GLOBAL ASYMPTOTIC CONTROLLABILITY IMPLIES INPUT-TO-STATE STABILIZATION[*]

MICHAEL MALISOFF[†], LUDOVIC RIFFORD[‡], AND EDUARDO SONTAG[§]

Abstract. The main problem addressed in this paper is the design of feedbacks for globally asymptotically controllable (GAC) control affine systems that render the closed-loop systems input-to-state stable (ISS) with respect to actuator errors. Extensions for fully nonlinear GAC systems with actuator errors are also discussed. Our controllers have the property that they tolerate small observation noise as well.

Key words. asymptotic controllability, Lyapunov functions, input-to-state stability, nonsmooth analysis

AMS subject classifications. 93B52, 93D15, 93D20

DOI. 10.1137/S0363012903422333

1. Introduction. The theory of input-to-state stability (ISS) forms the basis for much of the current research in mathematical control theory (see [15, 22, 23]). The ISS property was introduced in [19]. In the past decade, there has been a great deal of research done on the problem of finding ISS stabilizing control laws (see [7, 8, 9, 12]). This paper is concerned with the ISS of control systems of the form

$$(1.1) \qquad \dot{x} = f(x) + G(x)u,$$

where f and G are locally Lipschitz vector fields on $\mathbb{R}^n$, $f(0) = 0$, and the control u is valued in $\mathbb{R}^m$ (but see also section 5 for extensions for fully nonlinear systems). We assume throughout that (1.1) is globally asymptotically controllable (GAC), and we construct a feedback $K : \mathbb{R}^n \to \mathbb{R}^m$ for which

$$(1.2) \qquad \dot{x} = f(x) + G(x)K(x) + G(x)u$$

is ISS. As pointed out in [3, 24], a continuous stabilizing feedback K fails to exist in general. This fact forces us to consider discontinuous feedbacks K, so our solutions will be taken in the more general sense of sampling and Euler solutions for dynamics that are discontinuous in the state. By an Euler solution, we mean a uniform limit of sampling solutions, taken as the frequency of sampling becomes infinite (see section 2 for precise definitions). This will extend [19, 20], which show how to make C^o-stabilizable systems ISS to actuator errors. In particular, our results apply to the nonholonomic integrator (see [3, 10] and section 4 below) and other applications where

Brockett's condition is not satisfied, and which therefore cannot be stabilized by continuous feedbacks (see [21, 22, 25]).

Our results also *strengthen* [6], which constructed feedbacks for GAC systems that render the closed-loop systems globally asymptotically stable (GAS). Our main tool will be the recent constructions of semiconcave control Lyapunov functions (CLFs) for GAC systems from [16, 17]. Our results also apply in the more general situation where measurement noise may occur. In particular, our feedback K will have the additional feature that the *perturbed* system

$$(1.3) \qquad \dot{x} = f(x) + G(x)K(x+e) + G(x)u$$

is also ISS when the observation error $e : [0, \infty) \to \mathbb{R}^n$ in the controller is *sufficiently small*. In this context, the precise value of $e(t)$ is unknown to the controller, but information about upper bounds on the magnitude of $e(t)$ can be used to design the feedback. We will prove the following:

THEOREM 1.1. *If* (1.1) *is GAC, then there exists a feedback K for which* (1.3) *is ISS for Euler solutions.*

The preceding theorem characterizes the uniform limits of sampling solutions of (1.3) (see section 2 for the precise definitions of Euler and sampling solutions). From a computational standpoint, it is also desirable to know how frequently to sample in order to achieve ISS for sampling solutions. This information is provided in the following semidiscrete version of Theorem 1.1 for sampling solutions:

THEOREM 1.2. *If* (1.1) *is GAC, then there exists a feedback K for which* (1.3) *is ISS for sampling solutions.*

This paper is organized as follows. In section 2, we review the relevant background on CLFs, ISS, nonsmooth analysis, and discontinuous feedbacks. In section 3, we prove our main results. This is followed in section 4 by a comparison of our feedback construction with the known feedback constructions for C^o-stabilizable systems, and an application of our results to the nonholonomic integrator. We close in section 5 with an extension for fully nonlinear systems.

2. Definitions and main lemmas. Let $\mathcal{K}_\infty$ denote the set of all continuous functions $\rho : [0, \infty) \to [0, \infty)$ for which (i) $\rho(0) = 0$ and (ii) ρ is strictly increasing and unbounded. Note for future reference that $\mathcal{K}_\infty$ is closed under inverse and composition (i.e., if $\rho_1, \rho_2 \in \mathcal{K}_\infty$, then $\rho_1^{-1}, \rho_1 \circ \rho_2 \in \mathcal{K}_\infty$). We let $\mathcal{KL}$ denote the set of all continuous functions $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ for which (1) $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each $t \geq 0$, (2) $\beta(s, \cdot)$ is nonincreasing for each $s \geq 0$, and (3) $\beta(s, t) \to 0$ as $t \to +\infty$ for each $s \geq 0$.

For each $k \in \mathbb{N}$ and $r > 0$, we define

$$\mathcal{M}^k = \{\text{measurable } u : [0, \infty) \to \mathbb{R}^k : |u|_\infty < \infty\}$$

and $\mathcal{M}_r^k := \{u \in \mathcal{M}^k : |u|_\infty \leq r\}$, where $|\cdot|_\infty$ is the essential supremum. We let $\|u(s)\|_I$ denote the essential supremum of a function u restricted to an interval I. Let $|\cdot|$ denote the Euclidean norm, in the appropriate dimension, and

$$r\mathcal{B}_k := \{x \in \mathbb{R}^k : |x| < r\}$$

for each $k \in \mathbb{N}$ and $r > 0$. The closure of $r\mathcal{B}_k$ is denoted by $r\bar{\mathcal{B}}_k$, and $\text{bd}(S)$ denotes the boundary of any subset S in Euclidean space. We also set

$$\mathcal{O} := \{e : [0, \infty) \to \mathbb{R}^n\}, \quad \sup(e) = \sup\{|e(t)| : t \geq 0\}$$

for all $e \in \mathcal{O}$, and $\mathcal{O}_\eta := \{e \in \mathcal{O} : \sup(e) \leq \eta\}$ for each $\eta > 0$. For any compact set $\mathcal{F} \subseteq \mathbb{R}^n$ and $\varepsilon > 0$, we define the compact set

$$\mathcal{F}^\varepsilon := \{x \in \mathbb{R}^n : \min\{|x - p| : p \in \mathcal{F}\} \leq \varepsilon\},$$

i.e., the "ε-enlargement of $\mathcal{F}$." Given a continuous function

$$h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n : (x, u) \mapsto h(x, u)$$

that is locally Lipschitz in x uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}^m$, we let $\phi_h(\cdot, x_o, u)$ denote the trajectory of $\dot{x} = h(x, u)$ starting at $x_o \in \mathbb{R}^n$ for each choice of $u \in \mathcal{M}^m$. In this case, $\phi_h(\cdot, x_o, u)$ is defined on some maximal interval $[0, t)$, with $t > 0$ depending on u and x_o. Let C^k denote the set of all continuous functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ that have at least k continuous derivatives (for $k = 0, 1$). We use the following controllability notion, which was introduced in [18] and later reformulated in terms of $\mathcal{KL}$ functions in [22]:

DEFINITION 2.1. *We call the system* $\dot{x} = h(x, u)$ *globally asymptotically controllable* (GAC) *provided there are a nondecreasing function* $\sigma : [0, \infty) \to [0, \infty)$ *and a function* $\beta \in \mathcal{KL}$ *satisfying the following: for each* $x_o \in \mathbb{R}^n$, *there exists* $u \in \mathcal{M}^m$ *such that*
   (a) $|\phi_h(t, x_o, u)| \leq \beta(|x_o|, t)$ *for all* $t \geq 0$; *and*
   (b) $|u(t)| \leq \sigma(|x_o|)$ *for a.e.* $t \geq 0$.
In this case, we call σ *the* GAC modulus *of* $\dot{x} = h(x, u)$.

In our main results, the controllers will be taken to be discontinuous feedbacks, so the dynamics will be discontinuous in the state variable. Therefore, we will form our trajectories through sampling and through uniform limits of sampling trajectories, as follows. We say that $\pi = \{t_o, t_1, t_2, \ldots\} \subset [0, \infty)$ is a *partition* of $[0, \infty)$ provided $t_o = 0$, $t_i < t_{i+1}$ for all $i \geq 0$, and $t_i \to \infty$ as $i \to +\infty$. The set of all partitions of $[0, \infty)$ is denoted by Par. Let

$$F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n : (x, p, u) \mapsto F(x, p, u)$$

be a continuous function that is locally Lipschitz in x uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$. A *feedback* for F is defined to be any locally bounded function $K : \mathbb{R}^n \to \mathbb{R}^m$ for which $K(0) = 0$. In particular, we allow discontinuous feedbacks. The arguments x, p, and u in F are used to represent the state, feedback value, and actuator error, respectively.

Given a feedback $K : \mathbb{R}^n \to \mathbb{R}^m$, $\pi = \{t_o, t_1, t_2, \ldots\} \in$ Par, $x_o \in \mathbb{R}^n$, $e \in \mathcal{O}$, and $u \in \mathcal{M}^m$, the *sampling solution* for the initial value problem (IVP)

(2.1) $\dot{x}(t) = F(x(t), K(x(t) + e(t)), u(t))$,

(2.2) $x(0) = x_o$

is the continuous function defined by recursively solving

(2.3) $\dot{x}(t) \;=\; F(x(t), K(x(t_i) + e(t_i)), u(t))$

from the initial time t_i up to time $s_i = t_i \vee \sup\{s \in [t_i, t_{i+1}] : x(\cdot)$ is defined on $[t_i, s)\}$, where $x(0) = x_o$. In this case, the sampling solution of (2.1)–(2.2) is defined on the right-open interval from time zero up to time $\bar{t} = \inf\{s_i : s_i < t_{i+1}\}$. This sampling solution will be denoted by $t \mapsto x_\pi(t; x_o, u, e)$ to exhibit its dependence on $\pi \in$ Par,

$x_o \in \mathbb{R}^n$, $u \in \mathcal{M}^m$, and $e \in \mathcal{O}$, or simply by x_π, when the dependence is clear from the context. Note that if $s_i = t_{i+1}$ for all i, then $\bar{t} = +\infty$ (as the infimum of the empty set), so in that case, the sampling solution $t \mapsto x_\pi(t; x_o, u, e)$ is defined on $[0, \infty)$.

We also define the *upper diameter* and the *lower diameter* of a given partition $\pi = \{t_o, t_1, t_2, \ldots\}$ by

$$\overline{\mathbf{d}}(\pi) = \sup_{i \geq 0}(t_{i+1} - t_i), \quad \underline{\mathbf{d}}(\pi) = \inf_{i \geq 0}(t_{i+1} - t_i),$$

respectively. We let $\mathrm{Par}(\delta) := \{\pi \in \mathrm{Par} : \overline{\mathbf{d}}(\pi) < \delta\}$ for each $\delta > 0$. We will say that a function $y : [0, \infty) \to \mathbb{R}^n$ is an *Euler solution (robust to small observation errors)* of

$$(2.4) \qquad\qquad \dot{x}(t) = F(x(t), K(x(t)), u(t)), \quad x(0) = x_o$$

for $u \in \mathcal{M}^m$ provided there are sequences $\pi_r \in \mathrm{Par}$ and $e_r \in \mathcal{O}$ such that
   (a) $\overline{\mathbf{d}}(\pi_r) \to 0$;
   (b) $\sup(e_r)/\underline{\mathbf{d}}(\pi_r) \to 0$; and
   (c) $t \mapsto x_{\pi_r}(t; x_o, u, e_r)$ converges uniformly to y as $r \to +\infty$.
Note that the approximating trajectories in the preceding definition all use the same input u (but see Remark 2.4 for a more general notion of Euler solutions, which also involves sequences of inputs).

This paper will design feedbacks that make closed-loop GAC systems ISS with respect to actuator errors. More precisely, we will use the following definition:

DEFINITION 2.2. *We say that* (2.1) *is* ISS for sampling solutions *provided there are* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}_\infty$ *satisfying: For each* $\varepsilon, M, N > 0$ *with* $0 < \varepsilon < M$, *there exist positive* $\delta = \delta(\varepsilon, M, N)$ *and* $\kappa = \kappa(\varepsilon, M, N)$ *such that for each* $\pi \in \mathrm{Par}(\delta)$, $x_o \in M\bar{\mathcal{B}}_n$, $u \in \mathcal{M}_N^m$, *and* $e \in \mathcal{O}$ *for which* $\sup(e) \leq \kappa\underline{\mathbf{d}}(\pi)$,

$$(2.5) \qquad\qquad |x_\pi(t; x_o, u, e)| \leq \max\{\beta(M, t) + \gamma(N), \varepsilon\}$$

for all $t \geq 0$.

Roughly speaking, condition (2.5) says that the system is ISS, modulo small overflows, if the sampling is done "quickly enough," as determined by the condition $\pi \in \mathrm{Par}(\delta)$, but "not too quickly," as determined by the additional requirement that $\underline{\mathbf{d}}(\pi) \geq (1/\kappa)\sup(e)$. In the special case where the observation error $e \equiv 0$, the condition on $\underline{\mathbf{d}}(\pi)$ in Definition 2.2 is no longer needed; our results are new even for this particular case.

Notice that the bounds on e are in the supremum, not the essential supremum. It is easy to check that Definition 2.2 remains unchanged if we replace the right-hand side in (2.5) by $\beta(M, t) + \gamma(N) + \varepsilon$. We also use the following analogue of Definition 2.2 for Euler solutions:

DEFINITION 2.3. *We say that the system* (2.1) *is* ISS for Euler solutions *provided there exist* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}_\infty$ *satisfying: If* $u \in \mathcal{M}^m$ *and* $x_o \in \mathbb{R}^n$, *and if* $t \mapsto x(t)$ *is an Euler solution of* (2.4), *then*

$$(2.6) \qquad\qquad |x(t)| \leq \beta(|x_o|, t) + \gamma(|u|_\infty)$$

for all $t \geq 0$.

Remark 2.4. In the definition of Euler solutions we gave above, all of the approximating trajectories $t \mapsto x_{\pi_r}(t; x_o, u, e_r)$ use the same input $u \in \mathcal{M}^m$. A different way to define Euler solutions, which gives rise to a more general class of limiting solutions, is as follows: A function $y : [0, \infty) \to \mathbb{R}^n$ is a *generalized Euler solution* of (2.4) for $u \in \mathcal{M}^m$ provided there are sequences $\pi_r \in \mathrm{Par}$, $e_r \in \mathcal{O}$, and $u_r \in \mathcal{M}^m$ such that

(a) $\overline{\mathbf{d}}(\pi_r) \to 0$;
(b) $\sup(e_r)/\underline{\mathbf{d}}(\pi_r) \to 0$;
(c) $|u_r|_\infty \leq |u|_\infty$ for all r; and
(d) $t \mapsto x_{\pi_r}(t; x_o, u_r, e_r)$ converges uniformly to y as $r \to +\infty$.

We can then define ISS for generalized Euler solutions exactly as in Definition 2.3, by merely replacing "Euler solution" with "generalized Euler solution" throughout the definition. Our proof of Theorem 1.1 will actually show the following slightly more general result: If (1.1) is GAC, then there exists a feedback K for which (1.3) is ISS for generalized Euler solutions.

Our main tools in this paper will be nonsmooth analysis and nonsmooth Lyapunov functions. The following definitions will be used. Let Ω be an arbitrary open subset of $\mathbb{R}^n$. Recall the following definition:

DEFINITION 2.5. *Let $g : \Omega \to \mathbb{R}$ be a continuous function on Ω; it is said to be semiconcave on Ω provided for each point $x_o \in \Omega$, there exist $\rho, C > 0$ such that*

$$g(x) + g(y) - 2g\left(\frac{x+y}{2}\right) \leq C||x - y||^2$$

for all $x, y \in x_o + \rho\mathcal{B}_n$.

The *proximal superdifferential* (respectively, *proximal subdifferential*) of a function $V : \Omega \to \mathbb{R}$ at $x \in \Omega$, which is denoted by $\partial^P V(x)$ (resp., $\partial_P V(x)$), is defined to be the set of all $\zeta \in \mathbb{R}^n$ for which there exist $\sigma, \eta > 0$ such that

$$V(y) - V(x) - \sigma|y - x|^2 \leq \langle \zeta, y - x \rangle \quad (\text{resp., } V(y) - V(x) + \sigma|y - x|^2 \geq \langle \zeta, y - x \rangle)$$

for all $y \in x + \eta\mathcal{B}_n$. The *limiting subdifferential* of a continuous function $V : \Omega \to \mathbb{R}$ at $x \in \Omega$ is

$$\partial_L V(x) := \{q \in \mathbb{R}^n : \exists x_n \to x \quad \text{and} \quad q_n \in \partial_P V(x_n) \text{ s.t. } q_n \to q\}.$$

In what follows, we assume $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n : (x, u) \mapsto h(x, u)$ is continuous, that it is locally Lipschitz in x uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}^m$, and that $h(0, 0) = 0$. The following definition was introduced in [18] and reformulated in proximal terms in [22].

DEFINITION 2.6. *A control-Lyapunov function (CLF) for*

$$\dot{x} = h(x, u) \tag{2.7}$$

is a continuous, positive definite, proper function $V : \mathbb{R}^n \to \mathbb{R}$ for which there exist a continuous, positive definite function $W : \mathbb{R}^n \to \mathbb{R}$, and a nondecreasing function $\alpha : [0, \infty) \to [0, \infty)$, satisfying

$$\forall \zeta \in \partial_P V(x), \quad \inf_{|u| \leq \alpha(|x|)} \langle \zeta, h(x, u) \rangle \leq -W(x)$$

for all $x \in \mathbb{R}^n$. In this case, we call (V, W) a Lyapunov pair for (2.7).

Recall the following lemmas (see [17]):

LEMMA 2.7. *If (2.7) is GAC, then there exist a CLF V for (2.7) that is semiconcave on $\mathbb{R}^n \setminus \{0\}$ and a nondecreasing function $\alpha : [0, \infty) \to [0, \infty)$ that satisfy*

$$\forall \zeta \in \partial_L V(x), \quad \min_{|u| \leq \alpha(|x|)} \langle \zeta, h(x, u) \rangle \leq -V(x) \tag{2.8}$$

for all $x \in \mathbb{R}^n$.

LEMMA 2.8. *Let $V : \Omega \to \mathbb{R}$ be semiconcave. Then V is locally Lipschitz, and $\emptyset \neq \partial_L V(x) \subseteq \partial^P V(x)$ for all $x \in \Omega$. Moreover, for each compact set $Q \subset \Omega$, there exist constants $\sigma, \mu > 0$ such that $V(y) - V(x) - \sigma|y - x|^2 \leq \langle \zeta, y - x \rangle$ for all $y \in x + \mu \mathcal{B}_n$, all $x \in Q$, and all $\zeta \in \partial^P V(x)$.*

Notice that Lemma 2.8 allows the constants in the definition of $\partial^P V(x)$ to be chosen uniformly on compact sets.

Remark 2.9. In [17], the controls u take all their values in a given compact metric space U. The precise version of the CLF existence theorem in [17] is the same as our Lemma 2.7, except that the infimum in the decay condition (2.8) is replaced by the infimum over all $u \in U$. The version of Lemma 2.7 we gave above follows from a slight modification of the arguments of [16, 17], using the GAC modulus in the GAC definition (see Definition 2.1). The existence theory [16] for semiconcave CLFs is a strengthening of the proof that continuous CLFs exist for any GAC system (see [18]).

3. Proofs of theorems. Let V be a CLF satisfying the requirements of Lemma 2.7 for the dynamics

$$(3.1) \qquad\qquad h(x, u) = f(x) + G(x)u.$$

Define the functions $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ by

$$(3.2) \qquad \underline{\alpha}(s) = \min\{|x| : V(x) \geq s\} \text{ and } \overline{\alpha}(s) = \max\{|x| : V(x) \leq s\}.$$

One can easily check that

$$(3.3) \qquad \forall x \in \mathbb{R}^n, \quad \underline{\alpha}(V(x)) \leq |x| \text{ and } \overline{\alpha}(V(x)) \geq |x|.$$

Moreover, by reducing $\underline{\alpha}$, we may assume that $\underline{\alpha}(s) \leq s$ for all $s \geq 0$, while still satisfying (3.3).

Let $x \mapsto \zeta(x)$ be any selection of $\partial_L V(x)$ on $\mathbb{R}^n$ and let $\zeta(0) \in \mathbb{R}^n$ be arbitrary. For each $x \in \mathbb{R}^n$, we can choose $u = u_x \in \alpha(|x|)\mathcal{B}_m$ that satisfies the inequality in (2.8) for the dynamics (3.1) and $\zeta = \zeta(x)$. Define the feedback $K_1 : \mathbb{R}^n \to \mathbb{R}^m$ by $K_1(x) = u_x$ for all $x \neq 0$ and $K_1(0) = 0$. We use the functions

$$
\begin{aligned}
(3.4) \qquad & a(x) = \langle \zeta(x), f(x) + G(x)K_1(x) \rangle, \quad b_j(x) = \langle \zeta(x), g_j(x) \rangle \quad \forall j, \\
& K_2(x) = -V(x)(\operatorname{sgn}\{b_1(x)\}, \operatorname{sgn}\{b_2(x)\}, \ldots, \operatorname{sgn}\{b_m(x)\})^T,
\end{aligned}
$$

where g_j is the jth column of G for $j = 1, 2, \ldots, m$, and

$$
\operatorname{sgn}\{s\} \quad = \quad \begin{cases} 1, & s > 0, \\ -1, & s < 0, \\ 0, & s = 0. \end{cases}
$$

We remark that our results remain true, with minor changes in the proofs, if the factor $-V(x)$ in the definition of K_2 is replaced by $-W(x)$ for an arbitrary positive definite proper continuous function $W : \mathbb{R}^n \to \mathbb{R}$. In particular, $K := K_1 + K_2$ is a feedback for the dynamics

$$F(x, p, u) = f(x) + G(x)(p + u).$$

Moreover,

$$(3.5) \qquad\qquad a(x) \leq -V(x) < 0 \ \ \forall x \in \mathbb{R}^n \setminus \{0\}.$$

We next show that

$$(3.6) \qquad \dot{x}(t) = F(x(t), K(x(t) + e(t)), u(t))$$

is ISS for sampling solutions.

To this end, choose $\varepsilon, M, N > 0$ for which $0 < \varepsilon < M$. It clearly suffices to verify the ISS property (2.5) for $\varepsilon < 1$, since that would imply the property for *all* overflows $\varepsilon > 0$. Choose

$$(3.7) \qquad u \in \mathcal{M}_N^m, \;\; e \in \mathcal{O}_{\varepsilon/16}, \;\; x_o \in M\bar{\mathcal{B}}_n.$$

In what follows, x_π denotes the sampling solution for (3.6) for the choices (3.7) and $\pi \in \mathrm{Par}$, and $\tilde{x}_\pi$ is the (possibly discontinuous) function that is inductively defined by solving the IVP

$$\dot{x}(t) = f(x(t)) + G(x(t))[K(\tilde{x}_i) + u(t)], \;\; x(t_i) = \tilde{x}_i$$

on $[t_i, t_{i+1})$, where $\tilde{x}_i := x_i + e(t_i)$, $x_i := x_\pi(t_i)$, and $\pi = \{t_o, t_1, t_2, \ldots\}$. We later restrict the choice of π so that x_π and $\tilde{x}_\pi$ are defined on $[0, \infty)$. We will use the compact set

$$Q = \left\{ \left[\overline{\alpha} \circ \underline{\alpha}^{-1}(N + M) + 1 \right] \bar{\mathcal{B}}_n \right\} \setminus \varepsilon \mathcal{B}_n.$$

Notice that $Q, Q^{\varepsilon/2} \subseteq \mathbb{R}^n \setminus \{0\}$, and that $x_o \in Q^\varepsilon$. Using Lemma 2.8 and the semiconcavity of V on $\mathbb{R}^n \setminus \{0\}$, we can find $\sigma, \mu > 0$ such that

$$(3.8) \qquad V(y) - V(x) \leq \langle \zeta(x), y - x \rangle + \sigma |y - x|^2$$

for all $y \in x + \mu \mathcal{B}_n$ and $x \in Q^{\varepsilon/2}$. Let $\mathcal{L}_\varepsilon > 1$ be a Lipschitz constant for V on $Q^{\varepsilon/2}$, the existence of which is also guaranteed by Lemma 2.8. It follows from the definition of a CLF that

$$(3.9) \qquad \begin{aligned} \lambda_- &:= \min \left\{ V(p) : p \in Q^{\varepsilon/2} \right\}, \\ \lambda_+ &:= \max \left\{ V(p) : p \in Q^\varepsilon \right\} \end{aligned}$$

are finite positive numbers. Therefore, we can choose $\tilde{\varepsilon} \in (0, \varepsilon)$ for which

$$(3.10) \qquad \overline{\alpha} \left(p + \frac{\mathcal{L}_\varepsilon}{4} \tilde{\varepsilon} \right) \leq \overline{\alpha}(p) + \frac{\varepsilon}{8} \quad \forall p \in \left[0, \underline{\alpha}^{-1}(N) + \lambda_+ \right].$$

We can also find

$$(3.11) \qquad \delta = \delta(\varepsilon, M, N) \in \left(0, \frac{\tilde{\varepsilon}}{16 + \lambda_+ + 16\lambda_+} \right)$$

such that if

$$(3.12) \qquad \pi \in \mathrm{Par}(\delta), \;\; e \in \mathcal{O}_{\tilde{\varepsilon}/16}, \;\; x_i \in Q^\varepsilon,$$

and if $t \in [t_i, t_{i+1})$ is such that $x_\pi(s)$ and $\tilde{x}_\pi(s)$ remain in $Q^{2\varepsilon}$ for all $s \in [t_i, t]$, then

$$(3.13) \quad \max\{|x_\pi(t) - x_i|, |\tilde{x}_\pi(t) - \tilde{x}_i|\} \leq \min \left\{ \mu, \frac{\tilde{\varepsilon}}{16(1 + \mathcal{L}_\varepsilon)}, \sqrt{\frac{\lambda_-}{8\sigma}}(t - t_i) \right\}.$$

This follows from the local boundedness of K, f, and G. It follows from (3.13) that $\tilde{x}_\pi(t) \in Q^{\varepsilon/4}$ (resp., $x_\pi(t) \in Q^{\varepsilon/4}$) for all $t \in [t_i, t_{i+1})$ and all i such that $\tilde{x}_i \in Q$ (resp., $x_i \in Q$), since the trajectories cannot move the initial value more than $\frac{\varepsilon}{16}$ and there are no blow up times for the trajectories. In particular, (3.13) will show that x_π and $\tilde{x}_\pi$ are defined on $[0, \infty)$, since the argument we are about to give shows that $x_i \in Q^\varepsilon$ for all i. By reducing δ as necessary, we can assume

$$(3.14) \qquad \begin{aligned} \|\zeta(\tilde{x}_i) \cdot (F(\tilde{x}_i, K(\tilde{x}_i), u(s)) - f(\tilde{x}_\pi(s)) \\ - G(\tilde{x}_\pi(s))[u(s) + K(\tilde{x}_i)]) \|_{[t_i, t_{i+1})} \le \frac{\lambda_-}{8} \end{aligned}$$

for all i such that $\tilde{x}_i \in Q^{\varepsilon/2}$. This follows from the Lipschitzness of f and G on Q^ε. Having chosen δ to satisfy the preceding requirements, pick any $\pi \in \mathrm{Par}(\delta)$. It follows from (3.8) and (3.13) that

$$(3.15) \qquad \begin{aligned} V(\tilde{x}_\pi(t)) - V(\tilde{x}_i) &\le \langle \zeta(\tilde{x}_i), \tilde{x}_\pi(t) - \tilde{x}_i \rangle + \sigma |\tilde{x}_\pi(t) - \tilde{x}_i|^2 \\ &\le \langle \zeta(\tilde{x}_i), \tilde{x}_\pi(t) - \tilde{x}_i \rangle + \frac{\lambda_-}{8}(t - t_i) \end{aligned}$$

for all $t \in [t_i, t_{i+1})$ and all i such that $\tilde{x}_i \in Q^{\varepsilon/4}$. Moreover, if $\tilde{x}_i \in Q^{\varepsilon/4}$ and $t \in [t_i, t_{i+1})$, and if

$$(3.16) \qquad V(\tilde{x}_i) \ge N,$$

then

$$(3.17) \qquad \begin{aligned} \langle \zeta(\tilde{x}_i), \tilde{x}_\pi(t) - \tilde{x}_i \rangle &\le \left\langle \zeta(\tilde{x}_i), \int_{t_i}^t F(\tilde{x}_i, K(\tilde{x}_i), u(s)) \mathrm{d}s \right\rangle + \frac{\lambda_-}{8}(t - t_i) \quad \text{(by (3.14))} \\ &= (t - t_i)\langle \zeta(\tilde{x}_i), f(\tilde{x}_i) + G(\tilde{x}_i)K(\tilde{x}_i) \rangle \\ &\quad + \int_{t_i}^t \langle \zeta(\tilde{x}_i), G(\tilde{x}_i)u(s) \rangle \mathrm{d}s + \frac{\lambda_-}{8}(t - t_i) \\ &\le (t - t_i)a(\tilde{x}_i) - (t - t_i)V(\tilde{x}_i) \sum_{j=1}^m |b_j(\tilde{x}_i)| \\ &\quad + N(t - t_i) \sum_{j=1}^m |b_j(\tilde{x}_i)| + \frac{\lambda_-}{8}(t - t_i) \\ &\le (t - t_i)a(\tilde{x}_i) + \frac{\lambda_-}{8}(t - t_i) \quad \text{(by (3.16))} \\ &\le -(t - t_i)V(\tilde{x}_i) + \frac{\lambda_-}{8}(t - t_i) \quad \text{(by (3.5))}. \end{aligned}$$

Let

$$S = \{x \in \mathbb{R}^n : V(x) \le \underline{\alpha}^{-1}(N)\}.$$

Then $S \subset Q^\varepsilon$. Indeed, $x \in S$ implies

$$\underline{\alpha} \circ \overline{\alpha}^{-1}(|x|) \le \underline{\alpha} \circ \overline{\alpha}^{-1} \circ \overline{\alpha} \circ V(x) \le N,$$

and therefore $|x| \le \overline{\alpha} \circ \underline{\alpha}^{-1}(N)$. By further reducing ε, we can assume $(2\varepsilon)\mathcal{B}_n \subset S$. If $\tilde{x}_i \in Q^{\varepsilon/4}$ but $\tilde{x}_i \notin S$, then $V(\tilde{x}_i) \ge \underline{\alpha}^{-1}(N) \ge N$, so (3.9) and (3.15) give

$$\begin{aligned} V(\tilde{x}_\pi(t)) - V(\tilde{x}_i) &\le -(t - t_i)\frac{V(\tilde{x}_i)}{2} + (t - t_i)\frac{\lambda_-}{4} \\ (3.18) \qquad &\le -(t - t_i)\frac{V(\tilde{x}_i)}{4} \qquad \forall t \in [t_i, t_{i+1}). \end{aligned}$$

Let $\mathcal{L}_f$ and $\mathcal{L}_G$ be Lipschitz constants for f and G restricted to Q^ε, respectively. Define the constants

(3.19)
$$R = N + \sup\left\{|K(x)| : x \in Q^{\varepsilon/2}\right\},$$
$$L = \mathcal{L}_f + R\mathcal{L}_G, \quad \kappa = \kappa(\varepsilon, M, N) := \frac{\min\{\lambda_-, \varepsilon\}}{16\mathcal{L}_\varepsilon(e^{L\delta} + 1)}.$$

We will presently show that

(3.20)
$$\sup_{t_i \le t < t_{i+1}} |x_\pi(t) - \tilde{x}_\pi(t)| \le |e(t_i)|e^{L\delta} \quad \forall i \text{ s.t. } x_i \in Q^{\varepsilon/4}.$$

Using (3.20), we will now find $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ to satisfy the ISS estimate

(3.21)
$$|x_\pi(t)| \le \beta(|x_o|, t) + \gamma(N) + \varepsilon \quad \forall t \ge 0$$

which will prove Theorem 1.2.

To this end, assume $x_i \in Q$, but that $x_i \notin S^{\tilde{\varepsilon}/16}$. Then (3.13) implies $x_\pi(t)$ and $\tilde{x}_\pi(t)$ both remain in $Q^{\varepsilon/4}$ on $[t_i, t_{i+1})$. Moreover, $\tilde{x}_i \in Q^{\varepsilon/4} \setminus S$, by the choice of e in (3.12). Therefore, if $t \in [t_i, t_{i+1})$, and if

(3.22)
$$\sup(e) \le \kappa \underline{\mathbf{d}}(\pi),$$

then the choice of κ gives

$$
\begin{aligned}
V(x_{i+1}) - V(x_i) &= V(x_{i+1}) - V(\tilde{x}_\pi(t_{i+1}^-)) + V(\tilde{x}_\pi(t_{i+1}^-)) - V(\tilde{x}_i) \\
&\quad + V(\tilde{x}_i) - V(x_i) \\
&\le \mathcal{L}_\varepsilon |x_{i+1} - \tilde{x}_\pi(t_{i+1}^-)| - \frac{t_{i+1} - t_i}{4}V(\tilde{x}_i) \\
&\quad + \mathcal{L}_\varepsilon|e(t_i)| \ \text{(by (3.18))} \\
&\le \mathcal{L}_\varepsilon|e(t_i)|e^{L\delta} - \frac{t_{i+1} - t_i}{4}V(\tilde{x}_i) + \mathcal{L}_\varepsilon|e(t_i)| \ \text{(by (3.20))} \\
&\le \frac{\lambda_-}{16}(t_{i+1} - t_i) - \frac{t_{i+1} - t_i}{4}V(\tilde{x}_i) \ \text{(by (3.22))} \\
&\le -\frac{t_{i+1} - t_i}{8}V(\tilde{x}_i) \ \text{(by (3.9))} \\
&\le -\frac{t_{i+1} - t_i}{8}V(x_i) + \frac{t_{i+1} - t_i}{8}|e(t_i)|\mathcal{L}_\varepsilon \\
&\le -\frac{t_{i+1} - t_i}{8}V(x_i) + \frac{(t_{i+1} - t_i)^2}{16}\lambda_- \\
&\le -\frac{t_{i+1} - t_i}{16}V(x_i),
\end{aligned}
$$

(3.23)

where we use

$$t_{i+1} - t_i \ \le \ \overline{\mathbf{d}}(\pi) \ \le \ \delta < 1$$

to get the last inequality. Set

$$J(t) = \frac{16}{16 + t}$$

for all $t \ge 0$. One can easily check that Q^ε contains the set

$$S_V := \{p : V(p) \le \max\{V(q) : |q| \le M + N\}\}.$$

In fact, $p \in S_V$ implies

$$|p| \leq \overline{\alpha} \left(\max\{V(q) : |q| \leq M + N\} \right)$$
$$= \max\{\overline{\alpha} \circ \underline{\alpha}^{-1} \circ \underline{\alpha}(V(q)) : |q| \leq M + N\}$$
$$\leq \overline{\alpha} \circ \underline{\alpha}^{-1}(M + N).$$

In particular, $x_o \in S_V$. It follows from (3.23) that if none of $x_o, x_1, \ldots, x_j$ lies in $S^{\tilde{\varepsilon}/16}$, then

$$V(x_1) - V(x_0) \leq -\frac{t_1}{16} V(x_j),$$
$$V(x_2) - V(x_1) \leq -\frac{t_2 - t_1}{16} V(x_j),$$
$$\vdots$$
$$V(x_j) - V(x_{j-1}) \leq -\frac{t_j - t_{j-1}}{16} V(x_j).$$

Summing the preceding inequalities would then give

$$V(x_j) - V(x_o) \leq -\frac{t_j}{16} V(x_j), \quad \text{so} \quad V(x_j) \leq J(t_j) V(x_o).$$

Hence,

$$V(x_i) \leq J(t_i) V(x_o) \quad \text{for} \quad i = 0, 1, \ldots, j.$$

By the choice of δ in (3.11), it would then follow from (3.13) that

$$V(x_\pi(t)) \leq J(t) V(x_o) + \frac{\tilde{\varepsilon}}{8}$$

up to the least time t at which $x_\pi(t) \in S^{\tilde{\varepsilon}/16}$. Hence, for such t, the choice of $\tilde{\varepsilon}$ (see (3.10)) gives

$$|x_\pi(t)| \leq \overline{\alpha} \left(J(t) V(x_o) + \frac{\tilde{\varepsilon}}{8} \right),$$
$$\leq \overline{\alpha} \left(J(t) V(x_o) \right) + \frac{\varepsilon}{8}.$$

On the other hand, (3.23) also shows that if $x_\pi(t) \in S^{\tilde{\varepsilon}/8}$ for some t, then

(3.24) $$|x_\pi(s)| \leq \overline{\alpha} \circ \underline{\alpha}^{-1}(N) + \varepsilon \quad \forall s \geq t.$$

Indeed, let s_1 be the first sample time above such a time t. Assume $x_\pi(t) \notin \varepsilon \mathcal{B}_n$. By (3.13), $x_\pi(s_1) \in S^{\tilde{\varepsilon}/4}$ and $x_\pi(s_1) \notin \frac{\varepsilon}{2} \mathcal{B}_n$. Therefore, there exists $p \in S$ for which

$$V(x_\pi(s_1)) = V(x_\pi(s_1)) - V(p) + V(p)$$
$$\leq \mathcal{L}_\varepsilon \frac{\tilde{\varepsilon}}{4} + \underline{\alpha}^{-1}(N).$$

In fact, we can pick $p = x_\pi(s_1)$ if $x_\pi(s_1) \in S$ and $p \in \partial S$ otherwise, so $p \notin \frac{\varepsilon}{2} \mathcal{B}_n$. It follows from (3.13) and (3.23) that for the next sample time s_i, we either have $x_\pi(s_i) \in S^{\tilde{\varepsilon}/8}$, or else we have

$$V(x_\pi(s_i)) \leq \mathcal{L}_\varepsilon \frac{\tilde{\varepsilon}}{4} + \underline{\alpha}^{-1}(N).$$

In the first case,

$$|x_\pi(s_i)| \le \overline{\alpha} \circ \underline{\alpha}^{-1}(N) + \frac{\varepsilon}{8},$$

while in the second case,

$$|x_\pi(s_i)| \le \overline{\alpha}\left(\frac{\tilde{\varepsilon}\mathcal{L}_\varepsilon}{4} + \underline{\alpha}^{-1}(N)\right) \le \overline{\alpha} \circ \underline{\alpha}^{-1}(N) + \frac{\varepsilon}{8},$$

by the choice of $\tilde{\varepsilon}$. If $x_\pi(s_i) \notin S^{\tilde{\varepsilon}/16}$, then $V(x_\pi(s_{i+1})) \le V(x_\pi(s_i))$ (by (3.23)), so the preceding argument also gives

$$|x_\pi(s_{i+1})| \le \overline{\alpha} \circ \underline{\alpha}^{-1}(N) + \frac{\varepsilon}{8}.$$

By repeating this argument for subsequent sample times, the assertion (3.24) then follows from (3.13). Defining $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ by

$$(3.25) \qquad \beta(s,t) = \overline{\alpha}\left(\underline{\alpha}^{-1}(s)J(t)\right), \quad \gamma(s) = \overline{\alpha} \circ \underline{\alpha}^{-1}(s),$$

it follows that (3.21) holds for all $x_o \in M\bar{\mathcal{B}}_n$, $u \in \mathcal{M}_N^m$, $\pi \in \text{Par}(\delta)$, and $e \in \mathcal{O}$ for which $\sup(e) \le \kappa\underline{\mathbf{d}}(\pi)$. Therefore, Theorem 1.2 will follow once we check (3.20), which is a consequence of Gronwall's inequality.

To this end, notice that if $x_i \in Q^{\varepsilon/4}$, then

$$|x_\pi(t) - \tilde{x}_\pi(t)| \le |x_i - \tilde{x}_i| + \int_{t_i}^t \left(\mathcal{L}_f|x_\pi(s) - \tilde{x}_\pi(s)| + R\mathcal{L}_G|x_\pi(s) - \tilde{x}_\pi(s)|\right) \mathrm{d}s$$

for all $t \in [t_i, t_{i+1})$, where we are using the constants in (3.19). It follows from Gronwall's inequality that

$$|x_\pi(t) - \tilde{x}_\pi(t)| \le |x_i - \tilde{x}_i|e^{L|t_i - t_{i+1}|} \le |x_i - \tilde{x}_i|e^{L\overline{\mathbf{d}}(\pi)} \le |e(t_i)|e^{L\delta}$$

for all $t \in [t_i, t_{i+1})$, which is (3.20). This proves Theorem 1.2.

We turn next to Theorem 1.1. We need to show the ISS property (2.6) for all Euler solutions $x(t)$ of (2.4). We will actually prove the slightly stronger version of the theorem for generalized Euler solutions, as asserted in Remark 2.4. To this end, choose $u \in \mathcal{M}^m$, $x_o \in \mathbb{R}^n$, and $\varepsilon > 0$. Using our previous conclusion that (1.3) is ISS for sampling solutions, we can let

$$\delta_\varepsilon = \delta\left(\varepsilon, |x_o|, |u|_\infty\right) \quad \text{and} \quad \kappa_\varepsilon = \kappa\left(\varepsilon, |x_o|, |u|_\infty\right)$$

be the constants from Definition 2.2. Let $x(t)$ be a generalized Euler solution of (2.4), and let π_r, u_r, and e_r satisfy the requirements of the generalized Euler solution definition. It follows from the definition that there is an $\bar{r} \in \mathbb{N}$ such that

$$\overline{\mathbf{d}}(\pi_r) \le \delta_\varepsilon, \quad \sup(e_r) \le \kappa_\varepsilon\underline{\mathbf{d}}(\pi_r)$$

for all $r \ge \bar{r}$. It then follows from (3.21) that

$$(3.26) \qquad |x_{\pi_r}(t; x_o, u_r, e_r)| \le \beta(|x_o|, t) + \gamma(|u|_\infty) + \varepsilon$$

for all $t \ge 0$ and $r \ge \bar{r}$, where β and γ are in (3.25). The ISS condition (2.6) now follows by passing to the limit in (3.26) as $r \to \infty$, since $\varepsilon > 0$ was arbitrary. This concludes the proof of Theorem 1.1.

4. Stabilization of the nonholonomic integrator. In this section, we illustrate how the feedback constructed in section 3 can be used to stabilize Brockett's nonholonomic integrator control system (see [3, 10, 22]). We will also use the nonholonomic integrator to compare our feedback construction to the feedbacks from [19, 20]. The nonholonomic integrator was introduced in [3], as an example of a system that cannot be stabilized using continuous feedback. It is well known that if the state space of a system contains obstacles (e.g., if the state space is $\mathbb{R}^2 \setminus (-1, 1)^2$, and therefore has a topological obstacle around the origin), then it is impossible to stabilize the system using continuous feedback. In fact, this is a special case of a theorem of Milnor, which asserts that the domain of attraction of an asymptotically stable vector field must be diffeomorphic to Euclidean space, and therefore cannot be the complement $\mathbb{R}^2 \setminus (-1, 1)^2$ (see [21]).

Brockett's example illustrates how, even if we assume that the state evolves in Euclidean space, similar obstructions to stabilization may occur. These obstructions are not due to the topology of the state space, but instead arise from "virtual obstacles" that are implicit in the form of the control system (see [22]). Such obstacles occur when it is impossible to move *instantly* in some directions, even though it is possible to move *eventually* in every direction ("nonholonomy"). This gives rise to Brockett's criterion (see [3]), which is a necessary condition for the existence of a continuous stabilizer, in terms of the vector fields that define the system (see [21, 22, 25]). The nonholonomic integrator does not satisfy Brockett's criterion, and therefore cannot be stabilized by continuous feedbacks.

The physical model for Brockett's example is as follows. Consider a three-wheeled shopping cart whose front wheel acts as a castor. The state variable is $(x_1, x_2, \theta)^T$, where $(x_1, x_2)^T$ is the midpoint of the rear axle of the cart, and θ is the cart's orientation. The front wheel is free to rotate, but there is a "nonslipping" constraint that $(\dot{x}_1, \dot{x}_2)^T$ must always be parallel to $(\cos(\theta), \sin(\theta))^T$. This gives the equations

$$(4.1) \qquad \begin{aligned} \dot{x}_1 &= v_1 \cos(\theta), \\ \dot{x}_2 &= v_1 \sin(\theta), \\ \dot{\theta} &= v_2, \end{aligned}$$

where v_1 is a "drive" command and v_2 is a steering command. Using the feedback transformation

$$z_1 := \theta, \; z_2 := x_1 \cos(\theta) + x_2 \sin(\theta), \; z_3 := x_1 \sin(\theta) - x_2 \cos(\theta),$$
$$u_1 := v_2, \; u_2 := v_1 - v_2 z_3$$

followed by a second transformation brings the equations (4.1) into the form

$$(4.2) \qquad \begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1, \end{aligned}$$

which is called the nonholonomic integrator control system.

One can show (see [11]) that (4.2) is a GAC system. However, since Brockett's condition is not satisfied for (4.2), the system has no continuous stabilizer. While there does not exist a C^1 CLF for the system (4.2) (see [11]), it is now well known that every GAC system admits a continuous CLF (see [18]). In fact, it was shown in [10] that the nonholonomic integrator dynamics (4.2) has the nonsmooth CLF

$$(4.3) \qquad V(x) = \max\left\{ \sqrt{x_1^2 + x_2^2}, |x_3| - \sqrt{x_1^2 + x_2^2} \right\},$$

which is semiconcave outside the cone $x_3^2 = 4(x_1^2 + x_2^2)$ (see [17] for a detailed discussion of some special properties of this CLF). For the special case of the dynamics (4.2) and CLF (4.3), the feedback $K = K_1 + K_2$ we constructed in section 3 is as follows.

To simplify notation, we use the radius $r(x) := \sqrt{x_1^2 + x_2^2}$. We also use the sets

$$\begin{aligned} S_o &= \{x \in \mathbb{R}^3 : x_3 \neq 0,\ r(x) = 0\}, \\ S_+ &= \{x \in \mathbb{R}^3 : x_3^2 \geq 4r^2(x) > 0\}, \\ S_- &= \{x \in \mathbb{R}^3 : x_3^2 < 4r^2(x)\}, \end{aligned}$$

which form a partition of $\mathbb{R}^3 \setminus \{0\}$. Notice that $V(x) = r(x)$ on S_-, and also that $V(x) = |x_3| - r(x)$ on $\mathbb{R}^3 \setminus S_-$. To find our selection $\zeta(x) \in \partial_L V(x)$, we first choose $\zeta(0) = 0$, and $\zeta(x) = (0, -1, \text{sgn}\{x_3\})^T$ for all $x \in S_o$. Using the notation of (3.4), this gives

$$(4.4) \quad b(x) = \begin{cases} (\,-x_2\text{sgn}\{x_3\} - x_1/r(x),\ x_1\text{sgn}\{x_3\} - x_2/r(x)\,)^T, & x \in S_+, \\ (\,x_1/r(x),\ x_2/r(x)\,)^T, & x \in S_-, \\ (0, -1)^T, & x \in S_o, \end{cases}$$

and $b(0) = 0$. Notice that $1 \leq |b(x)|^2 \leq r^2(x) + 1$ for all $x \neq 0$. We also have

$$K_1(x) = \begin{cases} \mu_1(x)\,(\,-x_2\text{sgn}\{x_3\} - x_1/r(x),\ x_1\text{sgn}\{x_3\} - x_2/r(x)\,)^T, & x \in S_+, \\ -\,(x_1, x_2)^T, & x \in S_-, \\ (0, |x_3|)^T, & x \in S_o, \end{cases}$$

with $K_1(0) = 0$, where we have set

$$\mu_1(x) := \frac{r(x) - |x_3|}{r^2(x) + 1}.$$

In this case, we have taken

$$K_1(x) = -b(x)V(x)/|b(x)|^2$$

for $x \neq 0$, where $b(x)$ is defined in (4.4), and K_1 is continuous at the origin. On the other hand, our feedback K_2 from (3.4) becomes

$$K_2(x) = -\begin{cases} (\,\mu_2(x_1, -x_2, x),\ \mu_2(x_2, x_1, x)\,)^T, & x \in S_+, \\ r(x)\,(\,\text{sgn}\{x_1\},\ \text{sgn}\{x_2\}\,)^T, & x \in S_-, \\ |x_3|\,(0,\ -1)^T, & x \in S_0, \end{cases}$$

with $K_2(0) = 0$, where we have set

$$\mu_2(a, b, x) := (|x_3| - r(x))\,\text{sgn}\{\,b\,r(x)\,\text{sgn}\{x_3\} - a\,\}.$$

Since V is semiconcave on $\Omega := \mathbb{R}^3 \setminus \text{bd}(S_-)$, the argument from section 3 applies to sampling solutions that satisfy the additional requirement that $\tilde{x}_\pi(s) \in \Omega$ for all $s \geq 0$. It follows from the proof of Theorem 1.2 that the nonholonomic integrator system (4.2) can be stabilized for both actuator errors and small observation errors (for this restricted set of sampling solutions), using the combined feedback $K = K_1 + K_2$.

Remark 4.1. In this example, we chose to work with the CLF (4.3) because it has been explicitly proven in [10] to be a CLF for the control system (4.2). The

example illustrates how to extend our results to more general CLFs that may not be semiconcave on $\mathbb{R}^3 \setminus \{0\}$. For such cases, the ISS estimates hold for those sampling solutions that remain in the domain of semiconcavity of the CLF. On the other hand, we let the reader prove that the nonholonomic integrator system also has the CLF

$$\tilde{V}(x) = \left(\sqrt{x_1^2 + x_2^2} - |x_3| \right)^2 + x_3^2,$$

which is semiconcave on $\mathbb{R}^3 \setminus \{0\}$ (as the sum of the smooth function $x_1^2 + x_2^2 + 2x_3^2$ and a semiconcave function). Therefore, if we use $\tilde{V}$ to form our feedbacks, instead of the CLF (4.3), then our theorems apply directly, without any state restrictions on the sampling solutions.

Remark 4.2. The results in [19] designed feedbacks that make C^o-stabilizable systems ISS with respect to actuator errors. For the case of C^o-stabilizable systems, a smooth (i.e., C^∞) Lyapunov function is known to exist (see [1]). In [19], the system was rendered ISS using the feedback

(4.5) $$\hat{K}(x) := -L_G V(x) = -\nabla V(x) G(x),$$

where V is a smooth CLF for the dynamics (1.1). In that case, (4.5) is continuous at the origin. However, in the more general situation where the system is merely GAC, there may not exist a smooth Lyapunov function, so V must be taken to be nonsmooth. In this case, the use of the nonsmooth analogue

(4.6) $$\tilde{K}(x) := -\zeta(x) G(x)$$

of (4.5) (where $\zeta(x) \in \partial_L V(x)$ for all $x \neq 0$) could give rise to a feedback that would not be continuous at the origin. For example, if we use the nonholonomic integrator (4.2) and the CLF (4.3), then $\tilde{K}$ takes the values

$$\tilde{K}\left((\varepsilon, \varepsilon, 0)^T \right) = - \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)^T, \quad \tilde{K}\left((\varepsilon, \varepsilon, 3\sqrt{2}\varepsilon)^T \right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)^T + \varepsilon(1, -1)^T,$$

so $\tilde{K}$ is discontinuous at the origin. On the other hand, our choice of K_2 is automatically continuous at the origin.

Remark 4.3. Under the additional hypothesis that (1.1) satisfies the small control property (see [21]), the system can be stabilized by a feedback that is continuous at the origin (see [17]). More precisely, suppose there exists a semiconcave CLF V satisfying the following: For each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $0 < |x| \leq \delta$ implies

$$\exists u_x \in \varepsilon \mathcal{B}_m \quad \text{s.t.} \quad \forall \zeta \in \partial_P V(x), \quad \langle \zeta, f(x) + G(x) u_x \rangle \leq -V(x).$$

Then the system can be rendered GAS by a feedback that is continuous at the origin (see [17]). For the case of the nonholonomic integrator (4.2), the system is GAS under the feedback K_1, which is continuous at the origin, so our total feedback $K = K_1 + K_2$ is continuous at the origin as well.

5. ISS for fully nonlinear GAC systems. We conclude with an extension of our results for fully nonlinear GAC systems

(5.1) $$\dot{x} = f(x, u),$$

where we assume for simplicity that the observation error e in the controller is zero. We assume throughout this section that

$$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n : (x, u) \mapsto f(x, u)$$

is continuous and locally Lipschitz in x uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}^m$ and $f(0, 0) = 0$. It is natural to ask whether these hypotheses are sufficient for the existence of a continuous feedback $K(x)$ for which

$$(5.2) \qquad \dot{x} = f(x, K(x) + u)$$

is ISS for Euler solutions. However, one can easily construct examples for which such feedbacks cannot exist. Here is an example from [20] where this situation occurs. Consider the GAC system $\dot{x} = -x + u^2 x^2$ on $\mathbb{R}$. If $K(x)$ is any continuous feedback for which

$$(5.3) \qquad \dot{x} = -x + (K(x) + u)^2 x^2$$

is ISS, then $|K(x)| < x^{-1/2}$ for all $x > 0$. It follows that the solution of

$$\dot{x} = -x + (K(x) + 1)^2 x^2$$

starting at $x(0) = 4$ is unbounded. Therefore, there does not exist a continuous feedback K for which (5.3) is ISS. On the other hand, one *can* find a (possibly discontinuous) feedback that makes (5.1) ISS. We use the following weaker sense of ISS for fully nonlinear systems that was introduced in [20].

DEFINITION 5.1. *We say that* (5.1) *is* input-to-state stabilizable in the weak sense *provided there exist a feedback K, and an $m \times m$ matrix G of continuously differentiable functions which is invertible at each point, such that*

$$\dot{x} = F(x, K(x), u)$$

is ISS for sampling and Euler solutions, where $F(x, p, u) = f(x, p + G(x)u)$.

We will prove the following.

PROPOSITION 5.2. *If* (5.1) *is GAC, then* (5.1) *is also input-to-state stabilizable in the weak sense.*

Proof. We modify the proof from section 3. We define V, ζ, $\underline{\alpha}$, $\overline{\alpha}$, and K_1 as in the proof of Theorem 1.2, except we use the fully nonlinear dynamics $h = f$ from (5.1). Next we follow the proof of the main result in [20], with the following modifications. Define the (possibly discontinuous) function $\mathcal{D}$ by

$$(5.4) \qquad \mathcal{D}(s, r) = \sup \left\{ \langle \zeta(x), f(x, K_1(x) + p) \rangle + \frac{V(x)}{2} : |x| = s, |p| = r \right\}.$$

For any interval I of the form $[i, i+1]$, or of the form $[\frac{1}{i+1}, \frac{1}{i}]$, for $i \in \mathbb{N}$, one can find $r = r(i) > 0$ such that $s \in I$ implies $\mathcal{D}(s, b) < 0$ for all $b \in [0, r]$. This follows from the positive definiteness of V, the local Lipschitzness of f, and the local boundedness of $\partial_P V$ on compact subsets of $\mathbb{R}^n \setminus \{0\}$.

The argument of [20] therefore gives $\alpha_4 \in \mathcal{K}_\infty$ and a smooth, everywhere invertible matrix-valued function $G : \mathbb{R}^n \to \mathbb{R}^{m \times m}$ satisfying the following: If

$$(5.5) \qquad |x| > \alpha_4(|u(s)|_\infty),$$

then for a.e. $t \geq 0$,

$$\langle \zeta(x), f(x, K_1(x) + G(x)u(t)) \rangle + \frac{V(x)}{2} \leq \mathcal{D}(|x|, |G(x)u(s)|_\infty) < 0.$$

(See Remark 5.3 for a characterization of the set of matrices G for which ISS can be expected, in terms of $\mathcal{D}$.) We can evidently assume that $\alpha_4(s) \geq s$ for all $s \geq 0$ (e.g., by replacing $\alpha_4(s)$ by $\max\{\alpha_4(s), s\}$, which makes the condition (5.5) more restrictive). Fix M, N, $\varepsilon \in (0, M)$, $u \in \mathcal{M}_N^m$, and $x(t) = x_\pi(t)$ as before, with $e = 0$. Define the compact sets

$$S := \left\{ x \in \mathbb{R}^n : V(x) \leq \underline{\alpha}^{-1} \circ \alpha_4(N) \right\}, \quad Q = \left\{ (\overline{\alpha} \circ \underline{\alpha}^{-1}(M + \alpha_4(N)) + 1) \bar{\mathcal{B}}_n \right\} \setminus \varepsilon \mathcal{B}_n.$$

Notice that $S \subseteq Q^\varepsilon$. We choose $\tilde{\varepsilon}$ as before, and we choose $\delta = \delta(\varepsilon, M, N)$, satisfying (3.11), such that if $\bar{\mathbf{d}}(\pi) < \delta$, then

$$(5.6) \qquad |x_\pi(t) - x_i| \leq \min \left\{ \mu, \frac{\tilde{\varepsilon}}{16(1 + \mathcal{L}_\varepsilon)}, \sqrt{\frac{\lambda_-}{8\sigma}(t - t_i)} \right\}$$

for all indices i such that $x_i \in Q^\varepsilon$ and all $t \in [t_i, t_{i+1}]$, where σ and μ are as defined before, and $\lambda_- = \min\{V(x) : x \in Q^{\varepsilon/4}\}$. Reducing δ as necessary, we can assume

$$\|\zeta(x_i) \cdot [f(x_i, K_1(x_i) + G(x_i)u(s)) - f(x(s), K_1(x_i) + G(x(s))u(s))]\|_{[t_i, t_{i+1}]} \leq \frac{\lambda_-}{8}$$

for all indices i satisfying $x_i \in Q^{\varepsilon/2}$. Reasoning as in the earlier proof gives

$$V(x_\pi(t)) - V(x_i) \leq -(t - t_i)\frac{V(x_i)}{16} \qquad \forall t \in [t_i, t_{i+1}]$$

for all i such that $x_i \in Q^{\varepsilon/4} \setminus S$. The remainder of the proof is as before, except with $\overline{\alpha} \circ \underline{\alpha}^{-1}(N)$ replaced by $\overline{\alpha} \circ \underline{\alpha}^{-1}(\alpha_4(N))$, and with $\overline{\alpha} \circ \underline{\alpha}^{-1}(s)$ replaced by $\overline{\alpha} \circ \underline{\alpha}^{-1}(\alpha_4(s))$ in the definition of γ. This proves Proposition 5.2. $\square$

Remark 5.3. The statement of Proposition 5.2 is an existence result in terms of the invertible matrix G. However, we can strengthen the proposition by using the function $\mathcal{D}$ in (5.4) to characterize the class of G for which ISS can be expected, as follows. Following [20], we first choose strictly decreasing sequences $\{r_i\}$ and $\{r_i'\}$ of positive numbers such that $0 < r_{i+1} < r_i' < r_i$ for all $i \in \mathbb{N}$, and such that

$$\mathcal{D}(s, r) < 0 \qquad \forall (s, r) \in ([i, i+1] \times [0, r_i]) \cup ([1/(i+1), 1/i] \times [0, r_i'])$$

for all $i \in \mathbb{N}$. The existence of these sequences follows from the argument we gave in the proof of the proposition. Define $\rho : [0, \infty) \to [0, \infty)$ by setting

(ρ1) $\rho(s) = r_k$ for all $s \in [k, k+1)$ and $k \in \mathbb{N}$;
(ρ2) $\rho(s) = r_k'$ for all $s \in [1/(k+1), 1/k)$ and $k \in \mathbb{N}$; and
(ρ3) $\rho(0) = 0$.
We then choose any smooth function $g : [0, \infty) \to (0, \infty)$ satisfying

(g1) $g(s) = 1$ for all $s \in [0, 1]$;
(g2) $g(s) \leq \rho(s)/s$ for all $s \geq 2$; and
(g3) $g(s) \leq 1$ for all $s \geq 0$.
The existence of such a function g follows from exactly the same argument used in [20]. It then also follows from the argument of [20] that we can satisfy the conditions of the proposition by choosing $G(\xi) = g(|\xi|)I$.

Proposition 5.2 allows us to characterize GAC for fully nonlinear systems in terms of feedback equivalence, as follows. First recall that two systems $\dot{x} = f(x, u)$ and $\dot{x} = h(x, u)$, evolving on $\mathbb{R}^n \times \mathbb{R}^m$, are called *feedback equivalent* provided there exist a locally bounded function $K : \mathbb{R}^n \to \mathbb{R}^m$ and an everywhere invertible function $G : \mathbb{R}^n \to \mathbb{R}^{m \times m}$ for which

$$h(x, u) = f(x, K(x) + G(x)u)$$

for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. In this case, we also say $\dot{x} = f(x, u)$ is feedback equivalent to (2.1) with $e = 0$ and $F(x, p, u) = f(x, p + G(x)u)$. The following elegant statement follows directly from Proposition 5.2.

COROLLARY 5.4. *The fully nonlinear system* (5.1) *is GAC if and only if it is feedback equivalent to a system which is ISS for sampling and Euler solutions.*

Remark 5.5. Although, as shown by the counterexample (5.3), it is in general impossible to obtain input-to-state stabilization (in the nonweak sense) for systems that are not affine in controls, it is still the case that for some restricted classes of systems this objective can be attained, under appropriate neutral-stability assumptions on the dynamics. One such class is that of systems in which the input appears inside a saturation nonlinearity, such as $\dot{x} = f(x, u) = f_0(x) + g(x)\sigma(u)$. The papers [14] and [5] (see [26] for an application of these results to the recursive design of stabilizers for a large class of systems with saturation) as well as [4] and [13] dealt with such questions, for systems that are linear in the absence of the saturation (the f_0 and g vector fields are linear and constant, respectively), while [2] obtained similar results for more general nonlinear systems.

REFERENCES

[1] Z. ARTSTEIN, *Stabilization with relaxed controls*, Nonlinear Anal., 7 (1983), pp. 1163–1173.

[2] X. BAO AND Z. LIN, *On L_p input to state stabilizability of affine in control, nonlinear systems subject to actuator saturation*, J. Franklin Inst., 337 (2000), pp. 691–712.

[3] R. BROCKETT, *Asymptotic stability and feedback stabilization*, in Differential Geometric Control Theory, R. Brockett, R. Millman, and H. Sussmann, eds., Birkhäuser, Boston, 1983, pp. 181–191.

[4] Y. CHITOUR, *On the L^p stabilization of the double integrator subject to input saturation*, ESAIM Control Optim. Calc. Var., 6 (2001), pp. 291–331.

[5] Y. CHITOUR, W. LIU, AND E. SONTAG, *On the continuity and incremental-gain properties of certain saturated linear feedback loops*, Internat. J. Robust Nonlinear Control, 5 (1995), pp. 413–440.

[6] F. CLARKE, YU. S. LEDYAEV, E. SONTAG, AND A.I. SUBBOTIN, *Asymptotic controllability implies feedback stabilization*, IEEE Trans. Automat. Control, 42 (1997), pp. 1394–1407.

[7] R. FREEMAN AND P. KOKOTOVIĆ, *Inverse optimality in robust stabilization*, SIAM J. Control Optim., 34 (1996), pp. 1365–1391.

[8] M. KRSTIĆ, I. KANELLAKOPOULOS, AND P. KOKOTOVIĆ, *Nonlinear and Adaptive Control Design*, Wiley, New York, 1995.

[9] M. KRSTIĆ AND Z. LI, *Inverse optimal design of input-to-state stabilizing nonlinear controllers*, IEEE Trans. Automat. Control, 43 (1998), pp. 336–350.

[10] Y. LEDYAEV AND L. RIFFORD, *Robust Stabilization of the Nonholonomic Integrator*, unpublished.

[11] Y. LEDYAEV AND E. SONTAG, *A Lyapunov characterization of robust stabilization*, Nonlinear Anal., 37 (1999), pp. 813–840.

[12] D. LIBERZON, E. SONTAG, AND Y. WANG, *Universal construction of feedback laws achieving ISS and integral-ISS disturbance attenuation*, Systems Control Lett., 46 (2002), pp. 111–127.

[13] Z. LIN, *H_∞-almost disturbance decoupling with internal stability for linear systems subject to input saturation*, IEEE Trans. Automat. Control, 42 (1997), pp. 992–995.

[14] W. LIU, Y. CHITOUR, AND E. SONTAG, *On finite-gain stabilizability of linear systems subject to input saturation*, SIAM J. Control Optim., 34 (1996), pp. 1190–1219.

[15] L. PRALY AND Y. WANG, *Stabilization in spite of matched unmodelled dynamics and an equivalent definition of input-to-state stability*, Math. Control Signals Systems, 9 (1996), pp. 1–33.

[16] L. RIFFORD, *Existence of Lipschitz and semiconcave control-Lyapunov functions*, SIAM J. Control Optim., 39 (2000), pp. 1043–1064.

[17] L. RIFFORD, *Semiconcave control-Lyapunov functions and stabilizing feedbacks*, SIAM J. Control Optim., 41 (2002), pp. 659–681.

[18] E. SONTAG, *A Lyapunov-like characterization of asymptotic controllability*, SIAM J. Control Optim., 21 (1983), pp. 462–471.

[19] E. SONTAG, *Smooth stabilization implies coprime factorization*, IEEE Trans. Automat. Control, 34 (1989), pp. 435–443.

[20] E. SONTAG, *Further facts about input to state stabilization*, IEEE Trans. Automat. Control, 35 (1990), pp. 473–476.

[21] E. SONTAG, *Mathematical Control Theory, Deterministic Finite Dimensional Systems*, 2nd ed., Texts Appl. Math. 6, Springer-Verlag, New York, 1998.

[22] E. SONTAG, *Stability and stabilization: Discontinuities and the effect of disturbances*, in Nonlinear Analysis, Differential Equations, and Control, F. Clarke and R. J. Stern, eds., Kluwer, Dordrecht, The Netherlands, 1999, pp. 551–598.

[23] E. SONTAG, *The ISS philosophy as a unifying framework for stability-like behavior*, in Nonlinear Control in the Year 2000, Vol. 2, Lecture Notes in Control and Inform. Sci. 259, A. Isidori, F. Lamnabhi-Lagarrigue, and W. Respondek, eds., Springer-Verlag, London, 2000, pp. 443–468.

[24] E. SONTAG AND H. SUSSMANN, *Remarks on continuous feedback*, in Proceedings of the 19th Conference on Decision and Control, Albuquerque, NM, IEEE Control Systems Society, 1980, pp. 916–921.

[25] R. STERN, *Brockett's stabilization condition under state constraints*, Systems Control Lett., 47 (2002), pp. 335–341.

[26] H. SUSSMANN, E. D. SONTAG, AND Y. YANG, *A general result on the stabilization of linear systems using bounded controls*, IEEE Trans. Automat. Control, 39 (1994), pp. 2411–2425.