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GLOBAL STABILIZATION FOR SYSTEMS EVOLVING ON MANIFOLDS

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Abstract. We show that any globally asymptotically controllable system on any smooth manifold can be globally stabilized by a state feedback. Since we allow discontinuous feedbacks, we interpret the solutions of our systems in the "sample and hold" sense introduced by Clarke, Ledyaev, Sontag, and Subbotin (CLSS). We generalize their theorem which is the special case of our result for systems on Euclidean space. We apply our result to the input-to-state stabilization of systems on manifolds with respect to actuator errors, under small observation noise.

1. Introduction

This note is devoted to the study of completely nonlinear systems

$$\dot{x} = f(x, u), \quad x \in \mathcal{X}, \quad u \in \mathbf{U}, \tag{1.1}$$

evolving on arbitrary smooth manifolds $\mathcal{X}$ with inputs u in general locally compact metric spaces $\mathbf{U}$, where f is locally Lipschitz in x uniformly for u in compact sets, and jointly continuous in (x, u). We assume that (1.1) is globally asymptotically controllable (GAC) to a given compact weakly invariant nonempty set $\mathcal{A} \subseteq \mathcal{X}$ (see Sec. 3 below for the definition of GAC for systems on manifolds).

It is natural to inquire about the relationship between the GAC property for (1.1) and the existence of a feedback $k(x)$ such that the closed-loop system

$$\dot{x} = f(x, k(x)), \quad x \in \mathcal{X}, \tag{1.2}$$

is globally asymptotically stable to $\mathcal{A}$. For the special case where system (1.1) evolves on $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{A} = \{0\}$, this relationship has been well studied (see [7, 10, 12, 13, 22]). For that case, it is now well known that (1.1) does *not* in general admit a *continuous* stabilizing $k(x)$ (see [22, 24]). This negative result can also be seen from the Brockett criterion (see [7, 20])

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which states that a necessary condition for the existence of a continuous stabilizing feedback $k(x)$ for (1.1) with $k(0) = 0$ is that $(x, u) \mapsto f(x, u)$ be open at zero; see also [19, pp. 252–255] for a simple direct proof of Brockett's result using a homotopy. As a consequence, no totally nonholonomic mechanical system $\dot{x} = g_1(x)u_1 + \ldots + g_m(x)u_m$ on $\mathbb{R}^n$ with $m < n$ and $\mathrm{rank}[g_1(0), \ldots, g_m(0)] = m$ is stabilizable by a continuous state feedback (see [20]). On the other hand, if (1.1) is GAC to $\mathcal{A} = \{0\}$ on $\mathbb{R}^n$, then it can be stabilized by a continuous *time varying* feedback $u = k(t, x)$ provided (i) the system is completely controllable without drift or (ii) $n = 1$ (see [10, 22] and Remark 6.4 below).

However, if we allow *discontinuous* feedbacks, then we have the following positive result from [9] known as the Clarke–Ledyaev–Sontag–Subbotin (CLSS) theorem: *If* (1.1) *is GAC to* $\mathcal{A}$ *on* $\mathcal{X} = \mathbb{R}^n$, *then there exists a discontinuous feedback* $k(x)$ *for which* (1.2) *is globally asymptotically stable to* $\mathcal{A}$. Here and in the sequel, "discontinuous" means "not necessarily continuous in the state variable." The discontinuous feedback $k(x)$ produces a *discontinuous right-hand side* in (1.2), which requires a more general interpretation of solutions that can be applied to discontinuous dynamics. In [9], this issue is resolved by interpreting the trajectories of (1.2) as "sample and hold" (a.k.a. CLSS) solutions (see Definition 2.4 below). The CLSS solution concept has been used extensively in nonlinear control analysis and controller design including the input-to-state stabilization of systems relative to actuator errors under small observation noise (see [12, 13, 20] and Sec. 6 below). For example, CLSS solutions have been used to stabilize nonholonomic systems such as Brockett's example which are not stabilizable by continuous state feedbacks (see [12, 13]).

On the other hand, many important GAC systems evolve on manifolds other than $\mathbb{R}^n$ (e.g., stabilization of rigid bodies on the Lie group of rotations SO_3) and, therefore, are not tractable by the CLSS theorem. In fact, if (1.1) is GAC to a singleton $\mathcal{A} = \{p\}$ and admits a continuous stabilizing feedback $k(x)$, then a theorem of Milnor (see [14]) implies that $\mathcal{X}$ is diffeomorphic to the Euclidean space. This is since the existence of $k(x)$ would imply the existence of a smooth control-Lyapunov function on $\mathcal{X}$ that could be considered as a Morse function with a unique (possibly degenerate) critical point, and manifolds admitting such a Morse function are diffeomorphic to the Euclidean space (see [20]). Therefore, even if (1.1) is holonomic, there may still be topological obstacles to continuous global stabilization when $\mathcal{X} \neq \mathbb{R}^n$.

Motivated by these considerations, this note will extend the CLSS theorem to GAC systems on general smooth manifolds $\mathcal{X}$, proving the existence of a discontinuous feedback $k(x)$ rendering (1.2) globally stable to $\mathcal{A}$ in the sense of CLSS solutions. We follow the construction proposed in [21] which can be summarized as follows. We first embed $\mathcal{X}$ as a closed submanifold

$g(\mathcal{X})$ of a Euclidean space $\mathbb{R}^k$ for some k, e.g., using the Whitney embedding theorem. Then we extend the system to all of $\mathbb{R}^k$ so that

(a) points outside $g(\mathcal{X})$ can be controlled to a tubular neighborhood of $\mathcal{A}$,
(b) $g(\mathcal{X})$ is invariant for the extended system.

Then we apply the CLSS theorem to the extended system on $\mathbb{R}^k$ to design our feedback $k(x)$. The restriction of this feedback to $g(\mathcal{X})$ provides the desired stabilizer for the original system.

This note is organized as follows. In Sec. 2, we consider CLSS solutions and the CLSS theorem. We introduce the relevant definitions for stability on manifolds in Sec. 3. In Sec. 4, we prove our generalized CLSS theorem on the discontinuous stabilization of (1.1) on smooth manifolds. We illustrate our discontinuous feedback constructions in Sec. 5. Finally, in Sec. 6 we apply our results to the input-to-state stabilization of GAC systems on Riemannian manifolds with respect to actuator errors under small observation noise. This extends the corresponding results [12, 13, 18] on input-to-state stabilization for systems evolving on Euclidean space.

2. The CLSS theorem on Euclidean space

In this section, we give the main definitions and results from [9] on the stabilization of GAC systems on Euclidean space. Throughout this section, our state space is $\mathcal{X} = \mathbb{R}^n$. We extend this material to systems on smooth manifolds in the next sections. We consider a system (1.1) for which f is locally Lipschitz in x uniformly for $u \in \mathbf{U}$ in compact sets, and jointly continuous in x and u. Our input set $\mathbf{U}$ is a locally compact metric space with a metric $d_{\mathbf{U}}$ and a distinguished element $0 \in \mathbf{U}$, and we set $|u| = d_{\mathbf{U}}(u, 0)$ for each $u \in \mathbf{U}$. Let $\mathcal{U}$ denote the set of all *controls* for (1.1), i.e., the set of all measurable, locally essentially bounded functions $\mathbf{u} : \mathbb{R}_{\geq 0} \to \mathbf{U}$. The essential supremum of any control $\mathbf{u} \in U$ is denoted by $\|\mathbf{u}\|$, and $\mathcal{U}_N = \{\mathbf{u} \in \mathcal{U} : \|\mathbf{u}\| \leq N\}$ for each $N > 0$. Given $\xi \in \mathcal{X}$ and $\mathbf{u} \in U$, the maximal trajectory of (1.1) for the control $\mathbf{u}$ that satisfies $x(0) = \xi$ is denoted by $x(t, \xi, \mathbf{u})$ or, simply, by $x(t)$ when ξ and $\mathbf{u}$ are clear. We say that $x(t)$ is *well defined* provided it is defined for all $t \in \mathbb{R}_{\geq 0} := [0, \infty)$.

Let $\mathcal{A} \subseteq \mathcal{X}$. We say that $\mathcal{A}$ is *weakly invariant* (for (1.1)) provided that there exists $N > 0$ such that for any $\xi \in \mathcal{A}$ there is a control $\mathbf{u} \in \mathcal{U}_N$ such that the corresponding trajectory $x(t, \xi, \mathbf{u})$ is well defined and stays in $\mathcal{A}$. For example, $\mathcal{A} = \{0\}$ is weakly invariant if $f(0, \bar{a}) = 0$ for some $\bar{a} \in \mathbf{U}$. More generally, $\mathcal{A}$ could be a periodic orbit we want to stabilize. Let $|p|$ denote the Euclidean norm of any $p \in \mathcal{X}$. We denote by bd (respectively, clos) the boundary (respectively, closure) operator, and we define the distance $\operatorname{dist}(\mathcal{N}, x) = \inf\{|p - x| : p \in \mathcal{N}\}$ for any subset $\mathcal{N} \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. For any $x \in \mathcal{X}$, we denote by $|x|_{\mathcal{A}}$ the distance from x to $\mathcal{A}$. Therefore, $|x|_{\mathcal{A}} < \varepsilon$ means $x \in \mathcal{B}_\varepsilon(\mathcal{A}) := \{p \in \mathbb{R}^n : \operatorname{dist}(\mathcal{A}, p) < \varepsilon\}$.

Next, we state two equivalent definitions of globally asymptotic controllability. First, we state the well-known definition from [12, 20] in terms of comparison functions. Then we provide the original ε-δ formulation which we generalize to systems on manifolds in the next section. We use the following comparison function definitions from [20]. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ provided that α is continuous, strictly increasing, and satisfies $\alpha(0) = 0$; it is of class $\mathcal{K}_\infty$ provided that it is also unbounded. We say that $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{N}$ if α is nondecreasing, and of class $\mathcal{L}$ if $\alpha(s)$ is decreasing to 0 as $s \to +\infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if

(a) $\beta(s, \cdot) \in \mathcal{L}$ for every fixed s and
(b) $\beta(\cdot, t) \in \mathcal{K}$ for every fixed t.

We write $\beta \in \mathcal{KL}$ if β is of class $\mathcal{KL}$ and similarly for the other types of comparison functions.

Definition 2.1. Let $\mathcal{A} \subseteq \mathcal{X}$ be compact, nonempty, and weakly invariant for (1.1). System (1.1) is said to be *globally asymptotically controllable* (GAC) to $\mathcal{A}$ (on $\mathcal{X}$) if there exist $\beta \in \mathcal{KL}$ and $\sigma \in \mathcal{N}$ such that for each $\xi \in \mathcal{X}$, there exists a control $\mathbf{u}$ with $\|\mathbf{u}\| \leq \sigma(|\xi|_{\mathcal{A}})$ such that $x(t, \xi, \mathbf{u})$ is well defined and $|x(t, \xi, \mathbf{u})|_{\mathcal{A}} \leq \beta(|\xi|_{\mathcal{A}}, t)$ for all $t \geq 0$.

The following equivalent formulation of GAC has a natural generalization to systems on manifolds (see Definition 3.1 below). See [1] for the equivalence of our GAC definitions on $\mathbb{R}^n$.

Definition 2.2. Let $\mathcal{A} \subseteq \mathcal{X}$ be compact, nonempty, and weakly invariant for (1.1). System (1.1) is said to be *globally asymptotically controllable* (GAC) to $\mathcal{A}$ (on $\mathcal{X}$) if for all $\varepsilon_1, \varepsilon_2 > 0$ with $\varepsilon_1 < \varepsilon_2$, we have the following:

(1) There exist $T = T(\varepsilon_1, \varepsilon_2) > 0$ and $\delta = \delta(\varepsilon_1) > 0$ such that for each $\xi \in \mathcal{B}_{\varepsilon_2}(\mathcal{A})$, there exists a control $\mathbf{u}$ such that
  (a) $x(t, \xi, \mathbf{u})$ is well defined,
  (b) $x(t, \xi, \mathbf{u}) \in \mathcal{B}_{\varepsilon_1}(\mathcal{A})$ for all $t > T$,
  (c) if also $\xi \in \mathcal{B}_{\delta}(\mathcal{A})$, then $\mathbf{u}$ can be chosen so that $x(t, \xi, \mathbf{u}) \in \mathcal{B}_{\varepsilon_1}(\mathcal{A})$ for all $t \geq 0$.
(2) For every positive number $\varepsilon < \varepsilon_2$, there exists $N = N(\varepsilon) > 0$ such that if ξ from item (1) also satisfies $\xi \in \mathcal{B}_{\varepsilon}(\mathcal{A})$, then the control $\mathbf{u}$ from item (1) can be chosen so that $\mathbf{u} \in \mathcal{U}_N$.

Definition 2.3. A *feedback* for (1.1) is defined to be any locally bounded function $k : \mathcal{X} \to \mathbf{U}$.

In this note, we study the equivalence of (open loop) asymptotic controllability of (1.1) and the possibility of stabilizing the system to a weakly invariant set $\mathcal{A}$ via a state feedback. The novelty of our work lies in its applicability to systems on general smooth manifolds. Even for systems on

$\mathbb{R}^n$, it is often the case that a continuous stabilizing state feedback does not exist (see [12, 13, 20]). However, a discontinuous feedback is always possible to construct, provided that we use the Clarke–Ledyaev–Sontag–Subbotin (CLSS) definition of a "sample and hold" solution for a discontinuous dynamic. Next, we consider this generalized solution notion following the notation from [12, 13].

We define a *partition* (of $\mathbb{R}_{\geq 0}$) as a divergent sequence $\pi : 0 = t_0 < t_1 < t_2 < \dots$ and we call

$$\overline{\mathbf{d}}(\pi) = \sup_{i \geq 0}(t_{i+1} - t_i) \quad \text{(respectively, } \underline{\mathbf{d}}(\pi) = \inf_{i \geq 0}(t_{i+1} - t_i)\text{)}$$

the *upper* (respectively, *lower*) diameter of the partition $\pi = \{t_0, t_1, t_2, \dots\}$.

Definition 2.4. Let k be a feedback for system (1.1), $\xi \in \mathcal{X}$, and $\pi = \{t_i\}_{i \geq 0}$ be a partition. The *π-trajectory*

$$t \mapsto x_\pi(t, \xi, k)$$

for (1.1), ξ, π, and k is defined as the continuous function obtained by recursively solving

$$\dot{x}(t) = f(x(t), k(x(t_i)))$$

from the initial time t_i up to the maximal time

$$s_i = \max\Big\{t_i, \sup\big\{s \in [t_i, t_{i+1}] : x(\cdot) \text{ is defined on } [t_i, s)\big\}\Big\}, \tag{2.1}$$

where $x(0) = \xi$.[1] The domain of $x_\pi(\cdot, \xi, k)$ is $[0, t_{\max})$, where

$$t_{\max} = \inf\{s_i : s_i < t_{i+1}\}.$$

We say that $x_\pi(\cdot, \xi, k)$ is *well defined* if $t_{\max} = +\infty$.

The argument t_i in maximum (2.1) is needed to allow the possibility that $x(\cdot)$ is not defined at all on $[t_i, t_{i+1}]$ in which case the supremum in (2.1) alone would, by definition, give $-\infty$. The following notion of (global) stabilization for (1.1) was introduced in [9].

Definition 2.5. A feedback $k : \mathcal{X} \to \mathbf{U}$ is said to *s-stabilize* system (1.1) to $\mathcal{A}$ if for each pair (r, R) with $0 < r < R$, there exist $M = M(R) > 0$ with $\lim_{R \to 0} M(R) = 0$, $\delta = \delta(r, R) > 0$, and $T = T(r, R) > 0$ such that, for every π with $\overline{\mathbf{d}}(\pi) < \delta$ and $\xi \in \mathcal{B}_R(\mathcal{A})$, the π-trajectory $x(\cdot)$ for (1.1), initial value ξ, partition π, and feedback k is well defined and satisfies the following conditions:

(a) $|x(t)|_{\mathcal{A}} \leq r$ for all $t \geq T$ and
(b) $|x(t)|_{\mathcal{A}} \leq M(R)$ for all $t \geq 0$.

[1] The continuity requirement for $x(\cdot)$ amounts to stipulating that the final value on the previous subinterval is used as the initial value at the next subinterval.

The following result to be generalized was shown in [9] for $\mathcal{A} = \{0\}$ but can be shown for our general compact, nonempty, weakly invariant set $\mathcal{A} \subseteq \mathcal{X} = \mathbb{R}^n$ by similar arguments (e.g., using the existence results from [11] for locally Lipschitz Lyapunov functions for GAC systems and any compact set $\mathcal{A}$).

Theorem 1. *If* (1.1) *is GAC to $\mathcal{A}$ on $\mathcal{X} = \mathbb{R}^n$, then it admits a feedback that s-stabilizes the system to $\mathcal{A}$.*

The preceding result is called the *CLSS theorem*. Our main contribution is a generalized CLSS theorem for systems on smooth manifolds and is the subject of the next two sections. We provide related results on input-to-state stabilization on Riemannian manifolds in Sec. 6.

3. Stabilization on manifolds

We again consider system (1.1) but we assume from now on that the state space $\mathcal{X}$ for the system is an arbitrary smooth (i.e., C^∞) (second countable) manifold. Controls $\mathbf{u}$, as before, are measurable, locally essentially bounded functions $\mathbf{u} : \mathbb{R}_{\geq 0} \to \mathbf{U}$. We assume that

$$f : \mathcal{X} \times \mathbf{U} \to T_x(\mathcal{X}) : (x, u) \mapsto f(x, u) \tag{3.1}$$

is locally Lipschitz in x and jointly continuous in x and u, i.e.,

$$f(x, u) = \sum_i a_i(x, u) \frac{\partial}{\partial x_i},$$

where each $a_i : \mathcal{X} \times \mathbf{U} \to \mathbb{R}$ is locally Lipschitz in x uniformly for u in compact sets and jointly continuous, and $T_x(\mathcal{X})$ is the tangent space to $\mathcal{X}$ at x. We define the solutions $x(t, \xi, \mathbf{u})$ of (1.1) as before. Next, we generalize Definition 2.2 for GAC to manifolds.

Let $\mathcal{A}$ be a compact, nonempty, weakly invariant subset of $\mathcal{X}$ for (1.1), and let $\mathcal{PN}^{\mathcal{A}}$ be the set of all open precompact subsets of $\mathcal{X}$ containing $\mathcal{A}$. To extend the GAC definition to manifolds, we simply replace the ε-neighborhoods of $\mathcal{A}$ from Definition 2.2 with arbitrary sets in $\mathcal{PN}^{\mathcal{A}}$ as follows.

Definition 3.1. We say that (1.1) is *globally asymptotically controllable* (GAC) to $\mathcal{A}$ (on $\mathcal{X}$) if the following holds.

(1) Given any $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{PN}^{\mathcal{A}}$ with $\mathcal{E}_1 \subseteq \mathcal{E}_2$, there exist $T = T(\mathcal{E}_1, \mathcal{E}_2) > 0$ and $\Delta = \Delta(\mathcal{E}_1) \in \mathcal{PN}^{\mathcal{A}}$ such that for every $\xi \in \mathcal{E}_2$ there exists a control $\mathbf{u}$ such that
   - (a) $x(t, \xi, \mathbf{u})$ is well defined,
   - (b) $x(t, \xi, \mathbf{u}) \in \mathcal{E}_1$ for all $t > T$,
   - (c) if also $\xi \in \Delta$, then $\mathbf{u}$ can be chosen so that $x(t, \xi, \mathbf{u}) \in \mathcal{E}_1$ for all $t \geq 0$.

(2) For every set $\mathcal{N} \in \mathcal{PN}^{\mathcal{A}}$, there exists $N = N(\mathcal{N}) > 0$ such that if ξ from item (1) also satisfies $\xi \in \mathcal{N}$, then the control $\mathbf{u}$ from item (1) can be chosen with $\mathbf{u} \in \mathcal{U}_N$.

Throughout this section, we assume that our dynamic f is GAC to $\mathcal{A}$. Since our definitions of feedback and π-trajectory from Sec. 2 do not depend on the structure of the state space $\mathcal{X}$, they remain valid for systems on manifolds. We extend the definition of an s-stabilizing feedback to manifolds as follows.

Definition 3.2. A feedback $k : \mathcal{X} \to \mathbf{U}$ is said to *s-stabilize* system (1.1) to $\mathcal{A}$ if the following holds for all sets $\mathcal{R}_1$, $\mathcal{R}_2 \in \mathcal{PN}^{\mathcal{A}}$ with $\mathcal{R}_1 \subseteq \mathcal{R}_2$.

(1) There exist a set $\mathcal{M} = \mathcal{M}(\mathcal{R}_2) \subseteq \mathcal{X}$ and numbers $\delta = \delta(\mathcal{R}_1, \mathcal{R}_2) > 0$ and $T = T(\mathcal{R}_1, \mathcal{R}_2)$ such that, for any partition π with $\overline{\mathbf{d}}(\pi) < \delta$ and any ξ in $\mathcal{R}_2$, the π-trajectory $x(\cdot)$ for (1.1), the initial state ξ, and the feedback k is well defined and satisfies the following conditions:
   (a) $x(t) \in \mathcal{R}_1$ for all $t \geq T$,
   (b) $x(t) \in \mathcal{M}$ for all $t \geq 0$.
(2) For each set $\mathcal{E} \in \mathcal{PN}^{\mathcal{A}}$ there exists $\mathcal{D} \in \mathcal{PN}^{\mathcal{A}}$ such that if $\mathcal{R}_2 \subseteq \mathcal{D}$, then the set $\mathcal{M}$ in item (1) can be chosen so that $\mathcal{M} \subseteq E$.

Our goal is to show that the CLSS theorem remains true on any smooth manifold $\mathcal{X}$. To this end, we follow the strategy outlined in [21] which can be summarized as follows. We first embed the state space manifold $\mathcal{X}$ into some Euclidean space $\mathbb{R}^k$ (e.g., using the Whitney embedding theorem). Then we extend the dynamic to all of $\mathbb{R}^k$ so that

(a) the system is asymptotically controllable to a tubular neighborhood of $\mathcal{A}$ and
(b) $\mathcal{X}$ is a strongly invariant set under the extended system (see Lemma 3.5).

Next, we apply the CLSS theorem to the extended system. Thus, we obtain an s-stabilizing feedback on $\mathbb{R}^k$. When restricted to $\mathcal{X}$, this feedback will s-stabilize (1.1) to $\mathcal{A}$.

To make this construction precise, we use the following definitions and facts from differential topology (see [5, 6]). The following is known as the Whitney embedding theorem (see [6, p. 92]).

Lemma 3.3. *If $\mathcal{X}$ is an n-dimensional smooth manifold, then there exists an embedding $g : \mathcal{X} \to \mathbb{R}^{2n+1}$ for which $g(\mathcal{X})$ is a submanifold and a closed subset of $\mathbb{R}^{2n+1}$.*

By Lemma 3.3, we can assume that our state space $\mathcal{X}$ is a smooth submanifold of $\mathbb{R}^k$ with $\mathcal{X} \subseteq \mathbb{R}^k$ closed. The *normal bundle* $\Xi(\mathcal{X})$ of $\mathcal{X}$ in $\mathbb{R}^k$ is defined by

$$\Xi(\mathcal{X}) = \left\{ \langle x, q \rangle \in \mathcal{X} \times \mathbb{R}^k : q \perp T_x(\mathcal{X}) \right\}.$$

We define the projections

$$\pi_M : \Xi(\mathcal{X}) \to \mathcal{X}$$

by the rule $\pi_M(\langle x, q\rangle) = x$ and

$$\pi_N : \Xi(\mathcal{X}) \to \mathbb{R}^k$$

by the rule $\pi_N(\langle x, q\rangle) = q$, and $\theta(\langle x, q\rangle) := x + q$. For each smooth function $\omega : \mathcal{X} \to \mathbb{R}_{>0}$, the ω-*tube* $\Xi(\mathcal{X}, \omega)$ is defined by

$$\Xi(\mathcal{X}, \omega) = \{\langle x, q\rangle \in \Xi(\mathcal{X}) : |q| < \omega(x)\}.$$

The next result is known as the tubular neighborhood theorem.

Lemma 3.4. *Let $\mathcal{X}$ be a closed submanifold of $\mathbb{R}^k$. There exists a smooth function $\omega : \mathcal{X} \to \mathbb{R}_{>0}$ such that $\theta : \Xi(\mathcal{X}, \omega) \to \mathbb{R}^k : \langle x, q\rangle \mapsto x + q$ is a diffeomorphism onto an open neighborhood of $\mathcal{X}$ in $\mathbb{R}^k$.*

In particular, $\Xi(\mathcal{X}, \omega)$ is an open subset of $\mathbb{R}^k \times \mathbb{R}^k$. Choose functions ω and θ as in Lemma 3.4 for our state space manifold $\mathcal{X}$. Since $\mathcal{A} \subseteq \mathcal{X}$ is compact and ω is continuous, ω attains its minimum on $\mathcal{A}$. Let

$$\varepsilon = \frac{1}{2} \min_{x \in \mathcal{A}} \omega(x)$$

and for each set $\mathcal{A}_1 \subseteq \mathcal{X}$, define

$$\begin{aligned} \Xi(\mathcal{A}_1, \varepsilon) &= \{\langle x, q\rangle \in \Xi(\mathcal{X}) : x \in \mathcal{A}_1, |q| < \varepsilon\}, \\ \Xi(\mathcal{A}_1, \omega) &= \{\langle x, q\rangle \in \Xi(\mathcal{X}) : x \in \mathcal{A}_1, |q| < \omega(x)\}, \\ \mathbf{TN}_\varepsilon \mathcal{A}_1 &= \theta(\Xi(\mathcal{A}_1, \varepsilon)), \quad \mathbf{TN}_\omega \mathcal{A}_1 = \theta(\Xi(\mathcal{A}_1, \omega)). \end{aligned}$$

Note that if $\varepsilon \le \omega(x)$ for all $x \in \mathcal{A}_1$, then

$$\mathbf{TN}_\varepsilon \mathcal{A}_1 \subseteq \mathbf{TN}_\omega \mathcal{A}_1 \subseteq \mathbf{TN}_\omega \mathcal{X}.$$

Also, $\operatorname{clos} \mathbf{TN}_\varepsilon \mathcal{A}$ is a compact subset of $\mathbf{TN}_\omega \mathcal{X}$. Next, we consider the system

$$\dot{x} = f(x, u), \quad \dot{q} = qv, \quad \langle x, q\rangle \in \mathcal{X} \times \mathbb{R}^k, \quad \langle u, v\rangle \in \mathbf{U} \times \mathbb{R}, \tag{3.2}$$

whose (maximal) solution for the controls $\langle \mathbf{u}, \mathbf{v}\rangle$ starting from $\langle \xi, q_0\rangle$ we denote by $\langle x(t, \xi, \mathbf{u}), q(t, q_0, \mathbf{v})\rangle$, or by $\langle x(t), q(t)\rangle$ for brevity. If, for some initial state $\langle x(0), q(0)\rangle$ and controls $\langle \mathbf{u}, \mathbf{v}\rangle$, the trajectory $\langle x(t), q(t)\rangle$ of (3.2) stays in $\Xi(\mathcal{X}, \omega)$, then $y(t, y_0, \mathbf{u}, \mathbf{v}) = \theta(\langle x(t), q(t)\rangle)$ is the corresponding trajectory of

$$\begin{gathered} \dot{y} = f_1(y, u, v) := f(\pi_M(\theta^{-1}(y)), u) + \pi_N(\theta^{-1}(y))v, \\ y \in \mathbf{TN}_\omega \mathcal{X}, \quad \langle u, v\rangle \in \mathbf{U} \times \mathbb{R}, \end{gathered} \tag{3.3}$$

with the initial value $y_0 = y(0) = \theta(\langle x(0), q(0)\rangle)$. We denote this solution by $y(t)$ when no confusion would result. We also omit θ^{-1} inside the projections π_N and π_M in the sequel to simplify our notation. We (discontinuously) extend f_1 to $\mathbb{R}^k$ by defining it to be zero outside $\mathbf{TN}_\omega \mathcal{X}$.

Next, we extend our GAC system (1.1) to all of $\mathbb{R}^k$ as follows. Let $\mathcal{X}^\sharp \subseteq \mathbb{R}^k$ be any closed set contained in $\mathbf{TN}_\omega\mathcal{X}$ and containing $\mathcal{X}$ in its interior. Let $C_\omega \subseteq \mathbb{R}^k$ be any open set such that the following holds:

$$\mathbb{R}^k \setminus \mathbf{TN}_\omega\mathcal{X} \subseteq C_\omega \subseteq \operatorname{clos} C_\omega \subseteq \mathbb{R}^k \setminus \mathcal{X}^\sharp.$$

Then $\operatorname{bd} C_\omega \subseteq \mathbf{TN}_\omega\mathcal{X}$. Let $\phi : \mathbb{R}^k \to [0,1]$ be any smooth function such that

$$\phi(z) = \begin{cases} 1, & z \in \mathcal{X}^\sharp, \\ 0, & z \in \operatorname{clos} C_\omega, \end{cases} \tag{3.4}$$

which exists by a well-known separation result (see, e.g., [5, Exercise V.4.5]). Now define the system

$$\begin{gathered} \dot{z} = f_2(z,u,v,w) := f_1(z,u,v)\phi(z) + (1-\phi(z))w, \\ z \in \mathbb{R}^k, \quad \langle u,v,w\rangle \in \mathbf{U}\times\mathbb{R}\times\mathbb{R}^k, \end{gathered} \tag{3.5}$$

whose (maximal) solution starting from z_0 for given controls $\langle \mathbf{u},\mathbf{v},\mathbf{w}\rangle$ we denote by $z(t,z_0,\mathbf{u},\mathbf{v},\mathbf{w})$. Since $\phi \equiv 0$ in C_ω, we know that f_2 is locally Lipschitz in $z \in \mathbb{R}^k$. We use the following elementary observation.

Lemma 3.5. *Any trajectory $z(t)$ for f_2 starting from a point $\eta \in \mathcal{X}$ remains in $\mathcal{X}$ on its domain of definition and, therefore, is a trajectory of f. In other words, $\mathcal{X}$ is strongly invariant for f_2.*

Proof. Since $\langle x, 0\rangle \in \Xi(\mathcal{X},\omega)$ for all $x \in \mathcal{X}$, the uniqueness property for solutions of (3.3) in $\mathbf{TN}_\omega\mathcal{X}$ implies that all trajectories of f_1 starting in $\mathcal{X}$ remain in $\mathcal{X}$ and, therefore, are trajectories of f. On the other hand, trajectories $z(t)$ of f_2 starting in $\mathcal{X}$ are also trajectories of f_1 while they are in $\mathcal{X}^\sharp$ (by our choice (3.4) of ϕ), since f_1 and f_2 agree on $\mathcal{X}^\sharp$. By the uniqueness property for trajectories of f_1, $z(t)$ cannot enter $\mathcal{X}^\sharp \setminus \mathcal{X} \subseteq \mathbf{TN}_\omega\mathcal{X}$ and, therefore, stays in $\mathcal{X}$. Hence $z(t)$ is a trajectory of f_1, and also for f. ☐

This lemma forms the basis for our generalized CLSS theorem in the next section.

4. The CLSS theorem on manifolds

In this section, we prove the following generalized CLSS theorem for any smooth manifold $\mathcal{X}$ and any compact, nonempty, weakly invariant set $\mathcal{A} \subseteq \mathcal{X}$ for (1.1).

Theorem 2. *If* (1.1) *is GAC to $\mathcal{A}$ on the manifold $\mathcal{X}$, then it admits a feedback that s-stabilizes the system to $\mathcal{A}$.*

This follows from the following key lemma.

Lemma 4.1. *If system* (1.1) *is GAC to $\mathcal{A}$ on $\mathcal{X}$, then system* (3.5) *is GAC to* $\operatorname{clos} \mathbf{TN}_\varepsilon\mathcal{A}$ *on $\mathbb{R}^k$.*

We begin with proving Lemma 4.1. Fix $z_0 \in \mathbb{R}^k$, a precompact open set $\mathcal{B}$ containing clos $\mathbf{TN}_\varepsilon\mathcal{A}$, and an open set $\mathcal{A}_1 \in \mathcal{PN}^{\mathcal{A}}$ such that $\mathbf{TN}_\varepsilon\mathcal{A}_1 \subseteq \mathcal{B}$. First, we assume that $\eta := z_0 \in \mathbf{TN}_\omega\mathcal{X}$. Since (1.1) is GAC to $\mathcal{A}$, we can find a control $\mathbf{u} : [0,\infty) \to \mathbf{U}$ and constants $T_1 > 0$ and $p_1 > 0$ with $\|\mathbf{u}\| < p_1$ such that the trajectory $x(t) = x(t, \pi_M(\eta), \mathbf{u})$ of (1.1) is well defined and satisfies $x(t) \in \mathcal{A}_1$ for all $t \geq T_1$. This gives a compact set $\bar{B} \subseteq \mathcal{X}$ containing $\mathcal{A}$ such that $x(t) = x(t, \pi_M(\eta), \mathbf{u}) \in \bar{B}$ for all $t \geq 0$.

Since ω is positive and smooth on $\mathcal{X}$, there exist positive values

$$p_2 = 1 + \max_{x \in \bar{B}} |\nabla\omega(x)|, \quad p_4 = \min_{x \in \bar{B}} \omega(x), \tag{4.1}$$

and $p_3 > 0$ such that $|f(x,u)| < p_3$ for all $x \in \bar{B}$ and $|u| < p_1$. Then $p_4/2 \leq \varepsilon$, and

$$\left|\frac{d}{dt}\omega(x(t))\right| = |\nabla\omega(x(t)) \cdot f(x(t), \mathbf{u}(t))| \leq p_2 p_3 \quad \text{for almost all } t \geq 0. \tag{4.2}$$

In other words, p_2p_3 is an upper bound on the rate of change of the width $\omega(x(t))$ of $\mathbf{TN}_\omega\mathcal{X}$, as we move along the trajectory $x(t)$. Hence, to ensure that our stabilizing trajectory of (3.5) starting in $\mathbf{TN}_\omega\mathcal{X}$ stays there, we must design a control $\mathbf{v}$ so that the solution of (3.3) is pushed towards $\mathcal{X}$ faster than p_2p_3.

Since we have assumed that $\eta \in \mathbf{TN}_\omega\mathcal{X}$, we have $\langle \pi_M(\eta), \pi_N(\eta)\rangle \in \Xi(\mathcal{X}, \omega)$ and, therefore, $|\pi_N(\eta)| < \omega(\pi_M(\eta))$. Define $\mathbf{v} : [0, +\infty) \to \mathbb{R}$ by

$$\mathbf{v}(t) = \begin{cases} -\dfrac{p_2p_3}{p_4/4}, & t \in [0, T_2], \\ 0, & t > T_2, \end{cases} \tag{4.3}$$

where

$$T_2 = \max\left\{0, \frac{|\pi_N(\eta)| - p_4/4}{p_2p_3}\right\}.$$

Let $q(t)$ be the solution of $\dot{q} = \mathbf{v}q$ starting from $\pi_N(\eta)$. We set $y(t) = x(t) + q(t)$, where $x(t) = x(t, \pi_M(\eta), \mathbf{u})$ is defined above; then $y(t)$ has the domain $[0,\infty)$, and $\langle x(t), q(t)\rangle$ is a solution of (3.2) on $[0, +\infty)$. Next, we define

$$t' = \inf\{t \geq 0 : \langle x(t), q(t)\rangle \in \operatorname{bd}\Xi(\mathcal{X}, \omega)\},$$

so that $\langle x(t), q(t)\rangle \in \Xi(\mathcal{X}, \omega)$ on $[0, t')$. We show that $t' = +\infty$. This will show that $y(t)$ is a solution of (3.3) on all of $\mathbb{R}_{\geq 0}$. To this end, first note that:

(i) since the direction of $\mathbf{v}(t)q(t)$ is always opposite to that of $q(t)$ whenever $\mathbf{v}(t) \neq 0$, the function $|q(t)|$ is nonincreasing on $\mathbb{R}_{\geq 0}$;

(ii) at all points $t \in [0, T_2]$ for which $\dot{x}(t)$ exists and $|q(t)| \geq p_4/4$, the following holds:

$$\frac{d}{dt}|q(t)| = -\frac{p_2 p_3}{p_4/4}|q(t)| \leq -p_2 p_3 \leq -\left|\frac{d}{dt}\omega(x(t))\right| \leq \frac{d}{dt}\omega(x(t)). \tag{4.4}$$

By separately considering the case where $|q(t)|$ stays above $p_4/4$ on $[0, T_2]$ and using (4.3) and (4.4), one can easily verify that $|q(T_2)| \leq p_4/4$; this inequality is clear if $|q(t)|$ ever goes below $p_4/4$ on $[0, T_2]$, by (i). Hence, $|q(t)| \leq p_4/4$ for all $t \geq T_2$, by the choice of $\mathbf{v}$. Similarly, we can use (4.4), the definition of p_4, and the fact that $|\pi_N(\eta)| < \omega(\pi_M(\eta))$ to verify that

$$|q(t)| < \omega(x(t)) \quad \forall t \geq 0. \tag{4.5}$$

Suppose that $t' < \infty$. Then $\langle x(t'), q(t')\rangle \in \operatorname{bd}\Xi(\mathcal{X}, \omega)$. Since $\Xi(\mathcal{X})$ is closed and $\langle x(t), q(t)\rangle \in \Xi(\mathcal{X})$ on $[0, t')$, it follows from (4.5) that $\langle x(t'), q(t')\rangle \in \Xi(\mathcal{X}, \omega)$, contradicting the openness of $\Xi(\mathcal{X}, \omega)$. It follows that $t' = +\infty$ and, therefore, the solution $y(t) := y(t, \eta, \mathbf{u}, \mathbf{v})$ of system (3.3) maps all of $\mathbb{R}_{\geq 0}$ into $\mathbf{TN}_\omega\mathcal{X}$.

Finally, we define a control $\mathbf{w} : [0, \infty) \to \mathbb{R}^k$ as follows:

$$\mathbf{w}(t) = f_1(y(t, \eta, \mathbf{u}, \mathbf{v}), \mathbf{u}(t), \mathbf{v}(t)). \tag{4.6}$$

The control $\mathbf{w}$ cancels the effect of ϕ in (3.5) for states in $\mathbf{TN}_\omega\mathcal{X}$. In fact,

$$f_2(y(t, \eta, \mathbf{u}, \mathbf{v}), \mathbf{u}(t), \mathbf{v}(t), \mathbf{w}(t)) \equiv f_1(y(t, \eta, \mathbf{u}, \mathbf{v}), \mathbf{u}(t), \mathbf{v}(t)),$$

hence $y(t, \eta, \mathbf{u}, \mathbf{v}) \equiv z(t, \eta, \mathbf{u}, \mathbf{v}, \mathbf{w})$. By our choice of T_1, p_4, and $\mathbf{v}$, we have

(a) $x(t) = \pi_M(y(t, \eta, \mathbf{u}, \mathbf{v})) \in \mathcal{A}_1$ for all $t \geq T_1$,
(b) $|\pi_N(y(t, \eta, \mathbf{u}, \mathbf{v}))| < p_4/2 \leq \varepsilon$ for all $t > T_2$.

Therefore, it follows that

$$z(t, \eta, \mathbf{u}, \mathbf{v}, \mathbf{w}) = y(t, \eta, \mathbf{u}, \mathbf{v}) \in \mathbf{TN}_\varepsilon\mathcal{A}_1 \subseteq \mathcal{B} \quad \forall t > T,$$

where $T := \max\{T_1, T_2\}$. This proves the asymptotic controllability of (3.5) to our arbitrary neighborhood $\mathcal{B}$ of $\operatorname{clos}\mathbf{TN}_\varepsilon\mathcal{A}$ from any initial value in $\mathbf{TN}_\omega\mathcal{X}$. Next, we show that this controllability property also holds for initial values outside $\mathbf{TN}_\omega\mathcal{X}$.

Then, we assume that $z_0 \notin \mathbf{TN}_\omega\mathcal{X}$ and, therefore, $z_0 \in C_\omega$. We reduce this to the case where the initial value is in $\mathbf{TN}_\omega\mathcal{X}$. Let $p_5 = \operatorname{dist}(\mathcal{X}, z_0)$ and let $\eta_1 \in \mathcal{X}$ be such that $|\eta_1 - z_0| = p_5$. Define $\bar{\mathbf{w}}$ and $z(t)$ by the relations

$$\begin{gathered}\bar{\mathbf{w}}(t) = \frac{\eta_1 - z_0}{|\eta_1 - z_0|} \quad \forall t \geq 0; \quad z(t) = z_0 + t\frac{\eta_1 - z_0}{|\eta_1 - z_0|}, \\ 0 \leq t \leq \hat{t} := \inf\{t \geq 0 : z(t) \in \operatorname{bd} C_\omega\}.\end{gathered} \tag{4.7}$$

Then $z(t)$ is a solution of (3.5) starting from z_0 for any controls $\langle \mathbf{u}, \mathbf{v}\rangle$ and the choice $\mathbf{w} = \bar{\mathbf{w}}$, and $z(t) \in C_\omega$ on $[0, \hat{t})$. Also, $0 < \hat{t} \leq |\eta_1 - z_0|$, since if it were the case that $\hat{t} > |\eta_1 - z_0|$, then setting $t = |\eta_1 - z_0|$ in (4.7) would give $z(t) = \eta_1 \in \mathcal{X} \cap C_\omega$. This would contradict the fact that $C_\omega \subseteq \mathbb{R}^k \setminus \mathcal{X}$. In particular, we conclude that $\hat{t} < \infty$ and, therefore,

$$\eta := z(\hat{t}, z_0, \mathbf{u}, \mathbf{v}, \bar{\mathbf{w}}) \in \operatorname{bd} C_\omega \subseteq \mathbf{TN}_\omega \mathcal{X}.$$

For our precompact open set $\mathcal{B}$, we now construct the controls $\mathbf{u}$ from the controllability of (1.1), $\mathbf{v}$ as in (4.3), and $\mathbf{w}$ as in (4.6), driving this choice of η to $\mathcal{B}$. Let $\mathbf{u}^\sharp$ and $\mathbf{v}^\sharp$ be the concatenations of the zero functions on $[0, \hat{t})$, followed by $\mathbf{u}$ and $\mathbf{v}$, respectively. Let $\mathbf{w}^\sharp$ be the concatenation of $\bar{\mathbf{w}}$ on $t \in [0, \hat{t})$ from (4.7), followed by $\mathbf{w}$ from (4.6) for $t \geq \hat{t}$. The control vector $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle$ for (3.5) drives η to $\mathcal{B}$ in time T and, therefore, $z(t, z_0, \mathbf{u}^\sharp, \mathbf{v}^\sharp, \mathbf{w}^\sharp) \in \mathcal{B}$ for all $t > \hat{t} + T$. Since T and $\hat{t}$ are locally bounded functions of z_0 and $\mathcal{B}$, we conclude that conditions (1a) and (1b) from the GAC definition hold for (3.5) and the attractor $\operatorname{clos} \mathbf{TN}_\varepsilon \mathcal{A}$.

To establish condition (1c) of the GAC definition for (3.5), fix any precompact open set $E \subseteq \mathbb{R}^k$ containing $\operatorname{clos} \mathbf{TN}_\varepsilon \mathcal{A}$. We can find an open set $\mathcal{E}_1 \in \mathcal{PN}^{\mathcal{A}}$ and $\varepsilon' > \varepsilon$ such that $\varepsilon' < \omega(x)$ for all $x \in \mathcal{E}_1$, and such that $\mathbf{TN}_{\varepsilon'} \mathcal{E}_1 \subseteq E$. Next, we find a set $\Delta \in \mathcal{PN}^{\mathcal{A}}$ as in condition (1c) of Definition 3.1 for the GAC system (1.1), corresponding to $\mathcal{E}_1$. It follows that $\operatorname{clos} \mathbf{TN}_\varepsilon \mathcal{A} \subseteq \mathbf{TN}_{\varepsilon'} \Delta$. By reducing Δ, we can assume $\varepsilon' < \omega(x)$ for all $x \in \Delta$, and, therefore, $\mathbf{TN}_{\varepsilon'} \Delta \subseteq \mathbf{TN}_\omega \Delta \subseteq \mathbf{TN}_\omega \mathcal{X}$. We show that if $z_0 \in D := \mathbf{TN}_{\varepsilon'} \Delta$, then z_0 can be driven to $\mathcal{B}$ using system (3.3) and the vector of controls $\langle \mathbf{u}, \mathbf{v} \rangle$ as defined above, hence also by the extended system (3.5), while being kept inside $\mathbf{TN}_{\varepsilon'} \mathcal{E}_1 \subseteq E$ for all $t \geq 0$.

Let $z_0 \in D$. Since $\pi_M(z_0) \in \Delta$, we can arrange (by the choice of Δ) that $\mathbf{u}$ is such that $x(t, \pi_M(z_0), \mathbf{u}) \in \mathcal{E}_1$ for all $t \geq 0$. Next, we construct $\mathbf{v}$ defined by (4.3) for the initial state $z_0 \in \mathbf{TN}_{\varepsilon'} \Delta \subseteq \mathbf{TN}_\omega \mathcal{X}$. By (i), we know that $t \mapsto |\pi_N(y(t, z_0, \mathbf{u}, \mathbf{v}))|$ is nonincreasing. Thus, for all $t \geq 0$, we obtain

$$|\pi_N(y(t, z_0, \mathbf{u}, \mathbf{v}))| \leq |\pi_N(z_0)| < \varepsilon'$$

and, therefore, also $y(t, z_0, \mathbf{u}, \mathbf{v}) \in \mathbf{TN}_{\varepsilon'} \mathcal{E}_1 \subseteq E$, proving condition (1c) from the GAC definition for system (3.5).

It remains to verify that the concatenated controls $\mathbf{u}^\sharp$, $\mathbf{v}^\sharp$, and $\mathbf{w}^\sharp$ constructed above satisfy the boundedness requirement from Condition (2) of the GAC definition. That is, we need to verify that $\|\langle \mathbf{u}^\sharp, \mathbf{v}^\sharp, \mathbf{w}^\sharp \rangle\|$ is a locally bounded function of the initial state z_0. To do this, first note that the boundedness requirement on $\mathbf{u}^\sharp$ is satisfied since (1.1) is assumed to be GAC to $\mathcal{A}$ on $\mathcal{X}$. Next, $\|\mathbf{v}^\sharp\| \leq p_2 p_3 / (p_4/4)$, and (letting $\mathbf{u}$ be the second part of the concatenation $\mathbf{u}^\sharp$ and similarly for $\mathbf{v}$, as before)

$$\left\|\mathbf{w}^\sharp\right\| \leq 1 + \operatorname*{ess\,sup}_{t \geq 0} \left\{ |f(\pi_M(y(t, \eta, \mathbf{u}, \mathbf{v})), \mathbf{u}(t))| + |\pi_N(y(t, \eta, \mathbf{u}, \mathbf{v}))| \frac{p_2 p_3}{p_4/4} \right\}.$$

Here $f(\pi_M(y(t, \eta, \mathbf{u}, \mathbf{v})), \mathbf{u}(t))$ stays bounded since

(a) $\pi_M(y(t, \eta, \mathbf{u}, \mathbf{v})) = x(t, \pi_M(\eta), \mathbf{u}) \in \bar{B}$ for all $t \geq 0$,

(b) $\bar{B}$ and p_i are locally bounded functions of the state $\eta = z(\hat{t}, z_0, \mathbf{u}^\sharp, \mathbf{v}^\sharp, \mathbf{w}^\sharp)$.

Also, $|\pi_N(y(t, \eta, \mathbf{u}, \mathbf{v}))|$ stays bounded since it decreases from $|\pi_N(\eta)|$. Hence, condition (2) of the GAC definition holds. This completes the proof of Lemma 4.1.

Finally, we prove Theorem 2. The previous argument applied to $\eta \in \operatorname{clos} \mathbf{TN}_\varepsilon \mathcal{A}$ (with $\mathbf{u}$ chosen so that $x(t, \pi_M(\eta), \mathbf{u}) \in \mathcal{A}$ for all $t \geq 0$, which exists by the weak invariance of $\mathcal{A}$) shows that the compact set $\operatorname{clos} \mathbf{TN}_\varepsilon \mathcal{A}$ is weakly invariant for (3.5). Since (3.5) is GAC to $\operatorname{clos} \mathbf{TN}_\varepsilon \mathcal{A}$ on $\mathbb{R}^k$, the CLSS theorem (namely, Theorem 1 above) provides an s-stabilizing feedback $K(x)$ for (3.5). By Lemma 3.5, $\mathcal{X}$ is strongly invariant for f_2. It follows that the u-part of $K(x)$ stabilizes (1.1). This proves Theorem 2.

Remark 4.2. Given a GAC system (1.1) evolving on a manifold, we can also construct feedbacks that asymptotically stabilize the system in the sense of Euler or Carathéodory solutions from almost all initial values, or that stabilize certain Carathéodory solutions using discontinuous patchy feedbacks (see [2, 16] for the relevant definitions and results for systems on Euclidean space). The extension to manifolds is done by applying the corresponding feedback constructions of [2, 16] to the extended system on $\mathbb{R}^k$, which is GAC to $\operatorname{clos} \mathbf{TN}_\varepsilon \mathcal{A}$ on $\mathbb{R}^k$ by Lemma 4.1, and then restricting the feedback to the invariant set $\mathcal{X}$ to obtain the desired feedback for the original system, exactly as before. In the same way, we can use the results of [12] to design feedbacks for control-affine systems that render the systems input-to-state stable with respect to actuator errors, in the sense of Euler solutions.

5. Illustration

Next, we illustrate our stabilization approach using the system

$$\dot{x} = A_1(x)u_1 + A_2(x)u_2 \in T_x(\mathcal{X}), \quad x \in \mathcal{X}, \quad u = \langle u_1, u_2 \rangle \in \mathbb{R}^2, \qquad (5.1)$$

evolving on the sphere $\mathcal{X} := S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$. This simple example will illustrate how to construct stabilizing state feedbacks and Lyapunov functions on smooth manifolds. Even in this simple case, we will see the necessity for using *discontinuous* stabilizers. Our example is a modified version of the engineering examples in [8]. We choose the attractor $\mathcal{A} = \{\pm q\}$, where $q = \langle 0, 0, 1 \rangle$, but similar constructions apply for any $q \in S^2$. The vector fields A_1 and A_2 are chosen as follows. First, define

$$B_1(x) = q - (x \cdot q)x, \quad B_2(x) = x \times q,$$

which form an orthogonal basis for the tangent spaces $T_x(\mathcal{X}) = \operatorname{span}\{x\}^\perp$ on $S^2 \setminus \mathcal{A}$ (in terms of the cross product $\times$, the standard inner product $\cdot$, and the orthogonal complement $\perp$). Define the *geodesic distance* $\mathcal{G}$ on S^2 by $\mathcal{G}(x,x') := \arccos(x \cdot x')$ for $x, x' \in S^2$. We set $r = \langle 0,1,0 \rangle \in S^2$ and

$$\begin{aligned} V_q(x) &= \min\{\mathcal{G}(x,\bar{q}) : \bar{q} \in \{\pm q\}\}, \\ V_r(x) &= \max\{\mathcal{G}(x,\bar{r}) : \bar{r} \in \{\pm r\}\}, \end{aligned} \qquad x \in S^2. \tag{5.2}$$

Note the asymmetry between V_q and V_r. Roughly speaking, we use max in V_r to produce a component in our Lyapunov function that penalizes states near $\pm r$ (see (5.6)). Let $M_1 : S^2 \to [0,1]$ be any smooth function satisfying the following conditions:

(a) $M_1^{-1}(0) = \left\{x \in S^2 : \dfrac{x_1}{4} \leq x_2 \leq \dfrac{3x_1}{4} \text{ and } V_q(x) \geq \dfrac{\pi}{4}\right\}$,

(b) $M_1^{-1}(1) \supseteq \left\{x \in S^2 : x_2 \geq \dfrac{7x_1}{8} \text{ or } x_2 \leq \dfrac{x_1}{8} \text{ or } V_q(x) \leq \dfrac{\pi}{8}\right\}$.

We set

$$A_1(x) = M_1(x)B_1(x), \quad A_2(x) = B_2(x). \tag{5.3}$$

The factor M_1 in (5.3) introduces a set of zeros in A_1, consisting of a geodesic rectangle covering a part of the equator of S^2 in the quadrant

$$Q_{++} := \{x \in S^2 : x_1 > 0,\ x_2 > 0\}.$$

In particular, system (5.1) is not completely controllable. The fact that (5.1) is GAC to $\mathcal{A}$ follows since any initial value can be moved to $\mathcal{A}$ along the geodesic direction (i.e., "north" or "south" along a great circle through $\pm q$) using the vector field A_1, possibly by first using A_2 to move the state "west" out of $M_1^{-1}([0,1))$ (see below for a precise definition of these stabilizing trajectories). In fact, this global stabilization is done by the discontinuous feedback (5.5) constructed below. On the other hand, a simple continuous dependence and separation argument (e.g., the argument from the appendix in [24]) shows that the system has no Lipschitz stabilizing state feedback $K(x)$. (In fact, there is no continuous stabilizing feedback for this system either, since this and standard Lyapunov-function arguments would give a smooth Lyapunov function for the system and hence a stabilizing feedback that is smooth outside $\mathcal{A}$, which cannot exist by the same separation argument.)

The extension of (5.1) from the generalized CLSS theorem amounts to projecting onto the sphere as follows. The state space $\mathcal{X}$ is embedded into $\mathbb{R}^3$ by inclusion and $\Xi(\mathcal{X}) = \{\langle x, kx \rangle : x \in \mathcal{X},\ k \in \mathbb{R}\}$. We can choose $\omega(x) \equiv 1/4$ on $\mathcal{X}$. This gives the ω-tube and annular tubular neighborhood

$$\begin{aligned} \Xi(\mathcal{X},\omega) &= \{\langle x, kx \rangle \in \mathcal{X} \times \mathbb{R}^3 : |k| < 1/4\}, \\ \mathbf{TN}_\omega \mathcal{X} &= \theta(\Xi(\mathcal{X},\omega)) = \left\{x \in \mathbb{R}^3 : 3/4 < |x| < 5/4\right\}. \end{aligned}$$

In terms of the projection $\Pi_s(y) := y/|y|$ defined on $\mathbb{R}^3 \setminus \{0\}$, the corresponding system f_1 on $\mathbf{TN}_\omega\mathcal{X}$ is (see (3.3))

$$f_1(y,u,v) = M_1(\Pi_s(y))\,(q - \{\Pi_s(y)\cdot q\}\,\Pi_s(y))\,u_1 + \{\Pi_s(y)\times q\}\,u_2 \\ + (y - \Pi_s(y))\,v, \quad \langle u,v\rangle \in \mathbb{R}^2\times\mathbb{R},$$

which we (discontinuously) extend to $\mathbb{R}^3$ by setting $f_1 \equiv 0$ for states outside $\mathbf{TN}_\omega\mathcal{X}$. Next, we choose

$$\mathcal{X}^\sharp = \left\{ z\in\mathbb{R}^3 : \frac{7}{8} \le |z| \le \frac{9}{8}\right\},$$
$$C_\omega = \mathbb{R}^3 \setminus \left\{ z\in\mathbb{R}^3 : \frac{13}{16} \le |z| \le \frac{19}{16}\right\}.$$

Then the corresponding system f_2 on $\mathbb{R}^3$ can be defined by taking $\phi(z) = \Gamma(|z|^2)$ for any smooth function $\Gamma : [0,\infty) \to [0,1]$ such that

$$\Gamma \equiv \begin{cases} 1 & \text{on } [(7/8)^2, (9/8)^2], \\ 0 & \text{outside } ((13/16)^2, (19/16)^2). \end{cases}$$

Since f_2 is GAC to $\mathbf{TN}_{\omega/2}\mathcal{A}$, there exists a sample stabilizing feedback for f_2 whose restriction $K(x)$ to $\mathcal{X}$ stabilizes (5.1) to $\mathcal{A}$. This is the content of our generalized CLSS theorem.

The stabilizing feedback $K(x)$ and the corresponding control-Lyapunov function (CLF) can be explicitly constructed by the following variant of the argument from [8, Sec. 2]. We set $\mathcal{A}^\sharp = \{\pm q, \pm r\} \subseteq S^2$. Define

$$Y_x^{\bar p} := \Pi_s(\bar p - \{\bar p\cdot x\}x), \quad \bar p \in \mathcal{A}^\sharp, \quad x \ne \pm\bar p;$$

this gives the geodesic direction from x to $\bar p$. Note that

$$A_1(x)\cdot Y_x^q \equiv M_1(x)\sqrt{1-x_3^2}, \quad A_2(x)\cdot Y_x^q \equiv 0.$$

Also, $Y_x^{-\bar p} \equiv -Y_x^{\bar p}$ for all $\bar p \in \mathcal{A}^\sharp$. A straightforward calculation (see [8, Lemma 1]) shows that along any (open loop) trajectory of (5.1) that does not pass through $\mathcal{A}^\sharp$, we obtain

$$\frac{d}{dt}\mathcal{G}(x,\bar p) = -\dot x\cdot Y_x^{\bar p}, \quad \bar p\in\mathcal{A}^\sharp, \quad x\ne\pm\bar p. \tag{5.4}$$

We show that (5.1) can be globally stabilized to $\mathcal{A}$ by the (necessarily discontinuous) state feedback

$$K(x) = \begin{cases} \langle 1,0\rangle & \text{if } x_3 \ge 0 \text{ and } M_1(x) = 1, \\ -\langle 1,0\rangle & \text{if } x_3 < 0 \text{ and } M_1(x) = 1, \\ \langle 0,1\rangle & \text{if } M_1(x) < 1, \end{cases} \tag{5.5}$$

when the closed-loop trajectories are defined in the usual non-sampling sense in regions where K is constant. Then an easy argument shows that (5.5) also sample stabilizes (5.1). Before presenting our argument, we interpret

(5.5) in terms of the corresponding closed loop (nonsampling) trajectories. For values where $M_1(x) = 1$, the feedback K drives the state to $\mathcal{A}$ geodesically along a great circle through $\pm q$. On the other hand, any state where $M_1(x) < 1$ is driven towards $-r$ until the state reaches $M_1^{-1}(1)$ and then geodesically to $\mathcal{A}$.

We first analyze those nonsampling trajectories of the closed loop system for $K(x)$ that remain in regions where K stays constant (which are, in particular, Carathéodory solutions that are right continuous at $t = 0$), which we refer to simply as "closed loop trajectories" in the sequel. We use the following Lyapunov function construction. In terms of V_q and V_r in (5.2), we set

$$V(x) = V_q(x)[1 + V_r(x)], \quad x \in S^2. \tag{5.6}$$

Then V is continuous and nonnegative, and V is null only on $\mathcal{A}$. We will show that V is an *integral* Lyapunov function for (5.1) in the sense of [1]; this will imply that V is also a CLF in the usual Dini derivative sense used for example in [9].[2] Given a closed loop trajectory $x(t)$, we also let $\dot{V}(x)$ denote the derivative of $t \mapsto V(x(t))$ when it is defined. Along any trajectory $x(t)$ of the closed loop system that remains in $M_1^{-1}(1) \setminus \mathcal{A}^\sharp$ and that satisfies $x_2 > 0$ and $x_3 > 0$ everywhere, we obtain

$$V(x) = \mathcal{G}(x, q)[1 + \mathcal{G}(x, -r)]$$

and, therefore, (5.4) yields

$$\begin{aligned}
\dot{V}(x) &= -\dot{x} \cdot Y_x^q \, [1 + \mathcal{G}(x, -r)] - \dot{x} \cdot Y_x^{-r} \mathcal{G}(x, q) \\
&= -A_1(x) \cdot Y_x^q \, [1 + \mathcal{G}(x, -r)] + A_1(x) \cdot \Pi_s(r - \{x \cdot r\}x)\mathcal{G}(x, q) \\
&= -A_1(x) \cdot Y_x^q [1 + \mathcal{G}(x, -r)] - \frac{x_2 x_3}{|r - (x \cdot r)x|}\mathcal{G}(x, q) \\
&\leq -\sqrt{1 - (x \cdot q)^2},
\end{aligned}$$

by recalling that $|q - (x \cdot q)x| = \sqrt{1 - x_3^2}$, and this is zero only for $x \in \mathcal{A}$. Similar arguments show that

$$\dot{V}(x) \leq -\sqrt{1 - (x \cdot q)^2} \quad \text{if } x_2 \neq 0,\ x_3 \neq 0,\ M_1(x) = 1, \text{ and } x \notin \mathcal{A}, \tag{5.8}$$

[2] A *control-Lyapunov integral function* for (5.1) and $\mathcal{A}$ is defined to be any continuous function $V : \mathcal{X} \to [0, \infty)$ for which $V^{-1}(0) = \mathcal{A}$ and for which there exist a constant $N > 0$ and $\alpha_3 \in \mathcal{K}$ satisfying the following condition: For each $\xi \in \mathcal{X}$, there exists $u \in \mathcal{U}_N$ such that $x(t) := x(t, \xi, u)$ is well defined and satisfies (see [1,17])

$$V(x(t)) - V(\xi) \leq -\int_0^t \alpha_3(|x(s)|_{\mathcal{A}})ds \quad \forall t \geq 0. \tag{5.7}$$

We will verify decay condition (5.7) using closed loop trajectories and corresponding feedback controls $u(t) = K(x(t))$ for (5.1). Inequality (5.7) then gives the usual Dini derivative Lyapunov decay condition for V (e.g., from [9,11,23]) when we divide through by t and pass to the liminf. This last step uses the fact that $t \mapsto \dot{x}(t) = f(x(t), K(x(t)))$ is (right) continuous at $t = 0$ for each closed loop trajectory $x(t)$ of (5.1).

and

$$\dot{V}(x) = -\left(1 + \frac{\pi}{2}\right)\sqrt{1 - x_3^2}$$

along trajectories in $\{x_2 = 0, x_3 \neq 0\}$ (where $\mathcal{G}(x, \pm r) = \pi/2$). Note that $\dot{V}$ is continuous along closed loop trajectories in $M_1^{-1}(1)$ starting outside $\{x_3 = 0\}$, and that the closed loop trajectories starting in $M_1^{-1}(1)$ with $x_3(0) = 0$ also satisfy $x_3(t) \neq 0$ for all $t > 0$. This gives

$$V(x(t)) - V(x(0)) \leq -\int_0^t \sqrt{1 - x_3^2(s)}ds$$

along each closed loop trajectory starting in $M_1^{-1}(1)$.

On the other hand, along closed loop trajectories in $M_1^{-1}([0,1))$, we know that $V_r(x) = \mathcal{G}(x, -r)$ and, therefore,

$$\begin{aligned}\dot{V}(x) &= -\dot{x} \cdot Y_x^{\pm q}\,[1 + \mathcal{G}(x, -r)] - \dot{x} \cdot Y_x^{-r}\,V_q(x)\\ &= -A_2(x) \cdot Y_x^{-r}\,V_q(x) \quad (\text{since } A_2(x) \cdot Y_x^q \equiv 0)\\ &= (x \times q) \cdot \Pi_s(r - \{x \cdot r\}x)V_q(x) = -\frac{x_1\,V_q(x)}{\sqrt{x_1^2 + x_3^2}}\\ &=: -\mu(x)\end{aligned} \tag{5.9}$$

when $x_3 \neq 0$, by recalling that

$$|r - (r \cdot x)x| = \sqrt{1 - x_2^2} = \sqrt{x_1^2 + x_3^2}.$$

Note that $-\mu$ is bounded above by a negative constant in $M_1^{-1}([0,1))$. Also, $\dot{V}(x) \equiv -\pi/2$ along closed loop trajectories in $M_1^{-1}([0,1))$ along $x_3 = 0$, where $\mathcal{G}(x, \pm q) = \pi/2$. Therefore, reasoning exactly as before gives

$$V(x(t)) - V(x(0)) \leq -\int_0^t \mu(x(s))ds$$

along all closed loop trajectories $x(t)$ remaining in $M_1^{-1}([0,1))$. Finally, $B_1(\pm q) = B_2(\pm q) = 0$. Since $M_1^{-1}(1)$ is forward invariant for the closed loop trajectories, it follows that V satisfies the requirements for being a control-Lyapunov (integral) function for (5.1) and also a CLF for (5.1) in the usual Dini derivative sense of [9]. The fact that $K(x)$ also *sample* stabilizes (5.1) now follows since

(a) the sampling and (nonsampling) closed loop trajectories agree for initial points in $M_1^{-1}(1)$ and
(b) the equality $\dot{V}(x) = -\mu(x)$ holds throughout the quadrant Q_{++} if we use the control $u \equiv \langle 0, 1\rangle$ at all points in Q_{++}.

In fact, (a) implies that K sample stabilizes (5.1) for initial values in $M_1^{-1}(1)$ for all partitions π. Also, (b) implies that K sample stabilizes the dynamic for initial values in $M_1^{-1}([0,1))$ when $\overline{\mathbf{d}}(\pi)$ is sufficiently small for the sample control value to switch to $\pm\langle 1,0\rangle$ in $M_1^{-1}(1)$ but before the first time, the sample trajectory exits Q_{++}.

6. Further extensions

Next, we use our results to establish the input-to-state stabilizability (ISSability) of control affine systems

$$\dot{x} = h(x) + G(x)u, \quad x \in \mathcal{X}, \quad u \in \mathbf{U} = \mathbb{R}^m, \tag{6.1}$$

evolving on smooth Riemannian manifolds $\mathcal{X}$ with respect to actuator errors (but see Remark 6.3 below for an extension to completely nonlinear systems). We assume that (6.1) is GAC to a weakly invariant compact nonempty set $\mathcal{A} \subseteq \mathcal{X}$. In this context,

$$G(x)u = g_1(x)u_1 + \cdots + g_m(x)u_m$$

for locally Lipschitz vector fields $g_i : \mathcal{X} \to \mathbb{R}$. The stabilizers constructed in this section have the additional desirable feature that they are robust to small observation noise in the controllers. For *continuous* feedback stabilizers, small observation noise in the controllers can be tolerated. However, since our stabilizing feedback may need to be *discontinuous* (see Sec. 1), such noise terms can have a substantial effect on the dynamics. Therefore, the magnitude of the noise needs to be constrained in terms of the sampling frequency (see [12, 13, 20] and Definition 6.2 below).

To make our ISSability notion precise, we first introduce a Riemannian metric on $\mathcal{X}$ to quantify observation noise. Let $\mathcal{B}_r(y)$ denote the corresponding closed ball in $\mathcal{X}$ centered at $y \in \mathcal{X}$ of radius r. As before, a *feedback* for (6.1) is defined to be any locally bounded function $k : \mathcal{X} \to \mathbf{U}$. We introduce the set of functions $\mathcal{O} = \{e : [0,\infty) \to [0,\infty)\}$, which represent the observation errors in our controller; and for each $e \in \mathcal{O}$, we set $\sup(e) = \sup\{e(t) : t \geq 0\}$. We use the set of functions $\mathcal{O}(\eta) := \{e \in \mathcal{O} : \sup(e) \leq \eta\}$ for each $\eta > 0$. We denote by Par the set of all partitions and $\text{Par}(\delta) := \left\{\pi \in \text{Par} : \overline{\mathbf{d}}(\pi) < \delta\right\}$ for each $\delta > 0$. Our ISSability goal of this section is to find a feedback k such that

$$\dot{x}(t) = h(x(t)) + G(x(t))[k(\eta(t)) + \mathbf{u}(t)], \quad \eta(t) \in \mathcal{B}_{e(t)}(x(t)) \tag{6.2}$$

is input-to-state stable (ISS) for sampling solutions with respect to actuator errors $\mathbf{u}$ for small observation errors e.

Definition 6.1. Let k be a feedback for (6.1), $e \in \mathcal{O}$, $\mathbf{u} \in \mathcal{U}$, $\xi \in \mathcal{X}$, and $\pi = \{t_i\}_{i \geq 0}$ be any partition of $\mathbb{R}_{\geq 0}$. A *π-solution* of (6.2), the initial state ξ, the observation error $e \in \mathcal{O}$, and $\mathbf{u}$ is defined to be any continuous function $x(\cdot)$ obtained by recursively choosing any $\eta(t_i) \in \mathcal{B}_{e(t_i)}(x(t_i))$ and then solving

$$\dot{x}(t) = h(x(t)) + G(x(t))[k(\eta(t_i)) + \mathbf{u}(t)]$$

from the initial time $t = t_i$ up to time

$$s_i = \max\Big\{t_i, \sup\big\{s \in [t_i, t_{i+1}] : x(\cdot) \text{ is defined on } [t_i, s)\big\}\Big\}, \tag{6.3}$$

where $x(0) = \xi$.[3] The domain of $x(\cdot)$ is the interval $[0, t_{\max})$, where $t_{\max} = \inf\{s_i : s_i < t_{i+1}\}$. If $t_{\max} = +\infty$, $x(t)$ is said to be *well defined*.

Definition 6.2. Let k be a feedback for (6.1). We say that k renders (6.1) *sample-input-to-state stable* (*s-ISS*) *to* $\mathcal{A}$ provided that for each $\mathcal{R}_0 \in \mathcal{PN}^{\mathcal{A}}$ and each $N > 0$, there exists $\mathcal{R}_1 = \mathcal{R}_1(N) \in \mathcal{PN}^{\mathcal{R}_0}$ such that:

(1) for each $\mathcal{R}_2, \mathcal{R}_3 \in \mathcal{PN}^{\mathcal{R}_1}$ with $\mathcal{R}_2 \subseteq \mathcal{R}_3$, there exist $\mathcal{M} = \mathcal{M}(\mathcal{R}_3) \subseteq \mathcal{X}$ and positive numbers δ, T, and κ (depending on $\mathcal{R}_2$ and $\mathcal{R}_3$) such that if $\pi \in \mathrm{Par}(\delta)$, $\xi \in \mathcal{R}_3$, $\mathbf{u} \in \mathcal{U}_N$, and $e \in \mathcal{O}(\kappa\underline{\mathbf{d}}(\pi))$, then the corresponding π-solutions $x(t)$ for (6.2) starting at ξ are all well defined and satisfy
   (a) $x(t) \in \mathcal{R}_2$ for all $t \geq T$,
   (b) $x(t) \in \mathcal{M}$ for all $t \geq 0$.

(2) for each $\mathcal{E} \in \mathcal{PN}^{\mathcal{R}_1}$, there exists $D \in \mathcal{PN}^{\mathcal{R}_1}$ such that if the set $\mathcal{R}_3$ in item (1) is a subset of D, then the set $\mathcal{M}$ in item (1) can be chosen to be a subset of $\mathcal{E}$,

and for each $\mathcal{R}_0 \in \mathcal{PN}^{\mathcal{A}}$, there exists $N = N(\mathcal{R}_0) > 0$ such that conditions (1) and (2) hold with the choice $N = N(\mathcal{R}_0)$ and $\mathcal{R}_1 = \mathcal{R}_0$. In this case, we also say that (6.2) is *ISS for sampling solutions* and that (6.1) is *ISSable*.

This definition requires that the sampling be done sufficiently quickly so that $\pi \in \mathrm{Par}(\delta)$, but not so quickly that $\sup(e) > \kappa\underline{\mathbf{d}}(\pi)$. When $e \equiv 0$, the condition on $\underline{\mathbf{d}}(\pi)$ in Definition 6.2 is not needed. (See also [3] for stabilization under more general observation errors which are merely required to be small in total variation.) For $\mathcal{X} = \mathbb{R}^n$, one can easily verify that if (6.1) is sampling ISS in the sense defined in [12, 13] using some feedback k, then it is also ISSable in the sense of Definition 6.2 with the same feedback k. Then, for any compact nonempty weakly invariant set $\mathcal{A}$ for (6.1), we have the following theorem.

Theorem 3. *If* (6.1) *is GAC to* $\mathcal{A}$, *then there exists a feedback* $k(x)$ *rendering* (6.1) *s-ISS to* $\mathcal{A}$.

Proof. We indicate the changes needed in the proof of Theorem 2. As before, we first extend the dynamics $f(x, u) = h(x) + G(x)u$ to dynamics (3.5) defined on all of $\mathbb{R}^k$ that is GAC to $\operatorname{clos}\mathbf{TN}_\varepsilon\mathcal{A}$. By [11, Theorem 3.2], this extended dynamics admits a locally Lipschitz control-Lyapunov function

[3] As before, the continuity requirement for $x(\cdot)$ stipulates that the final value on the previous subinterval is used as the initial value at the next subinterval. Also, the t_i argument of (6.3) allows the possibility that $x(t)$ is not defined at all on $[t_i, t_{i+1}]$ (see Sec. 2).

(CLF) V (see [20] for background on CLFs). Using the argument from [15, Sec. 5], we can transform V into a (locally) semiconcave CLF for (3.5) on $\mathbb{R}^k \setminus \operatorname{clos} \mathbf{TN}_\varepsilon \mathcal{A}$. In [12], it was shown that control affine systems that are GAC to $\{0\}$ on $\mathbb{R}^k$ admit (possibly discontinuous) feedbacks, for which the corresponding closed loop systems are sampling ISS to $\{0\}$. Since (3.5) is again control affine, a slight variant of the argument from [12, Sec. 3] provides a feedback $K(x)$ rendering (3.5) s-ISS to $\operatorname{clos} \mathbf{TN}_\varepsilon \mathcal{A}$. Applying Lemma 3.5 as before shows that the u-part of $K(x)$ renders (6.1) s-ISS to $\mathcal{A}$. □

Remark 6.3. This theorem can be extended to cover completely nonlinear systems (1.1) on $\mathcal{X} \times \mathbb{R}^m$ if we reinterpret s-ISS in the following more general sense: a feedback k renders (1.1) *s-ISS to $\mathcal{A}$ in the weak sense* provided that there exists a smooth everywhere invertible matrix-valued function $G : \mathcal{X} \to \mathbb{R}^{m \times m}$ such that

$$\dot{x} = f(x, k(x) + G(x)u) \tag{6.4}$$

is s-ISS to $\mathcal{A}$. The s-ISS property for (6.4) is defined by taking $e \equiv 0$ in Definition 6.2, and the π-solutions of (6.4) are defined by recursively solving $\dot{x}(t) = f(x(t), k(x(t_i)) + G(x(t))\mathbf{u}(t))$ on successive intervals $[t_i, t_{i+1}]$ of the partition $\pi = \{t_i\}_{i \geq 0}$ and proceeding as in Definition 6.1 with $e \equiv 0$ (see [12] for details). In particular, the sampling is only done in the (possibly discontinuous) controller $k(x)$. Then we can prove the following for any smooth manifold $\mathcal{X}$ and $\mathbf{U} = \mathbb{R}^m$: *If* (1.1) *is GAC to a compact, nonempty, weakly invariant set $\mathcal{A}$, then there exists a feedback $k(x)$ rendering* (1.1) *s-ISS to $\mathcal{A}$ in the weak sense.* The proof combines the arguments from [12, Sec. 5] with our proof of Theorem 3 and is left to the reader.

Remark 6.4. As we noted in the Introduction, the GAC system (1.1) does not admit, in general, a continuous stabilizing state feedback. However, by [10], system (1.1) is stabilizable by a continuous *time varying* feedback $u = k(t, x)$ if it is completely controllable and drift-free (the latter condition is the requirement that $f(x, 0) \equiv 0$). In engineering applications, feedback laws are usually implemented via sampling. This motivated our construction of discontinuous state stabilizers $u = k(x)$ which we implemented using CLSS solutions. Yet another approach to stabilizing (1.1) is to look for a *dynamic* stabilizer. This means finding a locally Lipschitz regulator dynamic $\dot{z} = A(z, x)$ and a locally Lipschitz function $k(z, x)$ such that the interconnected system $\dot{x} = f(x, k(z, x))$, $\dot{z} = A(z, x)$ is globally asymptotically stable. (See [19] for an extensive discussion of dynamic stabilizers for *linear* systems.)

On the other hand, it turns out that a dynamic feedback for (1.1) may fail to exist, even if the system is completely controllable. An example from [24] where this occurs is

$$\dot{x} = f(x,u) = \begin{bmatrix} (4 - x_2^2)u_2^2 \\ e^{-x_1} + x_2 - 2e^{-x_1}\sin^2(u_1) \end{bmatrix}, \quad x \in \mathbb{R}^2,\ u \in \mathbb{R}^2. \tag{6.5}$$

The fact that (6.5) is completely controllable (and, therefore, GAC to $\mathcal{A} = \{0\}$) was shown in the appendix of [24], where it is also shown that it is impossible to choose paths converging to the origin so that this selection is continuous in the initial states. Since the flow mapping of any dynamic stabilizer would give a continuous choice of paths converging to the origin, no dynamic stabilizer for the system can exist, even if we omit the requirement that the state of the regulator converges to zero. In particular, we see that (6.5) cannot admit a continuous time varying feedback $u = k(t,x)$. This does not contradict the existence results [10] for time varying feedbacks since in this case, the system has a drift.

Remark 6.5. The feedback construction [12] used to prove Theorem 3 proceeds by first finding a semiconcave control-Lyapunov function (CLF) for the system and then adapting the feedback design from [18] to allow nonsmooth CLFs, observation noise, and discontinuous feedback. Semiconcave CLFs are known to exist for all (locally Lipschitz) GAC systems on Euclidean space and all compact nonempty weakly invariant attractors $\mathcal{A}$, by arguments from [15]. The semiconcavity property is intermediate between C^1 and local Lipschitzness. On the other hand, GAC systems will not, in general, admit *smooth* CLFs since their existence would imply the existence of continuous stabilizers $k(x)$, which are not the case in general (see [7,20]).

For a very different approach to ISS on manifolds (based on density functions) that gives rise to a sufficient condition for ISS-like behavior from almost all initial values (see [4]). The main ISS-like condition in [4] states the following: For a given Riemannian manifold $\mathcal{X}$ and a compact weakly invariant set $\mathcal{A} \subseteq \mathcal{X}$ for (1.1), we say that (1.1) is *weakly almost ISS to $\mathcal{A}$* provided that

(i) $\mathcal{A}$ is locally asymptotically stable for the system and
(ii) there exists $\gamma \in \mathcal{K}$ such that

$$\begin{gathered} \forall u \in \mathcal{U}\ \exists \mathcal{Z}_u \in \mathrm{Null}(\mathcal{X}) \ \text{ such that } \ \forall \xi \in \mathcal{X} \setminus \mathcal{Z}_u, \\ \liminf_{t \to +\infty} |x(t,\xi,u)|_{\mathcal{A}} \le \gamma(\|u\|), \end{gathered} \tag{6.6}$$

where $\mathrm{Null}(\mathcal{X})$ denotes the set of subsets of $\mathcal{X}$ of measure zero and $|\cdot|_{\mathcal{A}}$ denotes the distance to $\mathcal{A}$.

This condition differs from our ISS requirement mainly in its allowance of a null set of states that are not necessarily stabilized and in its use of Carathéodory solutions. An alternative and more intrinsic approach to feedback stabilization on manifolds would involve generalizing the concepts of set-valued differentials and semiconcave CLFs to manifolds and providing direct feedback constructions without first embedding into $\mathbb{R}^k$. The main difficulty that can be expected from this approach would be in defining the stabilizing feedback. We provided a first result in this direction in Sec. 5 above. We leave the development of this more intrinsic approach for another paper.

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