)-system. Elements of M_1^n denote the states of subsystems (or "cells") indexed by elements of C. All these subsystems have the same transition functions, and the next state of each depends on the present states of those cells in a certain relative position, weighted according to F. Inputs are simultaneously applied, at each instant, to every point in the pattern given by C. Each input has an effect over cells in a fixed "neighborhood" of subsystems, according to the form of G. Outputs correspond to an assignment of values to each point of C as a function of the states of the cells which lie in a certain neighborhood of the point. Thus (6.1) models cellular automata with linear local transitions.

As a particular case, assume $C: = \mathbf{Z}$ and $R: = \mathbf{R}$, the real numbers. Then an element of M_1^n indicates an assignment of n-vectors to each cell of a doubly infinite tape, and if we write $B = \mathbf{R}[x, x^{-1}]$, then x, x^{-1} act as the right and left shift on the tape. Elements of $A_1^n = (M_1((z^{-1})))^n \simeq M_1^n((z^{-1}))$ indicate the temporal behavior of states.

The situation in (6.1) applies also to space-time discretizations of certain kinds of linear partial differential equations. For an equation evolving in the plane, one would choose $C: = \mathbf{Z} \oplus \mathbf{Z}$; for the 2-torus, $C: = \mathbf{Z}_n \oplus \mathbf{Z}_n$ (for different n's), and so on.

(6.2) *Example.* Let $R: = \mathbf{R}$, $C: = \mathbf{Z}$, and let B, M_1 be as in (6.1). Define $M_2: = l^2(\mathbf{Z})$, the set of all f in M_1 with $\sum_{n=-\infty}^{\infty} |nf|^2 < \infty$. It is again trivial to verify faithfulness. The shifts are continuous operators, so (4.6.g) applies to give an admissible triple (A_2, B, α). An analogous construction holds for different groups C.

This setup applies to the case of *space* discretizations of partial differential equations, as in Brockett and Willems [2]. The same comments as for (6.1) apply here, with the only difference that the evolution now takes place in continuous-time.

(6.3) *Example.* Let R, C be defined as in (6.1), but restricted to the B-submodule M_3 of M_1 consisting of the functions $f: C \to R$ which are zero almost everywhere. Again (4.5.f) applies to give an admissible (A_3, B, α).

This example models the case in which at every instant only a finite number of cells are "active", and where controls can be applied simultaneously at a finite number of points only.

Observe that M_3 is isomorphic to B (acting on itself by right multiplication). So (A_3, B, α)-systems correspond to B-systems, in the sense of Part B. The study of partial difference equations via B-systems has been suggested by Johnston [7] and Wyman [24], among others.

(6.4) *Observations.* Assume C is finite in the previous examples. Then $M_1 = M_3$. Also, $M_1 = M_2$ for $R = \mathbf{R}$. Moreover, M_1 can be identified with R^n, where n is the cardinality of C, and the elements of B, as R-linear maps $M_1 \to M_1$, can be represented by $n \times n$ matrices over R. The present approach allows one to exploit the algebraic structure of the ring B.

(6.5) *Example.* A simple and natural noncommutative example of B appears when modelling the situation in which the contents of a cell c_0, after being acted upon by an element of B, are allowed to depend not only on previously stored values in a "neighborhood" of c_0, but also on stored values in a fixed distinguished set of cells. For simplicity take $G := \mathbf{Z}$ and distinguish the 0-th cell. More precisely, this amounts to taking B as the R-algebra generated by ($R[x, x^{-1}]$, π) where $\pi \in \text{End}\,(M_1)$ is the projection on the 0-th coordinate. The same setup is also useful in studying situations in which controls can only be applied at certain points of a system, for example, systems defined by equations of the type $x(t+1) = (xF)\,(t)+(u\pi)\,(t)$.

In all the cases in this section, the availability of a minimization algorithm for B-systems would give realizations where the state spaces of the cell-subsystems are smallest possible. In general, the advantages of the approach lie in the reduction of problems (realization, reachability, observability, etc.) to the corresponding problems for "finite-dimensional" discrete-time systems (which in turn may be studied using the theory of group algebras). This general method, rather than just the application of Theorem (5.4), is what should be stressed.

(7) Extensions.

Many notions (e.g., "invertibility") can be defined in the general context of admissible triples and can be characterized as in the classical cases given adequate restrictions on B. However, in order to allow consideration of time, initial conditions, observability and other system-theoretic notions, the definition of admissibility must be specialized in different ways. As an example, we sketch the case of reachability.

Let the "time-set" T be an ordered group and k a field. Restrict admissible triples to those for which A is a $k[T]$-submodule of k^T and $B[\alpha]$ is a commutative k-subalgebra of $\text{End}_{k[T]}(A)$, i.e., the operators are time-invariant.

A system (F, G, H) is (pointwise) *reachable* provided that for every x in k^n there exist ω in A^m and t in T such that $x = (\omega L)\,(t)$. It can be proved that the system is reachable if and only if there exists no matrix V in k^n with $GF^k V = 0$ for $k = 0, \cdots, n-1$.

The above gives a unified and algebraic proof of the classical reachability criterion for both discrete and continuous time systems, where $B = k$. Moreover, in the case that B is a polynomial ring over k, the above criterion is equivalent to a rank condition over k, and a new characterization of pointwise reachability is obtained for (retarded) delay-differential systems.

As suggestions for further work, we might mention further specializations allowing studies of other system-theoretic properties. Another interesting possibility is that of generalizing the present framework to include *bilinear* systems, where the map F is replaced by an affine map from A^m into $B^{n \times n}$. This approach would permit the treatment of *delay-differential bilinear* systems using the results on realization of rational power series on several non-commuting variables given by Fliess ([5], [6]).

E. Conclusions

Some essential aspects of the extension of linear system theory to the case of arbitrary coefficient rings were studied, isolating in particular a class of rings ($\mathscr{F}\mathscr{A}$) characterized by any of a number of properties which are well known for the commutative case. Another class ($\mathscr{F}\mathscr{O}$) was also defined by interchanging observability with reachability in the definitions. Algebraic characterizations were obtained in both cases, with the purpose of comparing them at a strictly algebraic level.

An approach to the use of system theory over rings as a tool for the study of more general systems was presented in Part D. Observe that, for example, (5.4) together with (4.6.b) and the discrete-time theory over **R** gives a rigorous and completely algebraic theory for existence and uniqueness of realizations of finite-dimensional constant continuous-time systems: the only nonalgebraic component is involved in the proof of admissibility, and then no knowledge is needed of the form of the solutions.

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