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A LYAPUNOV-LIKE CHARACTERIZATION OF ASYMPTOTIC CONTROLLABILITY*

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Abstract. It is shown that a control system in $\mathbf{R}^n$ is asymptotically controllable to the origin if and only if there exists a positive definite continuous functional of the states whose derivative can be made negative by appropriate choices of controls.

Key words. stabilization, controllability, Lyapunov functions, nonlinear control

1. Introduction. Lyapunov techniques have long been used in studying control problems for a system $\dot{x}(t)=f(x(t), u(t))$: Controlling so as to diminish the value of a suitable positive definite function is an obvious way of achieving stabilization, and feedback laws can be analyzed through the use of such a function—see for instance the books Barbashin [1970], Lefschetz [1965] and Letov [1961]. Sometimes one considers Lyapunov functions in conjunction with other techniques, like the analysis of sliding modes—see for instance Utkin [1977]; in these and other applications, the natural Lyapunov functions are often nondifferentiable.

In this paper we deal with the relation between the property of asymptotic controllability (every state can be driven, asymptotically, to a desired state "0", plus a local condition) and the existence of a positive definite continuous function V whose derivative can be made negative by appropriate choices of controls. If not only is the system asymptotically controllable but in fact there is a (suitable smooth) feedback law $K(\cdot)$ such that the closed loop system $\dot{x}(t)=f(x(t), K(x(t)))$ is asymptotically stable, then an inverse Lyapunov theorem can be applied to this closed loop system in order to obtain a V as above. Inverse Lyapunov results for classical (no control) systems have a long history themselves, with important contributions by Persidski, Malkin, Massera and others; a good reference is Hahn [1978]. In general, however, a continuous K fails to exist, even for very simple systems—see for instance the discussion in Sontag and Sussmann [1980]—so such an argument cannot be applied to conclude the existence of V.

The main result of this paper is that, for asymptotically controllable systems, a V as above always exists. We allow relaxed ("chattering") controls when testing the derivative of V. (Since relaxed directions belong to the convex hull of ordinary ones, the latter suffice in the C^1 case.) The proof will be based on a combination of some basic optimal control concepts and classical Lyapunov techniques (in particular, those of Zubov [1964]). For results somewhat related to this note, the reader may wish to consult the references Tokumaru et al. [1969] (gives a sufficient Lyapunov-like condition), Jacobson [1977] (gives a local necessary and sufficient criterion for a special class of systems), and Vinter [1980] (gives a time-varying functional characterizing nonreachability from a given point). Both the results and the techniques used here, however, are different from those in these references.

2. Definitions and statement of results. The systems to be studied are given by differential equations

$$\dot{x}(t)=f(x(t), u(t)) \tag{2.1}$$

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with states $x(t)$ in $X := \mathbf{R}^n$, and input values $u(t)$ in a locally compact metric space U for which the balls $\{\nu | d(\mu, \nu) \leqq r\}$ are compact for each ν in U (d = distance in U). A special element "0" is distinguished in U, and the state $x = 0$ of X is an equilibrium point, i.e., $f(0, 0) = 0$. The map f is locally Lipschitz in (x, u). The set of (*ordinary*) *controls* $\mathbf{U}$ is the set of measurable and locally essentially bounded functions $u: \mathbf{R}_+ \to U$. Here $\mathbf{R}_+$ denotes the set of nonnegative reals; sometimes (depending on the context) $\mathbf{R}_+$ denotes positive reals. By abuse of notation, we shall use the same terminologies for controls defined only on a finite interval; these may be extended arbitrarily outside the interval of interest. Solutions of (2.1) are assumed to be unique and to exist locally (in t) for all controls; suitable Carathéodory-type conditions insure this. (Note that we do *not* wish to impose the (somewhat unrealistic) assumption that the solutions are always defined for all $t \geqq 0$, i.e., that there are no finite escape times.)

Given the distinguished "zero" input value 0, let $|\mu| := d(\mu, 0)$ for μ in U. The set U_r consists of all those μ with $|\mu| \leqq r$, and $\mathbf{U}_r$ is the set of all measurable $u: \mathbf{R}_+ \to U_r$, seen as a subset of $\mathbf{U}$. The set of *generalized control values* is the set $W = W(U)$ of probability measures on U; the subset of those measures supported n U_r is W_r. The set W may be topologized using the weak topology, and one introduces then the space of *relaxed controls* $\mathbf{W}$ as the set of measurable functions $w: \mathbf{R}_+ \to W$. For the topology on $\mathbf{W}$ see the references below; we shall only need to know the continuous dependence facts mentioned later. The subspaces $\mathbf{W}_r$ correspond to the relaxed controls $w(t)$ which are in W_r a.e.; each of these subspaces is sequentially compact and—identifying μ in U with the Dirac measure concentrated at $\{\mu\}$—contains (*densely*) the corresponding $\mathbf{U}_r$. A *bounded* relaxed control w is one belonging to some $\mathbf{W}_r$; the infimum of the r for which w is in $\mathbf{W}_r$ is denoted by $\|w\|$. Note that, for ordinary controls, $\|u\|$ becomes the essential supremum of the values $|u(t)|$, t in $\mathbf{R}$. (The notation $\|x\|$ will be used also for the Euclidean norm on X, but this should cause no confusion.) For details on relaxed controls, see Warga [1972], or the (very clear) presentation in Gamkrelidze [1978]; the paper Arstein [1978] summarizes most of the needed facts.

There is a natural definition of solution of (2.1) when relaxed (rather than ordinary) controls are used; see the above references for details. The solution at time t for the initial condition $x(0) = \xi$ and control w will be denoted by $x(t; \xi, w)$ or just by $x(t)$ if both ξ and w are clear from the context. For any given ξ and w there is an open set $Y := I \times N \times M$ containing $(0, \xi, w)$ such that $x(t; \eta, v)$ is well-defined for any (t, η, v) in Y. Further, if this solution is known to be defined for $0 \leqq t \leqq T$, then the map $(t, \eta, v) \to x(t; \eta, v)$ is continuous on $[0, T] \times N \times M$, for some open N, M.

We are now ready to introduce our definitions and state the main result.

DEFINITION 2.2. The system (2.1) is *asymptotically* (null-) *controllable* (a.c., for short) if and only if the following properties hold:

(i) (global part) for each ξ in X there exists an (ordinary) control u such that $x(t) = x(t; \xi, u)$ is defined for all $t \geqq 0$ and $x(t) \to 0$ as $t \to \infty$;

(ii) (local part) for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for any state with $\|\xi\| < \delta$ there is a u as in (i) such that also $\|x(t)\| \leqq \varepsilon$ for all $t \geqq 0$;

(iii) (bounded controls) there exist positive η and k such that, if the ξ in (ii) satisfies also $\|\xi\| < \eta$, then the control u can be chosen with $\|u\| \leqq k$.

The above seems to be the obvious definition of a.c. if one is to model the usual uniform asymptotic stability notion in the controlled case. (Or intuitively, if the given system is to be seen as the open-loop part of some abstract closed-loop stable system.) The requirement (iii) of a bound on magnitudes of inputs required for controlling small states seems physically (and mathematically) reasonable. In fact, in order to model "regulation with internal stability" one adds the requirement that $u(t)\to 0$ as $t\to\infty$; see Sontag [1982]. Analogous results would hold in that case. Observe that both of the (local) parts (ii) and (iii) would hold, for instance, if $U=\mathbf{R}^m$, f is differentiable at $(0,0)$ and the linearization of (2.1) at the origin is stabilizable in the usual sense.

Given a function $V:X\to\mathbf{R}$, a state ξ, and a relaxed control w defined on an interval containing $t=0$, let

$$\dot V_w(\xi) := \varliminf_{t\to 0^+} \frac{1}{t}[V(x(t;\xi,w))-V(\xi)]. \tag{2.3}$$

Consider the following four properties of such a function V:

(2.4a) V is continuous;

(2.4b) $V(\xi)>0$ for $\xi>0$, and $V(0)=0$ (V is positive definite);

(2.4c) the set $\{\xi|V(\xi)<r\}$ is bounded, for each r (V is proper);

(2.4d) for each ξ in X there is a relaxed w with $\dot V_w(\xi)<0$, and there are positive numbers k and η such that w can be chosen with $\|w\|<k$ whenever $\|\xi\|<\eta$.

The following is an easy consequence of the above:

(2.4e) for each $\varepsilon>0$ there is a $\theta>0$ such that $V(\xi)<\theta$ implies $\|\xi\|<\varepsilon$.

In the next section we prove:

THEOREM 2.5. *The system* (2.1) *is asymptotically controllable if and only if there exists a V satisfying properties* (2.4a)–(2.4e).

The definition (2.3) of the (Dini) derivative along a trajectory is one immediate generalization of that used in the standard (no control) case; see for example Rouche et al. [1977]. (We could have used in in this definition a lim sup instead of a lim inf; in that case Theorem 2.5 *still holds*: The sufficiency statement becomes weaker, while the necessary part can be proved in exactly the same way.)

3. Proof of Theorem 2.5. We first establish the easy part:

A. Sufficiency. Let V, k, η be as in (2.4), and let $\varepsilon>0$. Take a θ as in (2.4e) such that $V(\xi)<\theta$ implies that $\|\xi\|<\min\{\eta,\varepsilon\}$. A state x will be called *nicely reachable* from a given state ξ if and only if there exists an (ordinary) control u and a time $T\geqq 0$ such that:

(3.1a) $x=x(T;\xi,u)$;

(3.1b) $V(x(t;\xi,u))<2V(\xi)$ for $0\leqq t\leqq T$;

(3.1c) if $V(\xi)<\theta$ then also $\|u\|<k$.

Let

$$\alpha(\xi) := \inf\{V(x)|x \text{ nicely reachable from } \xi\}. \tag{3.2}$$

Either $\alpha(\xi)=0$ for all ξ or $\alpha(\xi)\neq 0$ for some ξ.

Case I. $\alpha(\xi)=0$ for all ξ. Pick any ξ, and choose a ξ_1 nicely reachable and with $V(\xi_1)<V(\xi)/2$. Iterate the construction starting with ξ_1. One obtains in this way a sequence $\{\xi_i\}$ with $V(\xi_i)\to 0$ as $i\to\infty$ (hence also $\xi_i\to 0$) and such that $\xi_i=x(t_i;\xi,w)$ for an increasing sequence $\{t_i\}$ and a fixed w (obtained by concatenation). Let $T:=\sup\{t_i\}$; then $V(x(t))\to 0$ (for $x(t):=x(t;\xi,w)$) as $t\to T$. If $T<\infty$, extend w by

$w(t) := 0$ for $t \geqq T$; in any case one concludes that $x(t) \to 0$ as $t \to \infty$, with *all* $x(t)$ nicely reachable from ξ. This gives the first part of the a.c. definition. Pick now a $\delta > 0$ such that $\|x\| < \delta$ implies $V(x) < \theta/2$. If $\|\xi\| < \delta$ for the above ξ, then all $x(t)$ in the obtained trajectory satisfy $V(x(t)) < 2V(\xi) < \theta$. It follows that $\|x(t)\| < \varepsilon$, as required in (ii) of the a.c. definition. Finally, part (iii) is satisfied by construction, using the same k and any $\eta' > 0$ for which $\|\xi\| < \eta'$ implies $V(\xi) < \theta$.

Case II. $\alpha(\xi) > 0$ for some ξ. We shall derive a contradiction. Let $\{x_n\}$ be a sequence of nicely reachable states with $V(x_n) \to \alpha := \alpha(\xi)$. All these x_n belong to the compact set

$$\{x \mid V(x) \leqq V(\xi)\}; \tag{3.3}$$

replacing $\{x_n\}$ by an appropriate subsequence we may assume that

$$x_n \to \zeta, \qquad V(\zeta) = \alpha \leqq V(\xi). \tag{3.4}$$

By property (2.4d), there is then a sequence $t_i \to 0^+$ and a relaxed w with $V(x(t_i; \xi, w)) < V(\zeta) = \alpha$ for each t_i. Further, if $V(\xi) < \theta$ then also $V(\zeta) < \theta$ so one may pick such a w with $\|w\| < k$. It follows from the continuity of $V(x(t; \zeta, w))$ on t that there is an i such that (with $t := t_i$) also

$$V(x(s; \zeta, w)) < \tfrac{3}{2}V(\xi) \quad \text{for } 0 \leqq s \leqq t. \tag{3.5}$$

Thus for (w', ζ') sufficiently close to (w, ζ) it holds that

$$V(x(s; \zeta', w')) < 2V(\xi) \quad \text{for } 0 \leqq s \leqq t. \tag{3.6}$$

Pick an ordinary control $w' = u$ such that this holds and such that also

$$V(x(t; \zeta, u)) < \alpha. \tag{3.7}$$

If $V(\xi) < \theta$, require also that $\|u\| < k$. (Recall that ordinary controls are dense in $\mathbf{W}_k$.)

Let now $z_n := x(s; x_n, u)$. For large enough n, (3.4)–(3.6) give a z_n nicely reachable from ξ and with $V(z_n) < \alpha$. This contradicts the minimality of α.

B. Some bounding functions. We now start proving that a.c. implies the existence of a V as above. We shall need a sequence of basic lemmas. In order to simplify notations, $g(\pm\infty)$ will mean $\lim_{p \to \pm\infty} g(p)$, and $g(0) := \lim_{p \to 0^+} g(p)$ for a function defined on positive reals only. A fixed asymptotically controllable system is assumed given; the numbers k and η are as in the a.c. definition.

LEMMA 3.8. *There exist a positive number $p_0 < 1$ and maps $\tau, \phi, \mu : \mathbf{R}_+ \to \mathbf{R}$, $m : \mathbf{R} \to \mathbf{R}_+$ and $K : X \to \mathbf{U}$, where ϕ, μ and m are continuous, m is strictly decreasing, ϕ is strictly increasing and μ is nondecreasing, such that the following properties hold:*

a) $m(-\infty) = +\infty$, $m(+\infty) = 0$, $m(0) = 1$;
b) $p \leqq \phi(p)$ *for all* p, $\phi(0) = 0$;
c) $\tau(p) = 0$ *for* $0 \leqq p \leqq p_0$;
d) $\mu(p) = k$ *for* $0 \leqq p \leqq p_0$, $\mu(+\infty) = +\infty$;
e) *for each* $\xi \neq 0$, *with* $x(t) := x(t; \xi, K(\xi))$:
  (i) $\|K(\xi)\| \leqq \mu(\|\xi\|)$,
  (ii) $\|x(t)\| \leqq \phi(\|\xi\|)$ *for* $t \geqq 0$,
  (iii) $x(t) \to 0$ *as* $t \to \infty$,
  (iv) *for* $t \geqq \tau(\|\xi\|)$, $\|x(t)\| \leqq m(t - \tau(\|\xi\|))$.

Proof. Part 1. We shall first construct sequences of nonnegative numbers $\{\varepsilon_i\}$, $\{T_i\}$ and $\{b_i\}$, i in $\mathbf{Z}$, such that: $\{\varepsilon_i\}$ is strictly increasing, $\varepsilon_i \to 0$ (resp. $+\infty$) as $i + -\infty$ (resp. $+\infty$), and so that for each ξ for which $\|\xi\| < \varepsilon_i$ there is an (ordinary) control u satisfying $\|u\| < b_i$ and $\|x(t; \xi, u)\| < \varepsilon_{i+1}$ for all $t \geqq 0$ and also $\|x(t; \xi, u)\| < \varepsilon_{i-1}$ for $t \geqq T_i$.

Let $\varepsilon_0 := \frac{1}{2}$. By induction on Definition 2.2 one concludes the existence of a sequence of numbers $\varepsilon_i, i \leqq -1$, such that: for each ξ with $\|\xi\| \leqq \varepsilon_{i-1}$ there is a control u with $\|u\| \leqq k$, $\|x(t; \xi, u)\| < \varepsilon_i$ and $x(t) \to 0$ as $t \to \infty$; one may take the $\{\varepsilon_i\}$ strictly decreasing and approaching 0 as $i \to -\infty$. Further, one may assume that $\varepsilon_{-1} \leqq \eta$. Consider now a fixed $i < 0$ and take a ξ with $\|\xi\| \leqq \varepsilon_i$. There is then an u as above and some $T = T(\xi)$ such that $\|x(T)\| < \varepsilon_{i-2}$ for the corresponding solution. By continuity of $x(\cdot; \cdot, u)$ there is an open neighborhood $H(\xi)$ of ξ such that, for each z in $H(\xi)$, $\|z(t)\| < \varepsilon_{i+1}$ for $0 \leqq t \leqq T$ and $\|z(T)\| < \varepsilon_{i-2}$, where $z(t) := x(t; z, u)$. By construction of ε_{i-2}, there is for each such $z(T)$ a control v with $\|v\| < k$ (not necessarily the same u) such that

$$\|x(t; z(T), v)\| < \varepsilon_{i-1}, \qquad t \geqq 0. \tag{3.9}$$

Concatenating the restriction of u to $[0, T]$ with this v, one concludes that for each z in $H(\xi)$ there is some (ordinary) control with the resulting trajectory having $\|z(t)\| < \varepsilon_{i-1}$ for *all* $t \geqq T(\xi)$ while keeping $\|z(t)\| < \varepsilon_{i+1}$ for all t. (Note that the input to be applied in order to achieve this depends on the particular z; for the original u there may be no neighborhood on which this controllability is achieved.) The $H(\xi)$ cover the ball of radius ε_i; pick a finite subcover. Let $T_i :=$ largest of the $T(\xi)$ for this subcover. With all $b_i := k$ the sequences $\{\varepsilon_i\}, \{b_i\}, \{T_i\}$ satisfy the requirements for $i < 0$.

We now define the sequences for $i \geqq 0$, by induction on increasing i. Assume that ε_i, b_{i-1} and T_{i-1} have been already defined (recall for the first step that $\varepsilon_0 = \frac{1}{2}$). By property (i) in the definition of a.c., it follows that for each ξ with $\|\xi\| \leqq \varepsilon_i$ there exists some u and some $T = T(\xi)$ with $\|x(T)\| < \varepsilon_{i-2}$. By induction, it is possible to control $x(T)$ in such a way as to stay in the ball of radius ε_{i-1}. These further controls can be chosen with $\|v\| < b_{i-1}$. An argument like the one in the previous paragraph gives a fixed T_i such that each state ξ as above is controlled to $\|x(T_i)\| < \varepsilon_{i-1}$ by appropriate choice of controls. Further, all these controls are obtained by concatenating one of a finite number of controls u_j (finite subcover argument) with controls with $\|v\| < b_{i-1}$. Let b_i be larger than b_{i-1} and all $\|u_j\|$. To complete the induction step we need to define ε_{i+1}. Consider the set

$$\{x(t; \xi, u) \mid \|\xi\| \leqq \varepsilon_i, \|u\| \leqq b_i, 0 \leqq t \leqq T_i\}. \tag{3.10}$$

Since this set is compact, it is contained in the interior of some ball of radius ε_{i+1}. For simplicity of future arguments, we shall assume that the sum of the T_i, $i < 0$, is infinite; larger T_i's can always be chosen in the above constructions. This completes Part 1.

Part 2. Let $\phi: \mathbf{R}_+ \to \mathbf{R}$ be any continuous strictly increasing function such that $\phi(0) = 0$ and, for all i

$$\phi(p) > \varepsilon_{i+1} \quad \text{for } p \text{ in } [\varepsilon_{i-1}, \varepsilon_i]. \tag{3.11}$$

Let $p_0 := \varepsilon_{-1}$. Take $\mu: \mathbf{R}_+ \to \mathbf{R}$ to be any continuous nondecreasing function having $\mu(p) = b_0$ for $0 \leqq p \leqq p_0$, $\mu(+\infty) = +\infty$, and such that, for all i,

$$\mu(p) \geqq b_i \quad \text{for } p \text{ in } (\varepsilon_{i-1}, \varepsilon_i]. \tag{3.12}$$

Denote $t_0 := 0$ and $t_i := T_{-1} + \cdots + T_{-i}$ for $i > 0$. Let m be as in a) and satisfying, for $i \geqq 0$,

$$m(t) > \varepsilon_{-i} \quad \text{for } t \text{ in } [t_i, t_{i+1}]. \tag{3.13}$$

Let τ be the step function with value 0 for $p \leqq p_0$ and for $i \geqq 0$,

$$\tau(p) = T_0 + \cdots + T_i \quad \text{for } p \text{ in } (\varepsilon_{i-1}, \varepsilon_i]. \tag{3.14}$$

The open-loop "choice" function K is introduced merely for notational convenience; no smoothness of any kind is required. Given a state ξ, say with $\varepsilon_{i-1} < \|\xi\| \leqq \varepsilon_i$, find a control u_1 sending ξ to $\xi_1 = x(T_i; \xi, u)$ with $\|u_1\| < b_i$, $\|\xi_1\| < \varepsilon_{i-1}$, and all intermediate states with $\|x(t)\| < \varepsilon_{i+1}$. Repeat with ξ_1, finding a u_2, ξ_2. Iterating this construction, let $K(\xi)$ be the concatenation of all the u_j thus obtained. The corresponding trajectory satisfies $x(t) \to 0$ as $t \to \infty$. By construction, $\|K(\xi)\| < b_i$, which is less than $\mu(\|\xi\|)$ by (3.12). Also, $\|x(t)\| < \varepsilon_{i+1}$ for all $t \geqq 0$, and by (3.11) also $\|x(t)\| < \phi(\|\xi\|)$.

We now need only establish property (iv). Assume first that $\|\xi\| < \varepsilon_i$, with $i = -1$. Then $\tau(\|\xi\|) = 0$, and the above construction insures that $\|x(t)\| < \varepsilon_{-j}$ when t is in $[t_j, t_{j+1}]$. By (3.13), property (iv) holds. If $i < -1$, the trajectory has $\|x(t)\| < \varepsilon_{-j}$ when t is in $[t_j - L_i, t_{j+1} - L_i]$, where

$$L_i := T_{-1} + \cdots + T_{i+1}, \tag{3.15}$$

for any $j \geqq -i-1$, and $\|x(t)\| < \varepsilon_{i+1}$ for all $t \geqq 0$. If t is in $[t_j, t_{j+1}]$, $j < -i-1$, then $\|x(t)\| < \varepsilon_{-j}$ by the latter fact; if $j \geqq -i-1$ then $t \geqq t_j > t_j - L_i$ gives again that $\|x(t)\| < \varepsilon_{-j}$. Again by (3.13), property (iv) of (3.8c) holds. There remains the case $i \geqq 0$. In that case, after time

$$T = T_i + T_{i-1} + \cdots + T_0 = \tau(\|\xi\|) \tag{3.16}$$

a state $x(t)$ with $\|x(t)\| < \varepsilon_{-1}$ is reached, and after that the trajectory has states bounded by $m(t-T)$ (by the case $i = -1$).

LEMMA 3.17. *(The notations of the previous lemma still hold). There exist continuous strictly increasing functions $N, \psi, \nu: \mathbf{R}_+ \to \mathbf{R}_+$, with $N(0) = 0$, $N(+\infty) = +\infty$, such that the following properties hold. For any state ξ and for any relaxed control w, let*

$$R(\xi, w) := \int_0^\infty N(\|x(t; \xi, w)\|)\, dt + \max\{\|w\| - k, 0\} \tag{3.18}$$

if the solution is defined for all $t \geqq 0$, and $R(\xi, w) := +\infty$ otherwise. Then, for each ξ:

(a) *$R(\xi, K(\xi)) < \infty$;*
(b) *if $\|\xi\| \leqq p_0$ then $R(\xi, K(\xi)) \leqq \phi(\|\xi\|)$;*
(c) *if $R(\xi, w) < R(\xi, K(\xi))$ for some w, then $\|x(t; \xi, w)\| \leqq \psi(\|\xi\|)$ for all $t \geqq 0$ and $\|w\| \leqq \nu(\|\xi\|)$;*
(d) *for each $\alpha > 0$ there is a $\theta > 0$ such that if $R(\xi, w) < \alpha$ for some w, then $\|\xi\| \leqq \theta$;*
(e) *for each $\alpha > 0$ there is a $\beta > 0$ such that $\|\xi\| > \alpha$ implies that $R(\xi, w) > \beta$ for all w.*

Proof. Let $n := m^{-1}: \mathbf{R}_+ \to \mathbf{R}$ be the inverse function of m. Define the function

$$N_1(p) := p \exp[-n(p)]. \tag{3.19}$$

Both N_1 and $\exp[-n(\cdot)]$ are strictly increasing continuous functions, $N_1(0) = 0$, $N_1(+\infty) = +\infty$.

For each triple of positive numbers (a, b, c), choose the quantity $\gamma(a, b, c) > 0$ in such a way that the following property holds: if w is any control with $\|w\| \leqq a$, and if ξ_1, ξ_2 are any states with

$$\xi_2 = x(T; \xi_1, w) \tag{3.20}$$

for some $T > 0$ and the trajectory having

$$b \leqq \|x(t; \xi_1, w)\| \leqq c \quad \text{for } 0 \leqq t \leqq T, \tag{3.21}$$

then

$$\int_0^T N_1(\|x(t)\|)\,dt \geqq \gamma(a,b,c)\|\xi_1-\xi_2\|. \tag{3.22}$$

Such a quantity always exists, because the integrand is bounded below by $N_1(b)$ (so that the left side is at least $T\cdot N_1(b)$), while $\|\xi_1-\xi_2\|$ is bounded by TB, where B is a bound on the values $f(x,\mu)$ for $\|x\|\leqq c$ and $|\mu|\leqq a$.

We shall define inductively a nondecreasing sequence of maps N_j, $j\geqq 1$, starting with (3.19), and will then let (pointwise)

$$N := \lim_{j\to\infty} N_j. \tag{3.23}$$

This limit will be finite because the following property will hold by construction for each j:

$$p\leqq \phi(j) \quad \text{implies } N_j(p)=N_{j+1}(p) \quad (=N(p)). \tag{3.24}$$

(Note that $\phi(j)\to+\infty$ as $j\to\infty$.) Further, all the functions N_j (hence N itself) will be continuous (or even C^∞ if desired). Assume then that N_i has been defined for all $i\leqq j$, in such a way that (3.24) holds for $i=1,\cdots,j-1$. Introduce the quantities

$$r_j := N_j(\phi(j))\tau(j)+\phi(j), \tag{3.25}$$

$$L_j(\xi,w) := \int_0^\infty N_j(\|x(t;\xi,w)\|)\,dt \tag{3.26}$$

with $L_j(\xi,w):=+\infty$ if there is a finite escape time.

Take now any ξ with $\|\xi\|\leqq j$. When $t\geqq\tau(j)$, it follows from Lemma 3.8e(iv) that

$$\|x(t;\xi,K(\xi))\|<m(t-\tau(\|\xi\|))\leqq m(t-\tau(j)) \tag{3.27}$$

and hence also $\|x(t)\|<m(0)=1<\phi(1)$ for these t. Thus,

$$\int_{\tau(j)}^\infty N_j(\|x(t)\|)\,dt=\int_{\tau(j)}^\infty N_1(\|x(t)\|)\,dt \tag{3.28}$$

(by (3.24)), and by (3.27) this is less than

$$\phi(1)\int_{\tau(j)}^\infty \exp\left[-n(m(t-\tau(j)))\right]dt, \tag{3.29}$$

which equals $\phi(1)$. Since $\|x(t;\xi,K(\xi))\|<\phi(j)$ for all $t\geqq 0$ (again from Lemma 3.8e), one has the bound

$$L_j(\xi,K(\xi))\leqq r_j. \tag{3.30}$$

Define also

$$\gamma_j := \gamma(r_j+\mu(j)+k,\ \phi(j)+1,\ \phi(j)+2). \tag{3.31}$$

Let g_j be any continuous nonnegative function from $\mathbf{R}_+$ to $\mathbf{R}$ which vanishes outside the interval $[\phi(j),\phi(j)+3]$ and such that

$$g_j(p) := \frac{r_j+\mu_j}{\gamma_j} \tag{3.32}$$

for p in the interval $[\phi(j)+1,\phi(j)+2]$. The induction step is provided by the definition

$$N_{j+1}(p) := (1+g_j(p))N_j(p). \tag{3.33}$$

Note that (3.24) indeed holds. This completes the construction of N.

Finally, pick any ψ and ν continuous, increasing, and such that

$$\nu(p) > r_j + \mu(j) + k, \tag{3.34}$$

and

$$\psi(p) > \phi(j) + 2, \tag{3.35}$$

whenever p is in $[j-1, j]$.

We now prove that (a) to (e) hold. Define L as in (3.26) but using N in the integrand. Pick a ξ with $\|\xi\| < j$. The trajectory corresponding to the control $K(\xi)$ has $\|x(t)\| < \phi(j)$ for $t \geqq 0$, so, by (3.24), $L(\xi, K(\xi)) = L_j(\xi, K(\xi)) < r_j$. Thus

$$R(\xi, K(\xi)) \leqq r_j + \mu(j), \tag{3.36}$$

and in particular (a) holds. Assume that j was chosen so that $\|\xi\| > j-1$. Suppose now that w is a control for which $R(\xi, w) < R(\xi, K(\xi))$. By (3.36),

$$\|w\| \leqq r_j + \mu(j) + k < \nu(\|\xi\|), \tag{3.37}$$

as wanted in (c). Now assume that for the corresponding trajectory there would be some t with $\|x(t)\| > \psi(\|\xi\|) > \phi(j) + 2$. There are then times $t_1 < t_2$ such that $\|x(t_1)\| = \phi(j) + 1$, $\|x(t_2)\| = \phi(j) + 2$ and

$$\phi(j) + 1 \leqq \|x(t)\| \leqq \phi(j) + 2 \tag{3.38}$$

for $t_1 \leqq t \leqq t_2$. Hence

$$\begin{aligned} R(\xi, w) &\geqq \int_{t_1}^{t_2} N(\|x(t)\|)\, dt \geqq \int_{t_1}^{t_2} N_{j+1}(\|x(t)\|)\, dt \\ &\geqq (1 + g_j(\phi(j)+1)) \int_{t_1}^{t_2} N_1(\|x(t)\|)\, dt \geqq r_j + \mu(j) \geqq R(\xi, K(\xi)). \end{aligned} \tag{3.39}$$

This contradicts the choice of ξ and w. So (c) holds. To prove (b), let $\|\xi\| \leqq p_0$. Then $\tau(\|\xi\|) = 0$ and $\|K(\xi)\| \leqq k$, so, for $w = K(\xi)$,

$$R(\xi, K(\xi)) = \int_0^\infty N_1(\|x(t)\|)\, dt \leqq \phi(\|\xi\|) \int_0^\infty e^{-t}\, dt = \phi(\|\xi\|). \tag{3.40}$$

We now establish (d) and (e). Given $\alpha > 0$ choose any integer $j > 0$ so that $\alpha + k < \mu(j)$. If $R(\xi, w) < \alpha$ then $\|w\| < \mu(j)$. We claim that $\|\xi\| < \theta := \phi(j) + 2$. Otherwise, $\|\xi\| \geqq \phi(j) + 2$ implies that there exist $t_1 < t_2$ with $\|x(t_1)\| = \xi(j) + 2$, $\|x(t_2)\| = \xi(j) + 1$, and all $\|x(t)\|$ bounded by these values for $t_1 \leqq t \leqq t_2$. By an argument similar to the one used above, $R(\xi, w)$ can be proved to be larger than $r_j + \mu(j) > \alpha$, a contradiction. Thus (d) holds. Assume now that $\|\xi\| > \alpha$, and let $\gamma' := \gamma(k+1, \alpha/2, \alpha)$. Let w be given. If $\|w\| > k+1$ then $R(\xi, w) > 1$; otherwise $R(\xi, w) > \alpha\gamma'/2$. With

$$\beta := \min\left\{k+1, \frac{\alpha\gamma'}{2}\right\}, \tag{3.41}$$

(e) is also established.

C. The function *V*.

$$V(\xi) := \inf\{R(\xi, w) \mid w \text{ relaxed control}\}. \tag{3.42}$$

Note that $V(0)=0$, so by (3.18b) the function V is continuous at zero; V is always finite by (3.18a). By (3.18d), the sets $\{\xi | V(\xi)<\alpha\}$ are always bounded. By (3.18e), $V(\xi)>0$ for $\xi \neq 0$.

LEMMA 3.43. *Let $(\xi_n, w_n) \to (\xi, w)$ as $n \to \infty$, and assume that all $R(\xi_n, w_n)$ and $R(\xi, w)$ are finite. Then $R(\xi, w) \leqq \underline{\lim}\, R(\xi_n, w_n)$.*

Proof. Let $x_n(t) := x(t; \xi_n, w_n)$ and $x(t) := x(t; \xi, w)$. Then $N(\|x_n(t)\|)$ converges to $N(\|x(t)\|)$ for each t. By Fatou's lemma,

$$\int_0^\infty N(\|x(t)\|)\, dt \leqq \underline{\lim} \int_0^\infty N(\|x_n(t)\|)\, dt. \tag{3.44}$$

If the measures $w_j(t)$ are all supported in some U_r, for some subsequence $\{w_j\}$, then also $\|w\| \leqq r$. It follows that

$$\|w\| \leqq \underline{\lim}\, \|w_n\|. \tag{3.45}$$

Thus also

$$\max\{\|w\|-k, 0\} \leqq \underline{\lim}\, (\max\{\|w_n\|-k, 0\}), \tag{3.46}$$

and the conclusion follows from (3.44) and (3.46) and the elementary fact that always $\underline{\lim}\, a_n + \underline{\lim}\, b_n \leqq \underline{\lim}\, (a_n + b_n)$.

LEMMA 3.47. *For each ξ there is a w^* with $R(\xi, w^*) = V(\xi)$.*

Proof. Let $\{R(\xi, w_n)\}$ be a minimizing sequence. By (3.18d) we may assume that all $\|w_n\| \leqq \nu(\|\xi\|) =: r$ and

$$\|x_n(t)\| \leqq \psi(\|\xi\|) \text{ for } t \geqq 0. \tag{3.48}$$

By sequential compactness of $\mathbf{W}_r$, (a subsequence of) $\{w_n\}$ converges to a control w^* in $\mathbf{W}_r$. The solution $x(t) := x(t; \xi, w^*)$ is defined for all t, because (3.48) implies that $\|x(t)\|$ is bounded independently of t. By (3.43),

$$R(\xi, w^*) \leqq \lim R(\xi, w_n) = V(\xi), \tag{3.49}$$

so $V(\xi) = R(\xi, w^*)$, as wanted.

LEMMA 3.50. *V is lower semicontinuous.*

Proof. Let $\{\xi_n\}$ be a sequence converging to ξ, and write $V(\xi_n) = R(\xi_n, w_n)$. For a suitable subsequence one may assume that $w_n \to w$, for an appropriate w. (All $\|w_n\|$ are bounded by $\nu(\|\xi\|)+\delta$, some δ.) Thus

$$V(\xi) \leqq R(\xi, w) \leqq \underline{\lim}\, R(\xi_n, w_n) = \underline{\lim}\, V(\xi). \tag{3.51}$$

LEMMA 3.52. *V is continuous.*

Proof. We only need to establish upper semicontinuity. Pick $\varepsilon > 0$. Choose a positive $\delta < p_0$ so that

$$p < \delta \text{ implies } \phi(p) < \frac{\varepsilon}{2}. \tag{3.53}$$

Pick any state ξ, and let $R(\xi, w) = V(\xi)$. Since $x(t) := x(t; \xi, w)$ necessarily converges to zero, there is some T with $\|x(T)\| < \delta$. There is also a neighborhood H of ξ such that, for each z in H, and for the above w, T,

$$\int_0^T N(\|z(t)\|)\, dt < \int_0^T N(\|x(t)\|)\, dt + \frac{\varepsilon}{2} \tag{3.54}$$

for the corresponding solution with $z(0) = z$, and such that also $\|z(T)\| < \delta$. By (3.53) and (3.18b) (continuity at 0), $V(z) < V(\xi) + \varepsilon$. This proves upper semicontinuity.

We are only left with establishing (2.4d). Let ξ be any state, $V(\xi)=R(\xi, w)$. Take any $z=x(t)$ in the trajectory, and let w' be the translation of w by $(-t)$, so in particular $\|w'\| \leqq \|w\|$. It follows that

$$V(z) \leqq R(z, w') \leqq \int_t^\infty N(\|x(s)\|)\, ds + \max\{\|w'\| - k, 0\}. \tag{3.55}$$

So

$$\varlimsup_{t \to 0^+} \frac{1}{t}[V(z)-V(\xi)] < \lim_{t \to 0^+} \frac{1}{t}\left(-\int_0^t N(\|x(s)\|)\, ds\right) = -N(\|\xi\|) < 0. \tag{3.56}$$

Further, $\|w\|$ is bounded by $\nu(\|\xi\|)$. This completes the proof of the theorem.

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