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On the representation of switched systems with inputs by perturbed control systems

J.L. Mancilla-Aguilar[a], R. García[b], E. Sontag[c,*,1], Y. Wang[d,2]

[a]Department of Mathematics, Faculty of Engineering, University of Buenos Aires, Argentina
[b]Department of Physics & Mathematics of the Instituto Tecnológico de Buenos Aires; and the Faculty of Engineering of the University of Buenos Aires, Argentina
[c]Department of Mathematics, Rutgers University, New Brunswick, NJ, USA
[d]Department of Mathematical Sciences, Florida Atlantic University, FL, USA

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Abstract

This paper provides representations of switched systems described by controlled differential inclusions, in terms of perturbed control systems. The control systems have dynamics given by differential equations, and their inputs consist of the original controls together with disturbances that evolve in compact sets; their sets of maximal trajectories contain, as a dense subset, the set of maximal trajectories of the original system. Several applications to control theory, dealing with properties of stability with respect to inputs and of detectability, are derived as a consequence of the representation theorem.

Keywords: Switched systems; Differential inclusions; Relaxation theorems; Input-to-state stability; Lyapunov method

* Corresponding author. Department of Mathematics, Rutgers, The State University of New Jersey, Hill Center, 110 Frelinghuysen Rd, Piscataway, NJ 088548019, USA. Tel.: +1 7324453072; fax: +1 6306046398.

E-mail addresses: jmancil@fi.uba.ar (J.L. Mancilla-Aguilar), ragarcia@itba.edu.ar (R. García), sontag@math.rutgers.edu (E. Sontag), ywang@math.fau.edu (Y. Wang).

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1. Introduction

In the last decade, the study of the properties of switched systems described by

$$\dot{x}(t) = f_{\sigma(t)}(x(t), u(t)), \quad y(t) = h(x(t)), \tag{1}$$

with $\sigma : [0, +\infty) \to \Gamma$ an arbitrary *switching signal* and Γ an index set, has received a great deal of attention, mainly motivated by the rapidly development of the area of intelligent control (see [11] and references therein for details). In fact, switched systems (1) enable us, for example, to model the continuous portion of a hybrid system (see [4,14]).

Different stability properties for system (1) were studied and characterized in terms of Lyapunov functions (see [11,12,16,17]). In [16] in particular, it was proved that, under suitable hypotheses, the set of maximal trajectories of system (1) is dense in the set of maximal trajectories of an associated (non-switched) system with controls and perturbations. This amounts to a *representation* of the switched system by a system with controls and perturbations. By using this fact, different results about perturbed control systems described by differential equations could be extended to switched systems (1) ([8,16,17]).

On the other hand, although under mild regularity conditions, the differential equation (1) provides, for each initial condition and each switching signal, a complete description of the time evolution of the state $x(\cdot)$, a more robust model of behavior should take into account uncertainties caused by modeling errors and disturbances that are inevitable in any real-world control problem. This leads one to consider a more general model: switched systems described by forced differential inclusions

$$\dot{x}(t) \in F_{\sigma(t)}(x(t), u(t)), \qquad y(t) = h(x(t)). \tag{2}$$

In this work we obtain representations of switched systems (2) (assuming locally Lipschitz right-hand sides) by means of perturbed control systems described by ordinary differential equations, driven by inputs consisting of the controls of the original system and perturbations that evolve in compact sets.

The representations obtained are characterized by the facts that every maximal trajectory of a system (2) is also a maximal trajectory of the representing system, and that the set of trajectories of (2) is dense (see Remark 2.4 for the precise meaning) in the set of trajectories of the representing system. The latter statement is closely related to the relaxation theorems of differential inclusions which assert that, under suitable conditions, relaxed trajectories can be approximated by (regular) trajectories (see [7,9]). These approximation results allow one to convert the analysis of an inclusion to its relaxation or vice versa. An interesting application can be found in the recent work [15].

Our representation results in the current paper allow us to achieve two theoretical and conceptual simplifications: (1) all "uncertainty" about the system can be summarized into just one "disturbance" input, and (2) switching, which is in principle very hard to study (because spaces of switching signals do not have any completeness properties), can be understood in terms of arbitrary Lebesgue-measurable disturbances with values on a compact space.

As immediate applications of our results on representations, we extend previous results on Lyapunov characterizations of input/output stability and detectability properties for systems of differential equations to switched systems defined by differential inclusions. (Further corollaries will be explored in other papers; for instance, it is shown in [18] that Lyapunov characterizations can also be developed for systems whose switching functions are governed by digraphs.)

We wish to emphasize that many of the results that we present are new even in the very special cases of un-controlled differential inclusions (and even ordinary differential equations) with switching, or of differential inclusions without switching.

The paper is organized as follows. In Section 2 we present the basic notation, the class of switched systems that we address and the main results. Section 3 presents results on forced differential inclusions that are instrumental for the proof of the main result. In Section 4, a result on parametrization of set valued maps is presented and the main results are proved. In Section 5, we develop Lyapunov characterizations of input–output-to-state stability and several input-to-output stability properties. Finally, in Section 6, conclusions are given. An appendix contains several additional results as well as some technical lemmas needed in the main text.

2. Switched systems

We first introduce some notations and definitions that will be used in the sequel. We use $|\cdot|$ to denote the Euclidean norm for any given $\mathbb{R}^q$ and with $\mathscr{B}_q$ we denote the closed unit ball in $\mathbb{R}^q$.

Given a metric space E, we denote by $\mathscr{M}(E)$ the set of Lebesgue measurable functions $\eta : [0, +\infty) \to E$ that are *locally essentially bounded*. In case $E = \mathbb{R}^m$ we write $\mathscr{U}$ instead of $\mathscr{M}(\mathbb{R}^m)$. We say that a sequence $\{\eta_n, n \in \mathbb{N}\} \subset \mathscr{M}(E)$ is *locally equibounded* if for each compact interval $\mathscr{J} \subset [0, +\infty)$ there exists a compact subset $K \subseteq E$ such that for all $n \in \mathbb{N}$, $\eta_n(t) \in K$ for almost all $t \in \mathscr{J}$.

For a measurable function $\theta : [0, \infty) \to \mathbb{R}^l$, we denote by $\|\theta\|$ the (possibly infinite) L^∞_l-norm of θ and, for any $t \geqslant 0$, $\|\theta\|_{[0,t]}$ stands for the L^∞_l-norm of θ restricted to the interval $[0, t]$.

Let X be a metric space. We denote the distance from a point $\xi \in X$ to a set $A \subseteq X$ by $\operatorname{dist}(\xi, A)$. The *Hausdorff distance* between two nonempty closed subsets of X, A and B, is defined as $d_H(A, B) := \max\{\sup_{\xi\in B} \operatorname{dist}(\xi, A),\ \sup_{\eta\in A} \operatorname{dist}(\eta, B)\}$.

We let $\mathscr{K}(X)$ be the set of nonempty compact subsets of X and we recall that the Hausdorff distance d_H is a metric on $\mathscr{K}(X)$ and that $\mathscr{K}(X)$ is compact in the metric d_H if X is compact (see [19, p. 279]). For a normed space X, we still use $|\cdot|$ to denote the norm on X. For a subset A of X, co A and cl A stand for the convex hull and the closure of A respectively. We define $\|A\| := \sup\{|a| : a \in A\}$.

Given another metric space Z, we say that a set-valued map $G : Z \to \mathscr{K}(X)$ is continuous (Lipschitz) if it is continuous (Lipschitz) when the Hausdorff distance is considered in $\mathscr{K}(X)$. We use $\mathscr{C}(Z, \mathscr{K}(X))$ to denote the class of continuous maps from Z to $\mathscr{K}(X)$ equipped with the topology of uniform convergence on compact sets. It must be remarked that this topological space is metrizable when Z is a finite dimensional vector space. In order to see it, consider any norm $|\cdot|$ on Z and let C_r denote the closed ball centered at 0 with radius r in Z. Then C_r is compact. Let d_r be the metric for $\mathscr{C}(C_r, \mathscr{K}(X))$ given by

$d_r(f, g) = \sup_{x\in C_r} d_H(f(x), g(x))$. Then the metric d on $\mathscr{C}(Z, \mathscr{K}(X))$ defined by

$$d(f, g) = \sum_{j=1}^{\infty} \frac{d_j(f, g)}{1 + d_j(f, g)} 2^{-j},$$

induces the topology of uniform convergence on compact sets.

As usual, by a $\mathscr{K}$-function we mean a function $\alpha : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ that is strictly increasing and continuous, and satisfies $\alpha(0)=0$, by a $\mathscr{K}_\infty$-function one that is in addition unbounded, and we let $\mathscr{KL}$ be the class of functions $\mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ which are of class $\mathscr{K}$ on the first argument and decrease to zero on the second argument.

Before we introduce the class of switched systems with which we deal in this work, it is convenient to describe the class of functions that we will take as switching signals:

Definition 2.1. Given a nonempty set C, we say that a function $g : [0, +\infty) \to C$ is a *switching signal* if g verifies one of the two following conditions:

1. g is a piecewise constant function (i.e. the set of points where the function g has jumps is finite in each compact subinterval of $[0, +\infty)$, and g is constant between jumps) continuous from the right;
2. there exist a sequence of real numbers $\{t_k : k \in \mathbb{N}_0\}$ with $0 = t_0 < t_1 < \cdots < t_k < \cdots$ and $\lim_{k\to+\infty} t_k = T_g < +\infty$, a sequence $\{c_k \in C : k \in \mathbb{N}_0\}$ with $c_k \neq c_{k+1}$ for all $k \geqslant 0$ and a point $c^* \in C$ such that $g(t) = c_k$ for all $t_k \leqslant t < t_{k+1}$ and $g(t) = c^*$ for all $t \geqslant T_g$.

In what follows, we will denote by $\mathscr{S}(C)$ the family of all C-valued switching signals and with $\mathscr{S}_{\mathrm{pc}}(C)$ the subfamily of all C-valued piecewise constant switching signals.

Remark 2.2. The definition of switching signal that we use here is slightly more general than that usually considered in the literature on switched systems, where a switching signal means an element in $\mathscr{S}_{\mathrm{pc}}(C)$. Switching signals that do not belong to $\mathscr{S}_{\mathrm{pc}}(C)$ are usually related with the so-called Zeno-behavior of a hybrid system (see [26]) and due to this reason they will be here referred to as Zeno-switching signals.

As pointed out above, in this work we consider switched systems whose subsystems are described by forced differential inclusions. More precisely, given a family of locally Lipschitz set-valued maps

$$\mathscr{P} = \{F_\gamma \in \mathscr{C}(\mathbb{R}^n \times \mathbb{R}^m, \mathscr{K}(\mathbb{R}^n)) : \gamma \in \Gamma\}, \tag{3}$$

where Γ is an index set and, without loss of generality, $F_\gamma \neq F_{\gamma'}$ if $\gamma \neq \gamma'$, we consider the switched system with inputs

$$\dot{x} \in F_\sigma(x, u), \tag{4}$$

where x takes values in $\mathbb{R}^n$, $u \in \mathscr{U}$, and $\sigma \in \mathscr{S}(\Gamma)$.

Given an input $u \in \mathscr{U}$ and a switching signal $\sigma \in \mathscr{S}(\Gamma)$, we say that a locally absolutely continuous function $x : \mathscr{I} \to \mathbb{R}^n$ where $\mathscr{I} = [0, T]$ or $[0, T)$ with $0 < T \leqslant +\infty$ is a *trajectory* of (4) corresponding to $u \in \mathscr{U}$ and to $\sigma \in \mathscr{S}(\Gamma)$ if $\dot{x}(t) \in F_{\sigma(t)}(x(t), u(t))$ for almost all $t \in \mathscr{I}$. Observe that, due to the assumptions about F_γ, for each $\xi \in \mathbb{R}^n$, each $u \in \mathscr{U}$ and each $\sigma \in \mathscr{S}(\Gamma)$ there always exists a trajectory x corresponding to u and to σ that verifies $x(0) = \xi$ and that is defined on an interval $[0, T)$ for some small $T > 0$.

A trajectory $x : [0, T) \to \mathbb{R}^n$ corresponding to $u \in \mathscr{U}$ and to $\sigma \in \mathscr{S}(\Gamma)$ is called *maximal* if it does not have an extension which is a solution corresponding to u and to σ, i.e., either $T = +\infty$ or there does not exist a trajectory $z : [0, T') \to \mathbb{R}^n$ corresponding to u and to σ with $T' > T$ so that $z(t) = x(t)$ for all $t \in [0, T)$. Given a maximal trajectory x corresponding to $u \in \mathscr{U}$ and to $\sigma \in \mathscr{S}(\Gamma)$, we denote its domain by $[0, T_x)$. We write just T_x for simplicity, even though it is $T_{x,u,\sigma}$.

For any $\xi \in \mathbb{R}^n$, any $u \in \mathscr{U}$ and any $\sigma \in \mathscr{S}(\Gamma)$, we denote by $\mathscr{T}^s(\xi, u, \sigma)$ the collection of all the maximal trajectories x of (4) corresponding to u and to σ that satisfy $x(0) = \xi$.

2.1. Main results

In what follows we will establish one of the main results of this work, which asserts that under suitable hypotheses on the family $\mathscr{P}$ the set of maximal trajectories of system (4) is dense, in a sense that we will make precise, in the set of maximal trajectories of a system with inputs described by a system of differential equations:

$$\dot{z} = f(z, u, d, \omega), \tag{5}$$

where z takes values in $\mathbb{R}^n$, $u \in \mathscr{U}$, $d \in \mathscr{M}(D)$ with D a compact metric space, and $\omega \in \mathscr{M}(\mathscr{B}_n)$.

In order to establish the precise result, we consider for the family $\mathscr{P}$ the following hypotheses:

- C1: The family $\mathscr{P}$ is uniformly locally Lipschitz, i.e., for each $N \in \mathbb{N}$, there exists $l_N \geqslant 0$ such that

  $$d_H(F_\gamma(\xi, \mu), F_\gamma(\xi', \mu')) \leqslant l_N(|\xi - \xi'| + |\mu - \mu'|),$$

  for all $\xi, \xi' \in N\mathscr{B}_n$ and all $\mu, \mu' \in N\mathscr{B}_m$ (where we have used $r\mathscr{B}_q$ to denote the closed ball centered at 0 of radius r in $\mathbb{R}^q$).
- C2: The family $\mathscr{P}$ is pointwise equibounded, i.e., for each $(\xi, \mu) \in \mathbb{R}^n \times \mathbb{R}^m$ there exists $M_{(\xi,\mu)} \geqslant 0$ such that $\|F_\gamma(\xi, \mu)\| \leqslant M_{(\xi,\mu)}$ for all $\gamma \in \Gamma$.

One of the main results of the paper is the following:

Theorem 1. *Suppose that $\mathscr{P}$ verifies* C1 *and* C2. *Then there exist a compact metric space D, an injective function $\iota : \Gamma \to D$, and a continuous function $f : \mathbb{R}^n \times \mathbb{R}^m \times D \times \mathscr{B}_n \to \mathbb{R}^n$ such that*

1. *The set $\iota(\Gamma)$ is dense in D.*

2. *The map* $f(\cdot,\cdot,v,\cdot)$ *is locally Lipschitz uniformly on* $v \in D$, i.e., *for each compact subset K of* $\mathbb{R}^n \times \mathbb{R}^m$ *there is some constant* c_K *so that*

$$|f(\xi,\mu,v,\zeta) - f(\xi',\mu',v,\zeta')| \leqslant c_K(|\xi-\xi'| + |\mu-\mu'| + |\zeta-\zeta'|),$$

for all $(\xi,\mu),(\xi',\mu') \in K$, *all* $\zeta,\zeta' \in \mathscr{B}_n$ *and all* $v \in D$.
3. *For each* $\xi \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$ *and* $\gamma \in \Gamma$,

$$\text{co } F_\gamma(\xi,\mu) = \{f(\xi,\mu,\iota(\gamma),\zeta) : \ \zeta \in \mathscr{B}_n\}.$$

4. *Given* $u \in \mathscr{U}$, $\sigma \in \mathscr{S}(\Gamma)$ *and a maximal trajectory x of* (4) *corresponding to u and* σ, *there exists* $\omega \in \mathscr{M}(\mathscr{B}_n)$ *such that x is a maximal trajectory of* (5) *corresponding to u*, $d_\sigma = \iota \circ \sigma$ *and* ω.
5. *The trajectories of* (4) *are dense in the set of trajectories of* (5) *in the two following senses*:

   (a) *Given* $\varepsilon > 0$ *and a trajectory* $z : [0,T] \to \mathbb{R}^n$ *of* (5) *corresponding to* $u \in \mathscr{U}$, $d \in \mathscr{M}(D)$ *and* $\omega \in \mathscr{M}(\mathscr{B}_n)$, *there exist* $\sigma \in \mathscr{S}_{\text{pc}}(\Gamma)$ *and a trajectory x of* (4) *corresponding to u and* σ *with* $x(0) = z(0)$ *such that*

$$|z(t) - x(t)| < \varepsilon, \quad \forall t \in [0,T].$$

   (b) *Given a trajectory* $z : [0,T) \to \mathbb{R}^n$ *of* (5) *with* $T \leqslant \infty$ *corresponding to* $u \in \mathscr{U}$, $d \in \mathscr{M}(D)$ *and* $\omega \in \mathscr{M}(\mathscr{B}_n)$ *and a continuous function* $r : [0,T) \to \mathbb{R}_{>0}$, *there exist* $\sigma \in \mathscr{S}(\Gamma)$ *and a trajectory x of* (4) *corresponding to u and* σ *such that*

$$|z(t) - x(t)| < r(t), \quad \forall t \in [0,T).$$

   In the case when $T = \infty$, *the switching function* σ *can be chosen in* $\mathscr{S}_{\text{pc}}(\Gamma)$.

Remark 2.3. Parts 1–3 of Theorem 1 contain as a particular case a result obtained in [17] for switched systems described by differential equations (namely, Theorem 3.2 of [17]). In fact, if a family of functions $\{f_\gamma \in \mathscr{C}(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n) : \gamma \in \Gamma\}$ verifies the hypotheses of Theorem 3.2 of [17], then $\mathscr{P} = \{F_\gamma \in \mathscr{C}(\mathbb{R}^n \times \mathbb{R}^m, \mathscr{K}(\mathbb{R}^n)) : \gamma \in \Gamma\}$ with $F_\gamma(\cdot,\cdot) = \{f_\gamma(\cdot,\cdot)\}$ verifies C1 and C2. Consequently, there exist a compact metric space D, an injective function $\iota : \Gamma \to D$ and a continuous function $f : \mathbb{R}^n \times \mathbb{R}^m \times D \times \mathscr{B}_n \to \mathbb{R}^n$ that verify 1–3 of Theorem 1. From statement 3, it easily follows that $f(\xi,\mu,v,\zeta) = f(\xi,\mu,v,\zeta')$ for all $\zeta,\zeta' \in \mathscr{B}_n$, that is, the function $f(\xi,\mu,v,\cdot)$ is constant for any given ξ, μ and v. Hence, f can be considered as a function from $\mathbb{R}^n \times \mathbb{R}^m \times D$ to $\mathbb{R}^n$, and we recover Theorem 3.2 of [17].

Remark 2.4. Part 5(a) of Theorem 1 asserts that for fixed $T > 0$, $u \in \mathscr{U}$ and $\xi \in \mathbb{R}^n$, the set of trajectories of (4) corresponding to u and the initial state ξ with $\sigma \in \mathscr{S}_{\text{pc}}(\Gamma)$ that are defined on $[0,T]$ is dense in the set of trajectories of (5) corresponding to u and the initial state ξ with $d \in \mathscr{M}(D)$ and $\omega \in \mathscr{M}(\mathscr{B}_n)$ that are defined on $[0,T]$, when the topology in consideration is the topology of uniform convergence on the interval $[0,T]$.

On the other hand, part 5(b) states that for fixed $0 < T < +\infty$ ($T = \infty$ respectively) and $u \in \mathscr{U}$, the set of trajectories of (4) corresponding to u with $\sigma \in \mathscr{S}(\Gamma)$ ($\sigma \in \mathscr{S}_{\text{pc}}(\Gamma)$

respectively) that are defined on $[0, T)$ is dense in the set of trajectories of (5) corresponding to u with $d \in \mathscr{M}(D)$ and $\omega \in \mathscr{M}(\mathscr{B}_n)$ that are defined on $[0, T)$, when the topology under consideration is the $\mathscr{C}^0$ Whitney topology.

We point out that part 5(b) does not hold if we consider only piecewise constant switching signals (see Section 3.1). That is the reason why we must also consider Zeno-switching signals.

As a corollary of Theorem 1, we will also obtain the following representation theorem:

Theorem 2. *Suppose that* $\mathscr{P}$ *verifies* C1 *and* C2. *Then there exists a locally Lipschitz function* $f^* : \mathbb{R}^n \times \mathbb{R}^m \times \mathscr{B}_n \to \mathbb{R}^n$ *such that*

1. *For each* $\xi \in \mathbb{R}^n$ *and* $\mu \in \mathbb{R}^m$,

$$\operatorname{co}\operatorname{cl}\left(\bigcup_{\gamma\in\Gamma} F_\gamma(\xi,\mu)\right) = \{f^*(\xi,\mu,\zeta) : \zeta \in \mathscr{B}_n\}.$$

2. *Given* $u \in \mathscr{U}$, $\sigma \in \mathscr{S}(\Gamma)$ *and a maximal trajectory* x *of* (4) *corresponding to* u *and* σ, *there exists* $\omega \in \mathscr{M}(\mathscr{B}_n)$ *such that* x *is a maximal trajectory of*

$$\dot{x} = f^*(x, u, \omega). \tag{6}$$

3. *The trajectories of* (4) *are dense in the set of trajectories of* (6) *in the two following senses*:

   (a) *Given* $\varepsilon > 0$ *and a trajectory* $z : [0, T] \to \mathbb{R}^n$ *of* (6) *corresponding to* $u \in \mathscr{U}$ *and* $\omega \in \mathscr{M}(\mathscr{B}_n)$ *there exist* $\sigma \in \mathscr{S}_{\rm pc}(\Gamma)$ *and a trajectory* x *of* (4) *corresponding to* u *and* σ *such that* $x(0) = z(0)$ *and*

$$|z(t) - x(t)| < \varepsilon, \quad \forall t \in [0, T].$$

   (b) *Given a trajectory* $z : [0, T) \to \mathbb{R}^n$ *of* (6), *with* $T \leqslant \infty$, *corresponding to* $u \in \mathscr{U}$ *and* $\omega \in \mathscr{M}(\mathscr{B}_n)$ *and a continuous function* $r : [0, T) \to \mathbb{R}_{>0}$, *there exist* $\sigma \in \mathscr{S}(\Gamma)$ *and a trajectory* x *of* (4) *corresponding to* u *and* σ *such that*

$$|z(t) - x(t)| < r(t), \quad \forall t \in [0, T).$$

   In the case when $T = \infty$, *the switching signal* σ *can be choosen in* $\mathscr{S}_{\rm pc}(\Gamma)$.

The two results complement each other: while Theorem 1 provides a representation of the given switched differential inclusion in terms of a switched system of differential equations (driven by the same switching signal, provided that we identify σ with $\iota \circ \sigma$), Theorem 2 is more abstract, and it provides a representation in terms of a non-switched system of differential equations. This latter form, which looses the information about the switching signal, is of use for theoretical purposes, since it allows reducing questions about switched differential inclusions into questions about ordinary differential equations; examples are given later in the paper.

Next, we establish an association between the switched system (4) and a forced differential inclusion with two inputs. In order to do so, we will use the following result.

Lemma 2.5. *Suppose that* $\mathscr{P}$ *verifies* C1 *and* C2. *Then there exist a compact metric space* D, *an injective function* $\imath : \Gamma \to D$ *and a continuous set-valued map* $F : \mathbb{R}^n \times \mathbb{R}^m \times D \to \mathscr{K}(\mathbb{R}^n)$ *such that*

1. $\imath(\Gamma)$ *is dense in* D.
2. $F(\cdot, \cdot, v)$ *is locally Lipschitz uniformly with respect to* $v \in D$.
3. $F(\xi, \mu, \imath(\gamma)) = F_\gamma(\xi, \mu)$ *for all* $\xi \in \mathbb{R}^n$, *all* $\mu \in \mathbb{R}^m$ *and all* $\gamma \in \Gamma$.

Proof. Consider $\mathscr{P}$ as a subset of $\mathscr{C}(\mathbb{R}^n \times \mathbb{R}^m, \mathscr{K}(\mathbb{R}^n))$ equipped with the topology of the uniform convergence on compact sets. Condition C2 implies that for each $(\xi, \mu) \in \mathbb{R}^n \times \mathbb{R}^m$ there exists a compact set $K_{(\xi,\mu)} \subset \mathbb{R}^n$ such that $F_\gamma(\xi, \mu) \subseteq K_{(\xi,\mu)}$ for all $\gamma \in \Gamma$. Thus, for each pair (ξ, μ), $F_\gamma(\xi, \mu) \in \mathscr{K}(K_{(\xi,\mu)})$ for all $\gamma \in \Gamma$. Since $\mathscr{K}(K_{(\xi,\mu)})$ is compact, $\{F_\gamma(\xi, \mu) : \gamma \in \Gamma\}$ has compact closure in $\mathscr{K}(\mathbb{R}^n)$. Since condition C1 implies that the family $\mathscr{P}$ is also equicontinuous, it follows from the Arzelá–Ascoli Theorem (see [19, p. 290]) that $\overline{\mathscr{P}}$, the closure of $\mathscr{P}$ in $\mathscr{C}(\mathbb{R}^n \times \mathbb{R}^m, \mathscr{K}(\mathbb{R}^n))$, is compact. As $\mathscr{C}(\mathbb{R}^n \times \mathbb{R}^m, \mathscr{K}(\mathbb{R}^n))$ is metrizable, we have that $D := \overline{\mathscr{P}}$ is a compact metric space.

Now we define, for each $\gamma \in \Gamma$, $\imath(\gamma) := F_\gamma$ and, for $(\xi, \mu, v) \in \mathbb{R}^n \times \mathbb{R}^m \times D$, $F(\xi, \mu, v) := v(\xi, \mu)$. Clearly statements 1 and 3 of the lemma are verified.

As for the continuity of F, note that this function is the restriction to $\mathbb{R}^n \times \mathbb{R}^m \times D$ of the evaluation map $e_v : \mathbb{R}^n \times \mathbb{R}^m \times \mathscr{C}(\mathbb{R}^n \times \mathbb{R}^m, \mathscr{K}(\mathbb{R}^n)) \to \mathscr{K}(\mathbb{R}^n)$, and that this last map is continuous (see [19, p. 287]).

In order to prove assertion 2, it is sufficient to show that for each $N \in \mathbb{N}$, F is Lipschitz on $N\mathscr{B}_n \times N\mathscr{B}_m$ uniformly with respect to $v \in D$. Let $N \in \mathbb{N}$, $v \in D$ and (ξ, μ), $(\xi', \mu') \in N\mathscr{B}_n \times N\mathscr{B}_m$. Then there exists a sequence $\{v_k = \imath(\gamma_k), \gamma_k \in \Gamma, k \in \mathbb{N}\}$ such that $v_k \to v$ and, due to C1 and to the continuity of F,

$$
\begin{aligned}
d_H(F(\xi, \mu, v), F(\xi', \mu', v)) &= \lim_{k\to\infty} d_H(F_{\gamma_k}(\xi, \mu), F_{\gamma_k}(\xi', \mu')) \\
&\leqslant l_N(|\xi - \xi'| + |\mu - \mu'|).
\end{aligned}
$$

It follows that $F(\cdot, \cdot, v)$ is Lipschitz on $K = N\mathscr{B}_n \times N\mathscr{B}_m$ uniformly with respect to $v \in D$ with constant $c_K = l_N$. □

Now we associate with the switched system (4) the following system with two inputs:

$$
\dot{x} \in F(x, u, d), \tag{7}
$$

where x takes values in $\mathbb{R}^n$, $u \in \mathscr{U}$, $d \in \mathscr{M}(D)$ and where F and D are as in Lemma 2.5.

Given $\xi \in \mathbb{R}^n$, $u \in \mathscr{U}$ and $d \in \mathscr{M}(D)$, we denote with $\mathscr{T}(\xi, u, d)$ the collection of all the maximal solutions x of (7), corresponding to the inputs u and d, that satisfy $x(0) = \xi$.

Remark 2.6. The following facts readily follow from Lemma 2.5:

(i) For each $\sigma \in \mathscr{S}(\Gamma)$ ($\mathscr{S}_{\mathrm{pc}}(\Gamma)$ respectively), if $d_\sigma = \imath \circ \sigma$, then $d_\sigma \in \mathscr{S}(D)$ ($\mathscr{S}_{\mathrm{pc}}(D)$ respectively) and $\mathscr{T}^s(\xi, u, \sigma) = \mathscr{T}(\xi, u, d_\sigma)$.
(ii) For each $d \in \mathscr{S}(\imath(\Gamma))$ ($\mathscr{S}_{\mathrm{pc}}(\imath(\Gamma))$ respectively), there exists a unique switching signal $\sigma_d \in \mathscr{S}(\Gamma)$ ($\mathscr{S}_{\mathrm{pc}}(\Gamma)$ respectively) such that $\imath \circ \sigma_d = d$.

In this manner, any switched system (4) corresponding to a family $\mathscr{P}$ that verifies C1 and C2 can be viewed as a system described by the forced differential inclusion (7) with two inputs: one of them is the input to the switched system, and the other one is a switching signal that takes values in a dense subset of a compact metric space D.

3. Some results about forced differential inclusions

In what follows we consider the time-varying system with inputs described by

$$\dot{x} \in G(t, x, d), \tag{8}$$

where $t \geqslant 0$, x takes values in $\mathbb{R}^n$ and $d \in \mathscr{M}(D)$ with D a separable metric space. We also consider the relaxation of (8)

$$\dot{x} \in \operatorname{co} G(t, x, d). \tag{9}$$

Throughout the rest of this section we will suppose that the set-valued map $G : \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \times D \to \mathscr{K}(\mathbb{R}^n)$ verifies the following hypotheses:

(H_1) $G(\cdot, \xi, v)$ is measurable for each $(\xi, v) \in \mathbb{R}^n \times D$;
(H_2) $G(t, \cdot, \cdot)$ is continuous for each $t \geqslant 0$;
(H_3) for each compact subset $K \subset \mathbb{R}^n \times D$ there exists $L_K \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0}, \mathbb{R})$ such that for each $(\xi, v), (\xi', v) \in K$,

$$d_H(G(t, \xi, v), G(t, \xi', v)) \leqslant L_K(t)|\xi - \xi'|, \quad \forall t \geqslant 0;$$

(H_4) for each compact subset $K \subset \mathbb{R}^n \times D$ there exists $\alpha_K \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geqslant 0}, \mathbb{R})$ such that for each $(\xi, v) \in K$

$$\|G(t, \xi, v)\| \leqslant \alpha_K(t), \quad \forall t \geqslant 0.$$

Next, we will show that the set of trajectories of (8) generated by switching signals that take values in a dense subset D' of D are dense in the whole set of trajectories of its relaxation (9) when either one of the two following topologies are considered:

- The topology of the uniform convergence on a compact interval $[0, T]$ with $T > 0$;
- the $\mathscr{C}^0$ Whitney topology on a interval $[0, T)$ with $0 < T \leqslant +\infty$.

Theorem 3. *Suppose that the set-valued map* G: $\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \times D \to \mathscr{K}(\mathbb{R}^n)$, *where D is a separable metric space, satisfies the hypotheses* (H_1)–(H_4). *Let D' be a dense subset of*

D. Then, given a maximal trajectory $z : [0, T_z) \to \mathbb{R}^n$ *of* (9) *corresponding to* $d^* \in \mathscr{M}(D)$, *the following hold*:

1. *Given* $0 < T < T_z$ *and* $\varepsilon > 0$, *there exist a piecewise constant switching signal* $d \in \mathscr{S}_{\mathrm{pc}}(D')$ *and a maximal trajectory x of* (8) *corresponding to d such that* $x(0) = z(0)$ *and*

$$|x(t) - z(t)| < \varepsilon, \quad \forall t \in [0, T]. \tag{10}$$

2. *Given* $0 < T \leqslant T_z$ *and a continuous function* $r : [0, T) \to \mathbb{R}_{>0}$, *there exist a switching signal* $d \in \mathscr{S}(D')$ ($d \in \mathscr{S}_{\mathrm{pc}}(D')$ *if* $T = +\infty$) *and a maximal trajectory x of* (8) *corresponding to d such that*

$$|x(t) - z(t)| < r(t), \quad \forall t \in [0, T). \tag{11}$$

Remark 3.1. Theorem 3 (whose proof is given in Sections 3.2.1–3.2.2) remains valid, with the same proof, when the domain of the first variable of G is a finite interval $\mathscr{I}$ of the form $[0, a]$ or $[0, a)$ with $\mathbb{R}_{\geqslant 0}$ replaced by $\mathscr{I}$ in hypotheses (H_1)–(H_4).

Remark 3.2. Parts 1 and 2 of Theorem 3 can be considered extensions of the Filippov–Ważewski Theorem and Theorem 1 of [9], respectively, since they are particular cases of Theorem 3. To see it, just consider the case when D in Theorem 3 is a singleton.

3.1. A result on forward completeness of time-varying forced differential inclusions

In this short section we present a result about the forward completeness of the time-varying differential inclusion (8), which is a simple consequence of Theorem 3 part 2. We also show that part 2 of that theorem does not hold if we consider only piecewise constant switching signals.

Given a subclass of inputs $\mathscr{A} \subseteq \mathscr{M}(D)$, we will say that system (8) is forward complete with respect to $\mathscr{A}$ if every maximal trajectory of (8) which corresponds to an input $d \in \mathscr{A}$ and starts at time $t_0 = 0$, is defined for all $t \geqslant 0$. In case $\mathscr{A} = \mathscr{M}(D)$ we will say that the system is forward complete.

The following result follows readily from Theorem 3:

Theorem 4. *Suppose that the set-valued map* $G : \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \times D \to \mathscr{K}(\mathbb{R}^n)$, *with D a separable metric space, satisfies the hypotheses* (H_1)–(H_4). *Let* D' *be a dense subset of D. Then, the following statements are equivalent.*

1. *System* (8) *is forward complete*.
2. *System* (8) *is forward complete with respect to* $\mathscr{S}(D')$.

The implication 1. $\Rightarrow$ 2. is trivial. To prove the implication 2. $\Rightarrow$ 1., we quote an existence result for differential inclusions (e.g. [6], Theorem 7 on p. 85).

Lemma 3.3. *Let* $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ *be an open set containing* $(0, x_0)$, *and let G be a set-valued map so that* $G(t, x)$ *is nonempty and closed for* $(t, x) \in \Omega$. *Suppose that, for* $(t, x) \in \Omega$,

$\|G(t,x)\| \leqslant m(t)$ for some $m(\cdot) \in L^1_{\text{loc}}(\mathbb{R}_{\geqslant 0}, \mathbb{R})$, and that G is continuous in x, measurable in t. Then there exist $T > 0$ and a solution $x(\cdot)$ of

$$\dot{x} \in G(t,x), \quad x(0) = x_0$$

defined on the interval $[0, T]$.

Proof of Theorem 4. $2 \Rightarrow 1$**.** If system (8) is not forward complete, there exist an input $d \in \mathscr{M}(D)$ and a trajectory $z(\cdot)$ corresponding to d defined on a maximal interval $[0, T)$ with $T < \infty$. On the other hand, consider $r : [0, T) \to \mathbb{R} > 0$ defined by $r(t) = (t - T)^2$. By Theorem 3 part 2, there exist an input $d' \in \mathscr{S}(D')$ and a maximal trajectory $x(\cdot)$ of (8) defined on $[0, \infty)$ corresponding to d' such that $|z(t) - x(t)| < r(t)$ for all $t \in [0, T)$. Consequently, $\lim_{t\to T} z(t) = x(T)$. By Lemma 3.3, there exists some $\delta > 0$ such that $z(t)$ can be extended to $[0, T + \delta]$, which contradicts the maximality of T. □

Remark 3.4. From the proof of Theorem 4 it is clear that the statements of Theorem 4 still hold if we replace $\mathscr{S}(D')$ by any subclass of inputs $\mathscr{A} \subseteq \mathscr{M}(D)$ for which part 2 of Theorem 3 holds.

Now, we show that part 2 of Theorem 3 does not hold if we consider only inputs that belong to $\mathscr{S}_{\text{pc}}(D')$ instead of the whole $\mathscr{S}(D')$. Due to Remark 3.4, a system that is forward complete with respect to $\mathscr{S}_{\text{pc}}(D)$ but not forward complete serves as a counterexample.

Example 3.5. Consider the system with (d_1, d_2) as inputs

$$\dot{x} = d_1 f_1(x) + d_2 f_2(x), \tag{12}$$

where $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ $(i = 1, 2)$ is given by $f_1(\xi) = (1 + |\xi|^2)A_1\xi$, $f_2(\xi) = (1 + |\xi|^2)A_2\xi$, with

$$A_1 = \begin{bmatrix} 0 & 2 \\ -0.5 & -0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.1 & 0.5 \\ -2 & 0 \end{bmatrix}$$

and where $d = (d_1, d_2) \in \mathscr{M}(D)$ with $D = D' = \{(1, 0), (0, 1)\}$. Observe that

$$G(t, \xi, v) := \{v_1 f_1(\xi) + v_2 f_2(\xi)\}$$

verifies hypotheses (H_1)–(H_4).

Since the vector fields f_1 and f_2 are both globally asymptotically stable and therefore forward complete, it follows that the system (12) is forward complete with respect to $\mathscr{S}_{\text{pc}}(D)$. On the other hand, with the input $d = (d_1, d_2)$ defined by the feedback rule

$$d(t) = \begin{cases} (1, 0) & \text{if } x_1(t)x_2(t) \geqslant 0, \\ (0, 1) & \text{if } x_1(t)x_2(t) < 0, \end{cases} \tag{13}$$

there are trajectories with finite escape time. To see this, observe that the system (12) has the same phase portrait as the linear system

$$\dot{z} = d_1 A_1 z + d_2 A_2 z. \tag{14}$$

By explicitly solving the linear system (14), it can be seen that with the input give by (13), the trajectories of (14) are spirals running clockwise. Consider the trajectory starting at $(0, c)$ for some $c \neq 0$. Let $0 = t_0 < t_1 < t_2 < t_3 < \cdots$ be the switching times for $d(t)$ given by (13) (i.e., the time when the trajectory crosses the coordinate axes). Then, for $t \in [0, t_1]$, the trajectory of (14) with the initial state $z(0) = (0, c)$ is given by

$$z_1(t) = \frac{2c}{b}\,\mathrm{e}^{-at}\sin bt,$$
$$z_2(t) = \frac{-ac}{b}\,\mathrm{e}^{-at}\sin bt + c\mathrm{e}^{-at}\cos bt,$$

where $a = 0.05, b = \sqrt{1-(1/400)}$. From this it can be calculated that $t_1 = [\arctan(b/a)]/b \leqslant \pi/2$, $|z_1(t_1)| \geqslant 1.5|c|$ (and $z_2(t_1) = 0$).

Consequently, $|z(t_1)| \geqslant 1.5|z(t_0)|$. By symmetry (or by the same calculation), it can be shown that for each $k > 1$, $t_k - t_{k-1} = t_1$, $|z(t_k)| \geqslant 1.5|z(t_{k-1})|$. Furthermore, with
$r_0 := \min_{0\leqslant t\leqslant t_1} |\mathrm{e}^{-A_1 t}|^{-1}$ $(=\min_{t_1\leqslant t\leqslant t_2} |\mathrm{e}^{-A_2(t-t_1)}|^{-1})$,
one has $|z(t)| \geqslant r_0|z(t_{k-1})|$ for all $t \in [t_{k-1}, t_k]$.

We now consider the trajectory $x(t)$ of (12) starting at $(0, 1)$. By the uniqueness property of the trajectories, one has $x(t) = z(\varphi(t))$, where $\varphi(\cdot)$ is the solution of

$$\dot\varphi = 1 + |z(\varphi)|^2, \quad \varphi(0) = 0.$$

Let $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ be the crossing times of the trajectory $x(\cdot)$ with the coordinate axes. Then $t_k = \varphi(\tau_k)$.

Since for $t \in [\tau_{k-1}, \tau_k]$, $\dot\varphi(t) = (1 + |z(\varphi(t))|^2) \geqslant 1 + r_0^2|z(t_{k-1})|^2 \geqslant r_0^2 \cdot 1.5^{2(k-1)}$,
it follows that

$$\tau_k - \tau_{k-1} \leqslant \frac{t_k - t_{k-1}}{1.5^{2(k-1)} r_0^2} = 1.5^{-2k}\hat\tau,$$

where $\hat\tau = 1.5^2 t_1/r_0^2$. Let $T_k = \tau_1 + \tau_2 + \cdots + \tau_k$. Then $T_k \to \hat T$ for some $\hat T < \infty$. As $|x(\tau_k)| \geqslant 1.5^k$, it follows that $limsup_{t\to\hat T}|x(t)| = \infty$. This shows that the system (12) is not forward complete.

Remark 3.6. Note that system (14) also serves as an example to show that a system that switches between two globally asymptotically stable linear systems may fail to be asymptotically stable for some choices of the switching signals in $\mathscr{S}_{\mathrm{pc}}(D)$.

3.2. Proof of Theorem 3

The following lemma is needed in the proof of Theorem 3.

Lemma 3.7. *Let G be as in Theorem* 3 *and* $x : [0, T] \to \mathbb{R}^n$ *be a solution of* (8) *corresponding to* $d \in \mathscr{M}(D)$. *Let* $\{d_k,\ k \in \mathbb{N}\} \subset \mathscr{M}(D)$ *be a locally equibounded sequence such that* $\lim_{k\to\infty} d_k(t) = d(t)$ *a.e. on* $[0, +\infty)$. *Then, there exists a sequence* $\{x_k\}_{k\geqslant 1}$, *with* x_k *a maximal trajectory of* (8) *corresponding to* d_k *that verifies* $x_k(0) = x(0)$, *such that* x_k *is defined on* $[0, T]$ *for k large enough and in addition* $\{x_k\}$ *converges uniformly to x on* $[0, T]$.

Proof. Let $K \subset \mathbb{R}^n$ be a compact set such that $x(t) \in K$ for all $t \in [0, T]$. Pick a smooth function $\theta : \mathbb{R}^n \to \mathbb{R}$ such that

- $0 \leqslant \theta(\xi) \leqslant 1$ for all $\xi \in \mathbb{R}^n$;
- $\theta(\xi) = 1$ for all $\xi \in K + \mathscr{B}_n$;
- $\theta(\xi) = 0$, for all $\xi \in \mathbb{R}^n \setminus (K + 2\mathscr{B}_n)$

(where we have used $K + r\mathscr{B}_n$ to denote the set $\{\eta \in \mathbb{R}^n : d(\eta, K) \leqslant r\}$), and define the set-valued map $G^* : [0, T] \times \mathbb{R}^n \times D \to \mathscr{K}(\mathbb{R}^n)$ by

$$G^*(t, \xi, v) := \theta(\xi) G(t, \xi, v).$$

It easily follows that G^* verifies

(a) $G^*(\cdot, \xi, v)$ is measurable for each $\xi \in \mathbb{R}^n$ and each $v \in D$;
(b) $G^*(t, \cdot, \cdot)$ is continuous for each $t \in [0, T]$;
(c) for each compact subset $\Delta \subset D$ there exists $\beta_\Delta \in L^1([0, T], \mathbb{R})$ such that for all $\xi, \xi' \in \mathbb{R}^n$ and all $v \in \Delta$,

$$d_H(G^*(t, \xi, v), G^*(t, \xi', v)) \leqslant \beta_\Delta(t) |\xi - \xi'|, \quad \forall\, t \in [0, T];$$

(d) for each compact subset $\Delta \subset D$ there exists $\alpha_\Delta \in L^1([0, T], \mathbb{R})$ such that for all $\xi \in \mathbb{R}^n$ and all $v \in \Delta$,

$$\|G^*(t, \xi, v)\| \leqslant \alpha_\Delta(t), \quad \forall t \in [0, T]. \tag{15}$$

Then, if we consider for each $k \in \mathbb{N}$, $\hat{G}_k : [0, T] \times \mathbb{R}^n \to \mathscr{K}(\mathbb{R}^n)$ defined by

$$\hat{G}_k(t, \xi) := G^*(t, \xi, d_k(t)),$$

we have that $\hat{G}_k$ is measurable in t. In addition, from the local equiboundedness of $\{d_k\}$ and (c), it follows that for all $\xi, \xi' \in \mathbb{R}^n$,

$$d_H(\hat{G}_k(t, \xi), \hat{G}_k(t, \xi')) \leqslant \kappa(t) |\xi - \xi'|, \quad \forall t \in [0, T],$$

for a suitable $\kappa \in L^1([0, T], \mathbb{R})$.

Let $\rho_k(t) = \operatorname{dist}(\dot{x}(t), \hat{G}_k(t, x(t)))$. We have, for almost all $t \in [0, T]$,

$$\begin{aligned} \rho_k(t) &= \operatorname{dist}(\dot{x}(t), \hat{G}_k(t, x(t))) \\ &\leqslant \operatorname{dist}(\dot{x}(t), G(t, x(t), d(t))) + d_H(G(t, x(t), d(t)), \hat{G}_k(t, x(t))) \\ &= d_H(G^*(t, x(t), d(t)), G^*(t, x(t), d_k(t))). \end{aligned} \tag{16}$$

Thus, due to the continuity of G^* with respect to its third argument,

$$\lim_{k \to \infty} d_H(G^*(t, x(t), d(t)), G^*(t, x(t), d_k(t))) = 0, \quad \text{a.e. } t \in [0, T].$$

Consequently, $\lim_{k \to +\infty} \rho_k(t) = 0$ for almost all $t \in [0, T]$.

On the other hand, from (16) it follows that for almost all $t \in [0, T]$, $\rho_k(t) \leqslant \|G^*(t, x(t), d(t))\| + \|G^*(t, x(t), d_k(t))\|$. Taking into account the equiboundedness of $\{d_k\}$ and (15), we conclude that there exists $\alpha \in L^1([0, T], \mathbb{R})$ such that for each $k \in \mathbb{N}$, $\rho_k(t) \leqslant \alpha(t)$ for almost all $t \in [0, T]$.

From Lemma 8.3 of [5] we have that for each $k \in \mathbb{N}$ there exists a solution x_k of the initial value problem $\dot{x}_k \in \hat{G}_k(t, x_k),\ x_k(0) = x(0)$ such that

$$|x(t) - x_k(t)| \leqslant \varphi_k(t), \quad \forall t \in [0, T],$$

where φ_k is the solution of the initial value problem

$$\dot{\varphi}_k(t) = \kappa(t)\varphi_k(t) + \rho_k(t), \quad \varphi_k(0) = 0.$$

As $\rho_k(t) \to 0$ almost everywhere and $\{\rho_k\}$ is majorized by an integrable function, $\varphi_k \to 0$ uniformly on $[0, T]$. Consequently, $\{x_k\}$ converges uniformly to x on $[0, T]$.

The proof concludes by noticing that x_k is a solution of (8) corresponding to d_k for k large enough. In fact, there exists $N \in \mathbb{N}$ such that for all $k \geqslant N$, $|x_k(t) - x(t)| \leqslant 1$ for all $t \in [0, T]$. Thus, if $k \geqslant N$, for almost all $t \in [0, T]$ we have that

$$\dot{x}_k(t) \in \hat{G}_k(t, x_k(t)) = \theta(x_k(t))G(t, x_k(t), d_k(t)) = G(t, x_k(t), d_k(t)),$$

and hence x_k is a solution of (8) corresponding to d_k. □

3.2.1. Proof of part 1 of Theorem 3

From the Filippov–Wažewski Relaxation Theorem (cf. [5]) we can easily deduce the existence of a trajectory x of (8) corresponding to d^* so that $x(0) = z(0)$ and

$$|z(t) - x(t)| < \frac{\varepsilon}{2}, \quad \forall t \in [0, T].$$

On the other hand, due to the density of D' in D, there exists a locally equibounded sequence $\{d_k\}$ in $\mathscr{S}_{\mathrm{pc}}(D')$ such that $d_k \to d^*$ almost everywhere on $[0, +\infty)$. (First of all, since D is separable, by Remark C.1.2 of [21], there exists a locally equibounded sequence $\{\hat{d}_k\}$ in $\mathscr{S}_{\mathrm{pc}}(D)$ so that $\hat{d}_k \to d^*$ almost everywhere on $[0, +\infty)$. Next, for each $k \in \mathbb{N}$, we choose a piecewise constant switching signal $d_k \in \mathscr{S}_{\mathrm{pc}}(D')$ such that the distance from $d_k(t)$ to $\hat{d}_k(t)$ is less than $1/k$ for all $t \in \mathbb{R}_{\geqslant 0}$. The equiboundedness of $\{d_k, k \in \mathbb{N}\}$ can be proved by using arguments similar to those of Remark C.1.3 of [21]). By Lemma 3.7, there exists a sequence $\{x_k\}$, with x_k a maximal trajectory of (8) corresponding to d_k that verifies $x_k(0) = x(0)$, such that x_k is also defined on $[0, T]$ for k large enough and in addition $\{x_k\}$ converges uniformly to x on $[0, T]$. In consequence, for some k^* we have that

$$|x(t) - x_{k^*}(t)| < \frac{\varepsilon}{2}, \quad \forall t \in [0, T],$$

and, a posteriori, that

$$|z(t) - x_{k^*}(t)| < \varepsilon, \quad \forall t \in [0, T].$$

The proof of part 1 is completed by taking $d = d_{k^*}$. □

3.2.2. Proof of part 2 of Theorem 3

The proof that we give here is similar to the proofs of Lemma III 2 in [22] and its generalization in Theorem 1 in [9] and it uses similar techniques to those used in these papers.

Let $\{T_k\}_{k=0}^{\infty}$ be a strictly increasing sequence of times such that $T_0 = 0$ and $\lim_{k\to+\infty} T_k = T$.

We claim that there exist a sequence $\{d_k\}_{k=1}^{\infty} \subset \mathscr{S}_{\rm pc}(D')$, a sequence $\{\delta_k\}_{k=0}^{\infty}$ of positive real numbers and, for each nonnegative integer k, a sequence of points $\{\xi_j^k\}_{j=0}^{\infty}$ which satisfy the following:

(i) For each $k \geqslant 0$, $0 < \delta_k \leqslant \min\{r(t) : t \in [T_k, T_{k+1}]\}$;
(ii) for each $k \geqslant 0$, $\xi_j^k \in V_k := z(T_k) + \delta_k \mathscr{B}_n$ for all $j \geqslant 0$, where for $\xi \in \mathbb{R}^n$ and $\delta > 0$, we use $\xi + \delta\mathscr{B}_n$ to denote the closed ball centered at ξ with radius δ;
(iii) for any $k \geqslant 1$, if a subsequence $\{\xi_{j_l}^k\}_{l=1}^{\infty}$ converges, say to ξ^k, then the subsequence $\{\xi_{j_l}^{k-1}\}_{l=1}^{\infty}$ also converges, say to ξ^{k-1}, and there is a solution $\overline{x}_k : [0, T_k - T_{k-1}] \to \mathbb{R}^n$ of the initial value problem

$$\dot{x} \in G(T_{k-1} + t, x, d_k), \quad x(0) = \xi^{k-1}, \tag{17}$$

that verifies $\overline{x}_k(T_k - T_{k-1}) = \xi^k$ and

$$|\overline{x}_k(t) - z(T_{k-1} + t)| < r(T_{k-1} + t), \quad \forall t \in [0, T_k - T_{k-1}]. \tag{18}$$

Proof of the Claim. Let $\{r_k\}_{k=1}^{\infty}$ be the sequence of positive numbers defined by

$$r_k = \min\{r(t) : t \in [T_{k-1}, T_k]\}.$$

First, we will construct by induction a sequence of positive numbers $\{\delta_k\}_{k=0}^{\infty}$, a sequence of sets $\{V_k\}_{k=0}^{\infty}$, with $V_k = z(T_k) + \delta_k\mathscr{B}_n$, and a sequence of piecewise switching signals $\{d_k\}_{k=1}^{\infty} \subset \mathscr{S}_{\rm pc}(D')$ such that the following hold for each $k \geqslant 1$:

- $\delta_{k-1} \leqslant r_k$;
- there exists a continuous function $x_k : [0, T_k - T_{k-1}] \times V_k \to \mathbb{R}^n$ which satisfies the following:

  (a) for each $\eta \in V_k$, $x_k(\cdot, \eta)$ is a solution of

  $$\dot{x} \in G_k(t, x), \tag{19}$$

  with $G_k : [0, T_k - T_{k-1}] \times \mathbb{R}^n \to \mathscr{K}(\mathbb{R}^n)$ defined by $G_k(t, \xi) = -G(T_k - t, \xi, d_k(T_k - T_{k-1} - t))$, that verifies $x_k(0, \eta) = \eta$, and

  $$|z(T_k - t) - x_k(t, \eta)| \leqslant r_k, \quad \forall t \in [0, T_k - T_{k-1}];$$

  (b) $x_k(T_k - T_{k-1}, V_k) \subseteq V_{k-1}$.

We start the induction procedure by setting $\delta_0 = r_1$ and $V_0 = z(T_0) + \delta_0\mathscr{B}_n$. Assuming that we have already constructed δ_{k-1} for some positive k, we will construct δ_k, d_k, and x_k.

First note that if we consider the forced differential inclusion

$$\dot{x} \in \hat{G}_k(t, x, d), \tag{20}$$

with $\hat{G}_k : [0, T_k - T_{k-1}] \times \mathbb{R}^n \times D \to \mathscr{K}(\mathbb{R}^n)$ defined by $\hat{G}_k(t, \xi, v) = -G(T_k - t, \xi, v)$ and its relaxation

$$\dot{x} \in \operatorname{co} \hat{G}_k(t, x, d), \tag{21}$$

we have that $z_k : [0, T_k - T_{k-1}] \to \mathbb{R}^n$ defined by $z_k(t) = z(T_k - t)$ is a solution of (21) corresponding to the input $d_k^*(t) = d^*(T_k - t)$.

Note that $\hat{G}_k$ verifies (H_1)–(H_4) (with $[0, T_k - T_{k-1}]$ instead of $\mathbb{R}_{\geqslant 0}$). Then, from the first part of Theorem 3 (which holds if we replace $\mathbb{R}_{\geqslant 0}$ by $[0, T_k - T_{k-1}]$), there exist a piecewise constant switching signal $\hat{d}_k \in \mathscr{S}_{\mathrm{pc}}(D')$ and a trajectory x_k^* of (20) corresponding to $\hat{d}_k$ that verifies $x_k^*(0) = z_k(0) = z(T_k)$ and

$$|x_k^*(t) - z_k(t)| < \frac{\varepsilon_k}{2}, \quad \forall t \in [0, T_k - T_{k-1}],$$

where $\varepsilon_k := \min\{\delta_{k-1}, r_{k+1}\}$.

Let $d_k \in \mathscr{S}_{\mathrm{pc}}(D')$ be the piecewise constant switching signal defined by $d_k(t) = \hat{d}_k(T_k - T_{k-1} - t)$ for $t \in [0, T_k - T_{k-1}]$ and $d_k(t) = \hat{d}_k(0)$ for $t \geqslant T_k - T_{k-1}$. Then x_k^* is a solution of (19) which corresponds to d_k and verifies $x_k^*(0) = z(T_k)$. By applying Lemma 3.1 of [9] to system (19) (with x_k^* and $\varepsilon_k/2$ instead of z and ε respectively), it follows the existence of a $0 < \delta_k < \varepsilon_k$ and a continuous function $x_k : [0, T_k - T_{k-1}] \times V_k \to \mathbb{R}^n$ such that for each $\eta \in V_k$, $x_k(\cdot, \eta)$ is a solution of (19) with $x_k(0, \eta) = \eta$ and

$$|x_k^*(t) - x_k(t, \eta)| \leqslant \frac{\varepsilon_k}{2}, \quad \forall t \in [0, T_k - T_{k-1}].$$

In consequence, for every $\eta \in V_k$ and all $t \in [0, T_k - T_{k-1}]$,

$$\begin{aligned} |z(T_k - t) - x_k(t, \eta)| &= |z_k(t) - x_k(t, \eta)| \\ &\leqslant |z_k(t) - x_k^*(t)| + |x_k^*(t) - x_k(t, \eta)| \\ &< \frac{\varepsilon_k}{2} + \frac{\varepsilon_k}{2} = \varepsilon_k \leqslant r_{k+1}. \end{aligned}$$

In particular, taking $t = T_k - T_{k-1}$, we have that

$$|z(T_{k-1}) - x_k(T_k - T_{k-1}, \eta)| < \varepsilon_k \leqslant \delta_{k-1}.$$

In other words, $\{x_k(T_k - T_{k-1}, \eta) : \eta \in V_k\} \subseteq V_{k-1}$.

Next, we construct a sequence of points $\{\xi_j^k \in \mathbb{R}^n : k \geqslant 0, j \geqslant 0\}$ as follows. We set, for $j \geqslant 0$ and $k \geqslant j$,

$$\xi_j^k = z(T_k).$$

For $j \geqslant 0$ and $0 \leqslant k < j$, we obtain ξ_j^k by the recursive formula

$$\xi_j^k = x_{k+1}(T_{k+1} - T_k, \xi_j^{k+1}).$$

By construction, each $\xi_j^k \in V_k$.

We note that $\{\delta_k\}_{k=0}^{\infty}$ and $\{\xi_j^k : k \geqslant 0,\ j \geqslant 0\}$ verify (i) and (ii). It remains to verify that this construction satisfies (iii). Suppose that for some $k \geqslant 1$, $\{\xi_{j_l}^k\}_{l=1}^{\infty}$ converges to ξ^k. Taking into account that, by definition,

$$\xi_j^{k-1} = x_k(T_k - T_{k-1}, \xi_j^k), \quad \forall k \leqslant j$$

and that x_k is continuous, it follows that

$$\lim_{l\to\infty} \xi_{j_l}^{k-1} = x_k(T_k - T_{k-1}, \xi^k) := \xi^{k-1}.$$

Now, consider $\overline{x}_k : [0, T_k - T_{k-1}] \to \mathbb{R}^n$ defined by $\overline{x}_k(t) = x_k(T_k - T_{k-1} - t, \xi^k)$. Note that $\overline{x}_k$ is a solution of the initial value problem (17) which verifies $\overline{x}_k(T_k - T_{k-1}) = \xi^k$ and (18). The claim is thus proved.

We are now ready to show the existence of a switching signal $d \in \mathscr{S}(D')$ and a trajectory x of (8) corresponding to d that verify (11).

Pick any $c \in D'$ and define $d : [0, +\infty) \to D'$ by $d(t) = d_k(t - T_{k-1})$ for all $t \in [T_{k-1}, T_k)$ and $d(t) = c$ for all $t \geqslant T$. Clearly $d \in \mathscr{S}(D')$ and, in the case when $T = +\infty$, $d \in \mathscr{S}_{\mathrm{pc}}(\Gamma)$.

Since $\{\xi_j^k\}_{j=0}^{\infty} \subset V_k$ and V_k is compact for each $k \geqslant 0$, by using a diagonalization process, we deduce the existence of a subsequence $\{j_l\}_{l=1}^{\infty}$ of the sequence of the nonnegative integers such that $\{\xi_{j_l}^k\}$ converges, say to ξ^k, for each $k \geqslant 0$.

Then, from statement (iii) in the previous claim, there exists a solution $\overline{x}_k : [0, T_k - T_{k-1}] \to \mathbb{R}^n$ of (17) that verifies $\overline{x}_k(0) = \xi^{k-1}$, $\overline{x}_k(T_k - T_{k-1}) = \xi^k$ and that satisfies (18). Therefore, the function $x : [0, T) \to \mathbb{R}^n$ given by

$$x(t) = \overline{x}_k(t - T_{k-1}) \quad \text{when } t \in [T_{k-1}, T_k),$$

is a trajectory of (8) corresponding to d. In addition, it follows from (18) that

$$|x(t) - z(t)| < r(t), \quad \forall t \in [0, T). \quad \square$$

4. Proofs of Theorems 1 and 2

In this section we give the proofs of the main results Theorems 1 and 2. For this purpose, we first study the existence of parametrizations for set-valued maps that take nonempty convex compact values.

4.1. Parametrizations of set-valued maps

The following parametrization result for set-valued maps, whose proof is based on a construction developed in [20] for globally Lipschitz set-valued maps (see also [2]), asserts that under suitable conditions a set-valued map F^* admits a parametrization which is as regular as F^*.

Theorem 5. *Consider two metric spaces E, D and a continuous set-valued map $F^* : E \times D \to \mathscr{K}(\mathbb{R}^n)$ which takes convex values. Assume that for each compact subset $K \subseteq D$, $F^*(\cdot, v)$ is locally Lipschitz uniformly in $v \in K$. Then there exists a function $f : E \times D \times \mathscr{B}_n \to \mathbb{R}^n$ such that*

1. *f is continuous;*
2. *for each compact subset $K \subseteq D$, $f(\cdot, v, \cdot)$ is locally Lipschitz uniformly in $v \in K$;*
3. *$f(\xi, v, \mathscr{B}_n) = F^*(\xi, v)\ \forall e \in E, \forall v \in D$.*

Proof. Let $\mathscr{K}_c(\mathbb{R}^n)$ denote the family of all nonempty compact convex subsets of $\mathbb{R}^n$. For $K \in \mathscr{K}_c(\mathbb{R}^n)$, let $s_n(K)$ be the Steiner point of K. It follows from Theorem 9.4.1 of [2] that $s_n(K) \in K$ for all $K \in \mathscr{K}_c(\mathbb{R}^n)$ and that

$$|s_n(K) - s_n(L)| \leqslant n d_H(K, L), \quad \forall K, L \in \mathscr{K}_c(\mathbb{R}^n). \tag{22}$$

Let $P : \mathbb{R}^n \times \mathscr{K}_c(\mathbb{R}^n) \to \mathscr{K}_c(\mathbb{R}^n)$ be the map defined by

$$P(y, K) = K \cap (y + 2\,\mathrm{dist}\,(y, K)\mathscr{B}_n).$$

(It can be seen indeed that $P(y, K) \in \mathscr{K}_c(\mathbb{R}^n)$ for any $y \in E$ and any $K \in \mathscr{K}_c(\mathbb{R}^n)$.) Observe that $P(y, K) = \{y\}$ if and only if $y \in K$.

From Lemma 9.4.2 of [2] (see also Lemma 1 of [20]), we have that P is a Lipschitz map, more precisely,

$$\begin{aligned} &d_H(P(y, K), P(x, L)) \\ &\quad \leqslant 5(d_H(K, L) + |y - x|), \quad \forall x, y \in \mathbb{R}^n,\ \forall K, L \in \mathscr{K}_c(\mathbb{R}^n). \end{aligned} \tag{23}$$

Let $f : E \times D \times \mathscr{B}_n \to \mathbb{R}^n$ be defined by

$$f(\xi, v, y) = s_n(P(\|F^*(\xi, v)\| y, F^*(\xi, v))).$$

Proof of 3. First note that $P(\|F^*(\xi, v)\| y, F^*(\xi, v)) \subseteq F^*(\xi, v)$ for all $\xi \in E$ and $v \in D$. Since $s_n(K) \in K$ for all $K \in \mathscr{K}_c(\mathbb{R}^n)$, it follows that $f(\xi, v, y) \in F^*(\xi, v)$ for all $\xi \in E$, all $v \in D$ and all $y \in \mathscr{B}_n$. This implies that $f(\xi, v, \mathscr{B}_n) \subseteq F^*(\xi, v)$ for all $\xi \in E$ and all $v \in D$.

On the other hand, for each $\zeta \in F^*(\xi, v)$ there exists $y_0 \in \mathscr{B}_n$ such that $\zeta = \|F^*(\xi, v)\| y_0$. In consequence,

$$f(\xi, v, y_0) = s_n(P(\|F^*(\xi, v)\| y_0, F^*(\xi, v))) = s_n(P(\zeta, F^*(\xi, v))) = s_n(\{\zeta\}) = \zeta.$$

Thus $F^*(\xi, v) \subseteq f(\xi, v, \mathscr{B}_n)$ and statement 3 follows.

Proof of 1 and 2. From (22) and (23) we easily deduce that

$$\begin{aligned}&|f(\xi,v,y)-f(\xi',v',y')|\\&\quad=|s_n(P(\|F^*(\xi,v)\|y,F^*(\xi,v)))-s_n(P(\|F^*(\xi',v')\|y',\ F^*(\xi',v')))|\\&\quad\leqslant n d_H(P(\|F^*(\xi,v)\|y,F^*(\xi,v)),P(\|F^*(\xi',v')\|y',F^*(\xi',v')))\\&\quad\leqslant 5n[d_H(F^*(\xi,v),F^*(\xi',v'))+|\|F^*(\xi,v)\|-\|F^*(\xi',v')\||\cdot|y|\\&\qquad+\|F^*(\xi,v)\|\cdot|y-y'|].\end{aligned}$$

As $F^*(\xi,v)\subseteq F^*(\xi',v')+d_H(F^*(\xi,v),F^*(\xi',v'))\mathscr{B}_n$, it follows that

$$\|F^*(\xi,v)\|-\|F^*(\xi',v')\|\leqslant d_H(F^*(\xi,v),F^*(\xi',v')).$$

Then, by symmetry,

$$|\|F^*(\xi,v)\|-\|F^*(\xi',v')\||\leqslant d_H(F^*(\xi,v),F^*(\xi',v')).$$

In consequence, we have that for all $\xi,\xi'\in E$, all $v,v'\in D$ and all $y,y'\in\mathscr{B}_n$,

$$\begin{aligned}|f(\xi,v,y)-f(\xi',v',y')|\leqslant{}&5n[(1+|y|)d_H(F^*(\xi,v),F^*(\xi',v'))\\&+\|F^*(\xi,v)\|\cdot|y-y'|].\end{aligned}\tag{24}$$

Now, from (24), it easily follows that f is continuous and that $f(\cdot,v,\cdot)$ is locally Lipschitz uniformly with respect to $v\in K$ when K is a compact subset of D. □

Now, we are ready to prove Theorem 1 and afterwards Theorem 2.

4.2. Proof of Theorem 1

Consider a compact metric space D, a set-valued map $F:\mathbb{R}^n\times\mathbb{R}^m\times D\to\mathscr{K}(\mathbb{R}^n)$ and an injective function $\imath:\Gamma\to D$ as in Lemma 2.5. As $\imath(\Gamma)=D'$ is dense in D, part 1 of Theorem 1 holds.

Let $F^*:\mathbb{R}^n\times\mathbb{R}^m\times D\to\mathscr{K}(\mathbb{R}^n)$ be defined by

$$F^*(\xi,\mu,v)=\operatorname{co}F(\xi,\mu,v).$$

Since F is compact valued, F^* takes values in $\mathscr{K}_c(\mathbb{R}^n)$. Also observe that F^* has the same regularity as F, that is, F^* is continuous and locally Lipschitz in (ξ,μ) uniformly on $v\in D$. According to Theorem 5 (applied with $\xi=(\xi,\mu)$), there exists a continuous function $f:\mathbb{R}^n\times\mathbb{R}^m\times D\times\mathscr{B}_n\to\mathbb{R}^n$ so that $f(\cdot,\cdot,v,\cdot)$ is locally Lipschitz uniformly with respect to $v\in D$ and for all $\xi\in\mathbb{R}^n$, all $\mu\in\mathbb{R}^m$ and all $v\in D$ it holds that

$$f(\xi,\mu,v,\mathscr{B}_n)=F^*(\xi,\mu,v).\tag{25}$$

As for each $\xi\in\mathbb{R}^n$, $\mu\in\mathbb{R}^m$ and $\gamma\in\Gamma$, $F_\gamma(\xi,\mu)=F(\xi,\mu,\imath(\gamma))$, we have for all $\xi\in\mathbb{R}^n$, all $\mu\in\mathbb{R}^m$ and all $\gamma\in\Gamma$ that

$$\operatorname{co}F_\gamma(\xi,\mu)=\operatorname{co}F(\xi,\mu,\imath(\gamma))=F^*(\xi,\mu,\imath(\gamma))=f(\xi,\mu,\imath(\gamma),\mathscr{B}_n).$$

Parts 2 and 3 of the Theorem 1 are thus proven.

Before we prove the remaining statements of the theorem, it is convenient to note that, from (25) and Filippov's Lemma, it easily follows that the set of maximal trajectories of the relaxation of (7)

$$\dot{x} \in F^*(x, u, d) \tag{26}$$

coincides with the set of maximal trajectories of (5). More precisely, given $u \in \mathscr{U}$ and $d \in \mathscr{M}(D)$, we have that x is a maximal trajectory of (26) corresponding to u and d if and only if there exists $\omega \in \mathscr{M}(\mathscr{B}_n)$ such that x is a maximal trajectory of (5) corresponding to u, d and ω.

In order to prove part 4, let x be a maximal trajectory of (4) corresponding to $u \in \mathscr{U}$ and $\sigma \in \mathscr{S}(\Gamma)$. Then, according to Remark 2.6, x is a maximal trajectory of (7) and therefore of (26), corresponding to u and $d_\sigma = \imath \circ \sigma$ and hence, as was said above, there exists $\omega \in \mathscr{M}(\mathscr{B}_n)$ such that x is a maximal trajectory of (5) corresponding to u, d_σ and ω.

As for part 5(a), let $u \in \mathscr{U}$, $d \in \mathscr{M}(D)$, $\omega \in \mathscr{M}(\mathscr{B}_n)$ and let $z : [0, T] \to \mathbb{R}^n$ be a trajectory of (5) corresponding to u, d and ω. Pick $\varepsilon > 0$ and consider the set valued map $G : \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \times D \to \mathscr{K}(\mathbb{R}^n)$ defined by $G(t, \xi, v) = F(\xi, u(t), v)$. Observe that G satisfies (H_1)–(H_4).

As z is a trajectory of (26) corresponding to u and d, z is a trajectory of (9) corresponding to d. From Theorem 3 part 1, there exist a piecewise constant switching signal $d' \in \mathscr{S}_{\mathrm{pc}}(D')$ and a maximal trajectory x of (8) corresponding to d' such that $x(0) = z(0)$ and

$$|z(t) - x(t)| < \varepsilon, \quad \forall t \in [0, T].$$

On the other hand, from the definition of G, we have that x is a maximal trajectory of (7) corresponding to u and d'. The proof of this part concludes by noticing that, from Remark 2.6, we know that x is a maximal trajectory of (4) corresponding to u and a certain piecewise constant switching signal $\sigma_{d'} \in \mathscr{S}_{\mathrm{pc}}(\Gamma)$.

The proof of part 5(b) is the same as the proof of part 5(a) if we replace the interval $[0, T]$ and ε by the interval $[0, T)$ and $r(t)$ respectively, and use part 2 of Theorem 3 instead of part 1 of that theorem. □

4.3. *Proof of Theorem 2*

Let D, $\imath : \Gamma \to D$, and $f : \mathbb{R}^n \times \mathbb{R}^m \times D \times \mathscr{B}_n \to \mathbb{R}^n$ be as in Theorem 1 and define $\hat{F} : \mathbb{R}^n \times \mathbb{R}^m \to \mathscr{K}(\mathbb{R}^n)$ by

$$\hat{F}(\xi, \mu) = f(\xi, \mu, D, \mathscr{B}_n) = \{f(\xi, \mu, v, \zeta) : v \in D, \zeta \in \mathscr{B}_n\}.$$

We will consider the following differential inclusions with u as input:

$$\dot{x} \in \hat{F}(x, u) \tag{27}$$

and

$$\dot{x} \in \operatorname{co} \hat{F}(x, u). \tag{28}$$

Since *f* satisfies statement 2 of Theorem 1, it follows that $\hat{F}$, and hence the convex hull co $\hat{F}$ of $\hat{F}$, are both locally Lipschitz. Therefore, from Theorem 5, there exists a locally Lipschitz function $f^* : \mathbb{R}^n \times \mathbb{R}^m \times \mathscr{B}_n \to \mathbb{R}^n$, so that for all $\xi \in \mathbb{R}^n$ and all $\mu \in \mathbb{R}^m$ it holds that

$$\mathrm{co}\,\hat{F}(\xi,\mu) = f^*(\xi,\mu,\mathscr{B}_n) = \{f^*(\xi,\mu,\zeta) : \ \zeta \in \mathscr{B}_n\}. \tag{29}$$

We claim that f^* satisfies part 1 of Theorem 2, or equivalently, for all $\xi \in \mathbb{R}^n$ and all $\mu \in \mathbb{R}^m$,

$$\mathrm{co}\,\mathrm{cl}\left(\bigcup_{\gamma\in\Gamma} F_\gamma(\xi,\mu)\right) = \mathrm{co}\,\hat{F}(\xi,\mu). \tag{30}$$

Let $\xi \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$. By part 3 of Theorem 1,

$$\bigcup_{\gamma\in\Gamma} \mathrm{co}\,F_\gamma(\xi,\mu) = \{f(\xi,\mu,v,\eta) : v \in D', \eta \in \mathscr{B}_n\},$$

where $D' = \imath(\Gamma)$ is dense in D. By the continuity of *f* and the compactness of D and $\mathscr{B}_n$, one has

$$\mathrm{cl}\left(\bigcup_{\gamma\in\Gamma} \mathrm{co}\,F_\gamma(\xi,\mu)\right) = \{f(\xi,\mu,v,\eta) : v \in D,\ \ \eta \in \mathscr{B}_n\} = \hat{F}(\xi,\mu).$$

Therefore,

$$\mathrm{co}\,\mathrm{cl}\left(\bigcup_{\gamma\in\Gamma} \mathrm{co}\,F_\gamma(\xi,\mu)\right) = \mathrm{co}\,\hat{F}(\xi,\mu).$$

Taking into account that

$$\begin{aligned}\mathrm{co}\,\mathrm{cl}\left(\bigcup_{\gamma\in\Gamma} \mathrm{co}\,F_\gamma(\xi,\mu)\right) &= \mathrm{cl}\,\mathrm{co}\left(\bigcup_{\gamma\in\Gamma} \mathrm{co}\,F_\gamma(\xi,\mu)\right) = \mathrm{cl}\,\mathrm{co}\left(\bigcup_{\gamma\in\Gamma} F_\gamma(\xi,\mu)\right)\\ &= \mathrm{co}\,\mathrm{cl}\left(\bigcup_{\gamma\in\Gamma} F_\gamma(\xi,\mu)\right),\end{aligned}$$

we get (30). Part 1 of the theorem thus follows.

Part 2 is a straightforward consequence of part 1 and Filippov's Lemma. Let *x* be a maximal trajectory of (4) corresponding to $u \in \mathscr{U}$ and to certain $\sigma \in \mathscr{S}(\Gamma)$. Then, from (30), *x* is a maximal trajectory of (28) corresponding to *u*. From this fact, (29) and Filippov's Lemma, there exists $\omega \in \mathscr{M}(\mathscr{B}_n)$ so that *x* is a maximal trajectory of (6).

Finally we prove part 3. We note first that for a fixed $u \in \mathscr{U}$ and due to Filippov's Lemma, the maximal trajectories of (5) and those of (27) coincide, and that the same holds for the maximal trajectories of systems (6) and (28). That is, *x* is a maximal trajectory of (27) (resp. (28)) corresponding to *u* if and only if *x* is maximal trajectory of (5) (resp. (6)) corresponding to u and certain $d \in \mathcal{M}(D)$ and $\omega \in \mathcal{M}(\mathcal{B}_n)$ (resp. $\omega \in \mathcal{M}(\mathcal{B}_n)$). Combining this fact with parts 5(a)–(b) of Theorem 1, it will be enough to prove the following statements:

(i) Given a trajectory $z : [0, T] \to \mathbb{R}^n$ of (28) corresponding to $u \in \mathcal{U}$ and $\varepsilon > 0$, there exists a trajectory x of (27) corresponding to u satisfying $x(0) = z(0)$ such that

$$|z(t) - x(t)| < \varepsilon, \quad \forall t \in [0, T];$$

(ii) given a trajectory $z : [0, T) \to \mathbb{R}^n$ of (28), with $T \leqslant +\infty$, corresponding to $u \in \mathcal{U}$ and a continuous function $r : [0, T) \to \mathbb{R}_{>0}$, there exists a trajectory x of (27) corresponding to u such that

$$|z(t) - x(t)| < r(t), \quad \forall t \in [0, T).$$

Part (i) readily follows from the Filippov–Beważewski Theorem (or Theorem 3 part 1; see Remark 3.2). Part (ii) can be easily deduced from Theorem 1 of [9] (or part 2 of Theorem 3).

5. Stability properties of switched systems

In this section, we consider several stability properties for switched systems with input and output as in the following:

$$\dot{x}(t) \in F_{\sigma(t)}(x(t), u(t)), \quad y = h(x(t)), \tag{31}$$

where x, u, σ and F_γ are still as in Section 2, the output map $h : \mathbb{R}^n \to \mathbb{R}^p$ is locally Lipschitz, and $h(0)=0$. The results on parametrization given in Section 2 allow the extension of several important previous results for systems given by differential equations too switched systems defined by differential inclusions. (Many of these extensions are novel even for the very special cases of switched systems of differential equations, or for non-switched differential inclusions.) In this section, we treat the notion of input–output-to-state stability and a few notions on input-to-output stability.

Throughout this section, we assume that the collection $\mathcal{P}$ (see (3)) satisfies assumptions C1 and C2.

5.1. Uniform input–output-to-state stability

The following definition is based on the work [10]:

Definition 5.1. Given a subclass $\mathcal{S}^*$ of $\mathcal{S}(\Gamma)$, we say that the system (31) is *uniformly input–output-to-state stable with respect to* $\mathcal{S}^*$ (uIOSS w.r.t. $\mathcal{S}^*$) if there exist $\beta \in \mathcal{KL}$, $\theta \in \mathcal{K}$ and $\gamma \in \mathcal{K}$ such that for all $\xi \in \mathbb{R}^n$, all $u \in \mathcal{U}$, all $\sigma \in \mathcal{S}^*$ and all $x \in \mathcal{T}^s(\xi, u, \sigma)$,

$$|x(t)| \leqslant \beta(|\xi|, t) + \theta(\|y\|_{[0,t]}) + \gamma(\|u\|), \quad \forall t \in [0, T_x). \tag{32}$$

Observe that it results in the same definition if one replaces $\|u\|$ in (32) by $\|u\|_{[0,t]}$.

When applying the uIOSS definition to systems with zero output map (i.e., $h \equiv 0$), one recovers the standard input-to-state stability notion, but extended to switched systems defined by differential inclusions:

Definition 5.2. The system (31) is *uniformly input-to-state stable with respect to* $\mathscr{S}^*$ (uISS w.r.t. $\mathscr{S}^*$) if there exist $\beta \in \mathscr{KL}$ and $\gamma \in \mathscr{K}$ such that for all $\xi \in \mathbb{R}^n$, all $u \in \mathscr{U}$, all $\sigma \in \mathscr{S}^*$ and all $x \in \mathscr{T}^s(\xi, u, \sigma)$,

$$|x(t)| \leqslant \beta(|\xi|, t) + \gamma(\|u\|), \quad \forall t \in [0, T_x). \tag{33}$$

For system (31), let $f^* : \mathbb{R}^n \times \mathbb{R}^m \times \mathscr{B}_n \to \mathbb{R}^n$ be as in Theorem 2, and consider the corresponding parametrization of (31):

$$\dot{x}(t) = f^*(x(t), u(t), \omega(t)), \quad y(t) = h(x(t)). \tag{34}$$

For such a system, the uIOSS property means that an estimate as in (32) holds for any maximal trajectory starting at ξ with any $u \in \mathscr{U}$ and any $\omega \in \mathscr{M}(\mathscr{B}_n)$ over the maximal interval (c.f. [10]).

Lemma 5.3. *The system* (31) *is uIOSS w.r.t.* $\mathscr{S}_{\mathrm{pc}}(\Gamma)$ *if and only if the corresponding system* (34) *is uIOSS.*

Proof. The statement that uIOSS property of (34) implies the uIOSS property of (31) w.r.t. $\mathscr{S}_{\mathrm{pc}}(\Gamma)$ follows immediately from statement 2 of Theorem 2.

Suppose system (31) is uIOSS w.r.t. $\mathscr{S}_{\mathrm{pc}}(\Gamma)$ with the decay estimate (32). Let $z : [0, T_z) \to \mathbb{R}^n$ be a solution of (34) with some $u \in \mathscr{U}$ and some $\omega \in \mathscr{M}(B_n)$. Pick any $\varepsilon > 0$ and $0 < T < T_z$. Let $r = \|z\|_{[0,T]}$, and let $\delta > 0$ be such that $|h(\xi) - h(\zeta)| < \varepsilon$ for all $\xi \in (r+1)\mathscr{B}_n$ and all $\zeta \in (r+1)\mathscr{B}_n$ such that $|\xi - \zeta| < \delta$. Without loss of generality, we assume that $\delta \leqslant \max\{1, \varepsilon\}$.

By 3(a) of Theorem 2, there exists some trajectory $x(\cdot)$ of (31) corresponding to u and some $\sigma \in \mathscr{S}_{\mathrm{pc}}(\Gamma)$ with $x(0) = z(0)$ such that

$$|z(t) - x(t)| < \delta, \quad \forall t \in [0, T]. \tag{35}$$

By the choice of δ, one has

$$|h(x(t)) - h(z(t))| < \varepsilon, \quad \forall t \in [0, T]. \tag{36}$$

With the uIOSS estimate (32) for (31), we get

$$|x(t)| \leqslant \beta(|x(0)|, t) + \theta(\|h(x)\|_{[0,t]}) + \gamma(\|u\|), \quad \forall t \in [0, T].$$

Since $z(0) = x(0)$, it follows from (35)–(36) that

$$|z(t)| \leqslant \beta(|z(0)|, t) + \theta(\|h(z)\|_{[0,t]} + \varepsilon) + \gamma(\|u\|) + \varepsilon, \quad \forall t \in [0, T].$$

Since $T < T_z$ and $\varepsilon > 0$ can be chosen arbitrarily, we get

$$|z(t)| \leqslant \beta(|z(0)|, t) + \theta(\|h(z)\|_{[0,t]}) + \gamma(\|u\|), \quad \forall t \in [0, T_z).$$

This shows that the system (34) is uIOSS. □

Applying the Lyapunov results on IOSS obtained in [10, Theorem 2.4] to system (34), we get the following:

Theorem 6. *Assume for system* (31) $F_\gamma(0,0)=\{0\}$ *for all* $\gamma \in \Gamma$. *Then the system is uIOSS w.r.t.* $\mathscr{S}_{\mathrm{pc}}(\Gamma)$ *if and only if there exists a smooth* ($\mathscr{C}^\infty$) *function* $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ *such that*

- *There exist* $\mathscr{K}_\infty$*-functions* α_1, α_2 *such that*

$$\alpha_1(|\xi|) \leqslant V(\xi) \leqslant \alpha_2(|\xi|), \quad \forall \xi \in \mathbb{R}^n; \tag{37}$$

- *there exist a* $\mathscr{K}_\infty$*-function* α *and* $\mathscr{K}$*-functions* σ_1, σ_2 *such that*

$$DV(\xi)v \leqslant -\alpha(|\xi|) + \sigma_1(|h(\xi)|) + \sigma_2(|\mu|), \quad \forall \xi \in \mathbb{R}^n,$$

  for all $\mu \in \mathbb{R}^m$, *all* $\gamma \in \Gamma$, *and all* $v \in F_\gamma(\xi,\mu)$.

A function V as in Theorem 6 is called a *common uIOSS-Lyapunov function* for (31).

Remark 5.4. In Theorem 6 we assume that $F_\gamma(0,0)=\{0\}$ for all $\gamma \in \Gamma$ because in [10] the authors consider by hypothesis systems as in (34) that verify $f^*(0,0,\zeta)=0$ for all $\zeta \in \mathscr{B}_n$.

Remark 5.5. To be precise, in order to apply [10, Theorem 2.4] in the above proof, one should assume that the output map h is $\mathscr{C}^1$. But one can relax this condition to only requiring h to be locally Lipschitz. To see this point, suppose h is a locally Lipschitz function. Find a smooth function $\hat{h} : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ so that for some $\mathscr{K}_\infty$-functions ρ_1, ρ_2,

$$\rho_1(|h(\xi)|) \leqslant \hat{h}(\xi) \leqslant \rho_2(|h(\xi)|), \quad \forall \xi \in \mathbb{R}^n.$$

(see Lemma B.1 in the appendix for details). Replacing the output map h in (31) with $\hat{h}$, one gets the following system with a smooth output map:

$$\dot{x}(t) \in F_{\sigma(t)}(x(t),u(t)), \quad y=\hat{h}(x(t)). \tag{38}$$

It can be seen that the system (31) is uIOSS if and only if the system (38) is uIOSS; and V is a uIOSS-Lyapunov function for (31) if and only if V is a uIOSS-Lyapunov function for (38). Hence, one can establish Theorem 6 for (31) by first showing the existence of uIOSS-Lyapunov functions for (38) with the $\mathscr{C}^1$ output map $y=\hat{h}(x)$.

Applying Theorem 6 to the special case with $h \equiv 0$, we get the following uISS-Lyapunov result, which can be viewed as an extension of the Lyapunov results obtained in [17] for switched systems defined by differential equations to switched systems defined by differential inclusions.

Theorem 7. *The system is uISS w.r.t.* $\mathscr{S}_{\mathrm{pc}}(\Gamma)$ *if and only if there exists a smooth* ($\mathscr{C}^\infty$) *function* $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ *such that*

- *for some* $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$, (37) *holds*; *and*

- *for some $\mathcal{K}_\infty$-function α and $\mathcal{K}$-functions σ the following holds*:

$$DV(\xi)v \leqslant -\alpha(|\xi|) + \sigma(|\mu|),$$

for all $\mu \in \mathbb{R}^m$, all $\gamma \in \Gamma$, and all $v \in F_\gamma(\xi, \mu)$.

5.2. Uniform input-to-output stability

Following the work in [23], in what follows we will introduce some notions on input-to-output stability properties for switched systems defined by differential inclusions, but first we need introduce some terminology. Given a subclass $\mathcal{S}^*$ of $\mathcal{S}(\Gamma)$, we say that a system as in (31) is forward complete with respect to $\mathcal{S}^*$ if every trajectory $x \in \mathcal{T}^s(\xi, u, \sigma)$ corresponding to any $\xi \in \mathbb{R}^n$, any $u \in \mathcal{U}$ and any $\sigma \in \mathcal{S}^*$ is defined for all $t \in [0, +\infty)$. When the system is forward complete w.r.t. $\mathcal{S}(\Gamma)$ we just say it is forward complete. We observe that the forward completeness property of (31) with $\sigma \in \mathcal{S}_{\rm pc}(\Gamma)$ does not guarantee the forward completeness property of (31) w.r.t. $\mathcal{S}(\Gamma)$ (c.f. Example 3.5).

Definition 5.6. Let $\mathcal{S}^*$ be a subclass of $\mathcal{S}(\Gamma)$. A forward complete system w.r.t. $\mathcal{S}^*$ as in (31) is

1. *Uniformly input to output stable with respect to $\mathcal{S}^*$* (uIOS w.r.t. $\mathcal{S}^*$) if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that for all $\xi \in \mathbb{R}^n$, all $u \in \mathcal{U}$, all $\sigma \in \mathcal{S}^*$ and all $x \in \mathcal{T}^s(\xi, u, \sigma)$,
$$|y(t)| \leqslant \beta(|\xi|, t) + \gamma(\|u\|), \quad \forall\, t \geqslant 0; \tag{39}$$
2. *uniformly output-Lagrange input to output stable with respect to $\mathcal{S}^*$* (uOLIOS w.r.t. $\mathcal{S}^*$) if it is uIOS w.r.t. $\mathcal{S}^*$ and there exist some $\mathcal{K}$-functions σ_1, σ_2 such that for all $\xi \in \mathbb{R}^n$, all $u \in \mathcal{U}$, all $\sigma \in \mathcal{S}^*$ and all $x \in \mathcal{T}^s(\xi, u, \sigma)$,
$$|y(t)| \leqslant \max\{\sigma_1(|h(\xi)|), \sigma_2(\|u\|)\}, \quad \forall t \geqslant 0; \tag{40}$$
3. *uniformly state-independent input-to-output stable with respect to $\mathcal{S}^*$* (uSIIOS w.r.t. $\mathcal{S}^*$) if there exist some $\beta \in \mathcal{KL}$ and some $\gamma \in \mathcal{K}$ such that for all $\xi \in \mathbb{R}^n$, all $u \in \mathcal{U}$, all $\sigma \in \mathcal{S}^*$ and all $x \in \mathcal{T}^s(\xi, u, \sigma)$,
$$|y(t)| \leqslant \beta(|h(\xi)|, t) + \gamma(\|u\|), \quad \forall t \geqslant 0. \tag{41}$$

Note that when h is the identity function, the properties uIOS, uOLIOS and uSIIOS coincide with the uISS property.

As in Section 5.1, we consider the parametrization (34) for system (31), where f^* is as in Theorem 2.

For a forward complete system as in (34), the uIOS (uOLIOS, uSIIOS, respectively) property means that (39) ((40), (41) respectively) holds for any trajectory starting at ξ with any $u \in \mathcal{U}$ and any $\omega \in \mathcal{M}(\mathcal{B}_n)$.

Following the same idea as in the proof of Lemma 5.3, and taking into account that system (34) is forward complete whenever (31) is forward complete (cf. Theorem 2 part 3(b)), one can prove the following uIOS analogue of Lemma 5.3.

Lemma 5.7. *Suppose the system* (31) *is forward complete. Then it is uIOS, uOLIOS, uSIIOS respectively w.r.t.* $\mathcal{S}_{\text{pc}}(\Gamma)$ *if and only if the corresponding system* (34) *is uIOS, uOLIOS, uSIIOS, respectively*.

To present the Lyapunov characterizations of the output stability properties, we need to introduce the following:

Definition 5.8. A system as in (31) is *uniformly bounded input bounded state stable with respect to* $\mathcal{S}^*$ (uBIBS w.r.t. $\mathcal{S}^*$) if there exist some nondecreasing functions σ_1 and σ_2 such that for all $\xi \in \mathbb{R}^n$, all $u \in \mathcal{U}$, all $\sigma \in \mathcal{S}^*$ and all $x \in \mathcal{T}^s(\xi, u, \sigma)$,

$$|x(t)| \leqslant \max\{\sigma_1(|\xi|),\ \sigma_2(\|u\|)\}, \quad \forall t \geqslant 0. \tag{42}$$

A system as in (34) is uBIBS if there exist some nondecreasing functions σ_1 and σ_2 such that (42) holds for any trajectory of (34) starting at ξ with any $u \in \mathcal{U}$ and any $\omega \in \mathcal{M}(\mathcal{B}_n)$.

An entirely analogous proof to that of Lemma 5.3 gives the following result:

Lemma 5.9. *The system* (31) *is uBIBS w.r.t.* $\mathcal{S}_{\text{pc}}(\Gamma)$ *if and only if the corresponding system* (34) *is uBIBS*.

Remark 5.10. From Lemma 5.9 and Theorem 2 part 2 it follows that if a system as in (31) is uBIBS w.r.t. $\mathcal{S}_{\text{pc}}(\Gamma)$ then it is forward complete.

In [24], some Lyapunov characterizations on several input-to-output stability properties were developed for systems as in (34) in the special case when the disturbance term ω does not appear in the system. It is straightforward to generalize the Lyapunov results in [24] to the more general case when a system as in (34) is subject to disturbances taking values in compact sets. See Appendix A for more details.

Combining Lemmas 5.7, 5.9, and A.2 and taking into account Remark 5.10, we obtain the following:

Theorem 8. *Suppose the system* (31) *is uBIBS w.r.t.* $\mathcal{S}_{\text{pc}}(\Gamma)$.

1. *The system is uIOS w.r.t.* $\mathcal{S}_{\text{pc}}(\Gamma)$ *if and only if there exist a smooth function* $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ *such that*
   - *for some* $\alpha_1 \in \mathcal{K}_\infty,\ \alpha_2 \in \mathcal{K}_\infty$,

$$\alpha(|h(\xi)|) \leqslant V(\xi) \leqslant \alpha_2(|\xi|), \quad \forall \xi \in \mathbb{R}^n; \tag{43}$$

   - *for some* $\chi \in \mathcal{K}$ *and* $\alpha_3 \in \mathcal{KL}$, *the following holds for all* $\xi \in \mathbb{R}^n$, *all* $\mu \in \mathbb{R}^m$, *all* $\gamma \in \Gamma$, *and all* $v \in F_\gamma(\xi, \mu)$:

$$V(\xi) \geqslant \chi(|\xi|) \Rightarrow DV(\xi)v \leqslant -\alpha_3(V(\xi), |\xi|). \tag{44}$$

2. *The system is uOLIOS w.r.t.* $\mathscr{S}_{\rm pc}(\Gamma)$ *if and only if there exists a smooth function* $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ *such that*

   - *for some* $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$,

$$\alpha_1(|h(\xi)|) \leqslant V(\xi) \leqslant \alpha_2(|h(\xi)|), \quad \forall\, \xi \in \mathbb{R}^n. \tag{45}$$

   - *for some* $\chi \in \mathscr{K}_\infty$ *and some* $\alpha_3 \in \mathscr{KL}$, (44) *holds for all* $\xi \in \mathbb{R}^n$, *all* $\mu \in \mathbb{R}^m$, *all* $\gamma \in \Gamma$, *and all* $v \in F_\gamma(\xi, \mu)$.

3. *The system is uSIIOS w.r.t.* $\mathscr{S}_{\rm pc}(\Gamma)$ *if and only if there exists a smooth function* $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ *such that*

   - *for some* $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$, (45) *holds; and*
   - *there exist* $\alpha_3 \in \mathscr{K}_\infty$ *and* $\chi \in \mathscr{K}_\infty$ *such that for all* $\xi \in \mathbb{R}^n$, *all* $\mu \in \mathbb{R}^m$, *all* $\gamma \in \Gamma$, *and all* $v \in F_\gamma(\xi, \mu)$,

$$V(\xi) \geqslant \chi(|\mu|) \;\Rightarrow\; DV(\xi)v \leqslant -\alpha_3(|\xi|). \tag{46}$$

As indicated in Remark A.3, the uBIBS condition is not needed in the uSIIOS case. Instead, the uBIBS condition can be replaced by the forward completeness property. Hence, for the uSIIOS case, we have the following:

Proposition 5.11. *Suppose the system* (31) *is forward complete. Then the system is uSIIOS w.r.t.* $\mathscr{S}_{\rm pc}(\Gamma)$ *if and only if there exists a smooth function* $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ *such that*

- *for some* $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$, (45) *holds; and*
- *there exist* $\alpha_3 \in \mathscr{K}_\infty$ *and* $\chi \in \mathscr{K}_\infty$ *such that for all* $\xi \in \mathbb{R}^n$, *all* $\mu \in \mathbb{R}^m$, *all* $\gamma \in \Gamma$, *and all* $v \in F_\gamma(\xi, \mu)$, *property* (46) *holds*.

6. Conclusions

In this paper we have studied the representation of switched systems given by differential inclusions, by perturbed control systems described by differential equations, whose inputs are the original control and perturbations that take values in compact sets. We have obtained, under suitable hypotheses, representations whose sets of maximal trajectories contain, as a dense subset (both in the topology of uniform convergence in a compact interval and in the Whitney topology, according to the switching signals involved) the set of maximal trajectories of the original switched system. As immediate applications, we have extended previous results on Lyapunov characterizations for the input–output-to-state stability and input-to-output stability properties to switched systems defined by differential inclusions.

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Appendix A. Input-to-output stability properties

In this section, we discuss how the Lyapunov results developed in [24] can be generalized to systems with disturbances taking values in compact metric spaces.

Consider a system whose dynamics depend on two types of inputs, which we call respectively *controls* and *disturbances*:

$$\dot{x}(t) = f(x(t), u(t), w(t)), \quad y(t) = h(x(t)), \tag{47}$$

where the state $x(\cdot)$ and the input $u(\cdot)$ are the same as in the previous sections. The disturbances are measurable functions $w : \mathbb{R}_{\geqslant 0} \to \Omega$ with Ω a compact metric space. The function $f : \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}^n$ is continuous, and locally Lipschitz in (x, u) uniformly on w and $h : \mathbb{R}^n \to \mathbb{R}^p$ is locally Lipschitz and vanishes at 0.

Definition A.1. A forward-complete system as in (47) is:

- *uniformly input to output stable* (uIOS) if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that
$$|y(t, \xi, u, w)| \leqslant \beta(|\xi|, t) + \gamma(\|u\|), \quad \forall t \geqslant 0; \tag{48}$$
- *uniformly output-Lagrange input to output stable* (uOLIOS) if it is uIOS and there exist some $\mathcal{K}$-functions σ_1, σ_2 such that
$$|y(t, \xi, u, w)| \leqslant \max\{\sigma_1(|h(\xi)|), \sigma_2(\|u\|)\}, \quad \forall t \geqslant 0; \tag{49}$$
- *uniformly state-independent input-to-output stable* (uSIIOS) there exist some $\beta \in \mathcal{KL}$ and some $\gamma \in \mathcal{K}$ such that
$$|y(t, \xi, u, w)| \leqslant \beta(|h(\xi)|, t) + \gamma(\|u\|), \quad \forall t \geqslant 0. \tag{50}$$

In each case, we interpret the estimates as holding for all inputs *u* and initial states $\xi \in \mathbb{R}^n$ and all disturbances w.

Observe that it results in an equivalent definition if one replaces (48) by

$$|y(t, \xi, u, w)| \leqslant \max\{\beta(|\xi|, t), \gamma(\|u\|)\}, \quad \forall t \geqslant 0. \tag{51}$$

We say that a system (47) is *uniformly bounded input bounded state* stable (uBIBS) for short, if it is forward complete and, for some nondecreasing function σ, the following estimate holds for all solutions:

$$|x(t, \xi, u, w)| \leqslant \max\{\sigma(|\xi|), \sigma(\|u\|)\}, \quad \forall t \geqslant 0, \ \forall \xi, \ \forall u, \ \forall w. \tag{52}$$

In [24], some results on Lyapunov characterizations on the output stability properties were obtained for systems as in (47) for the disturbance free case, that is, the case when the disturbance $w(\cdot)$ is not present. Following the same proofs as in [24], one can show that the Lyapunov results obtained in [24] also hold for more general systems with disturbance:

Lemma A.2. *Suppose the system* (47) *is uBIBS.*

1. *The system is uIOS if and only if there exist a smooth function* $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, *some* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$, *and* $\alpha_3 \in \mathcal{KL}$ *such that*

$$\alpha_1(|h(\xi)|) \leqslant V(\xi) \leqslant \alpha_2(|\xi|), \quad \forall \xi \in \mathbb{R}^n \tag{53}$$

and

$$V(\xi) \geqslant \chi(|\mu|) \Rightarrow DV(\xi) f(\xi, \mu, v) \leqslant -\alpha_3(V(\xi), |\xi|), \quad \forall \xi \in \mathbb{R}^n, \ \forall \mu \in \mathbb{R}^m, \ \forall v \in \Omega. \tag{54}$$

2. *The system is uOLIOS if and only if it admits a Lyapunov function V as in the uIOS case satisfying* (54) *for some* $\chi \in \mathcal{K}_\infty$ *and some* $\alpha_3 \in \mathcal{KL}$, *and with* (53) *strengthened to*

$$\alpha_1(|h(\xi)|) \leqslant V(\xi) \leqslant \alpha_2(|h(\xi)|), \quad \forall \xi \in \mathbb{R}^n, \tag{55}$$

for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.

3. *The system is uSIIOS if and only if it admits a Lyapunov function as in the uIOS case with* (53) *strengthened to* (55) *for some* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, *and* (54) *strengthened to*

$$V(\xi) \geqslant \chi(|\mu|) \Rightarrow DV(\xi) f(\xi, \mu, v) \leqslant -\alpha_3(V(\xi)), \quad \forall \xi \in \mathbb{R}^n, \ \forall \mu \in \mathbb{R}^m, \ \forall v \in \Omega, \tag{56}$$

for some $\chi \in \mathcal{K}$ *and* $\alpha_3 \in \mathcal{K}$.

Remark A.3. As in the work [24], the uBIBS condition is not needed in the case of uSIIOS, that is, part 3 of Lemma A.2 holds true for forward complete systems that are not necessarily uBIBS.

A.1. Uniform output stability properties

Definition A.4. For a system free of the control signals as in

$$\dot{x}(t) = f(x(t), w(t)), \quad y(t) = h(x(t)), \tag{57}$$

we say that the system is uniformly output stable (uOS) if it is forward complete and for some $\beta \in \mathcal{KL}$ it holds that

$$|y(t, \xi, w)| \leqslant \beta(|\xi|, t), \quad \forall t \geqslant 0, \ \forall \xi, \ \forall w. \tag{58}$$

If, in addition, there exists $\sigma \in \mathcal{K}$ such that

$$|y(t, \xi, w)| \leqslant \sigma(|h(\xi)|), \quad \forall t \geqslant 0 \tag{59}$$

holds for all trajectories of the system with $w \in \mathcal{M}(\Omega)$, then the system is *output-Lagrange uniformly output stable* with respect to $w \in \mathcal{M}(\Omega)$. Finally, if (58) is strengthened to

$$|y(t,\xi,w)| \leqslant \beta(|h(\xi)|,t), \quad \forall t \geqslant 0, \tag{60}$$

holding for all trajectories of the system with respect to $w \in \mathcal{M}(\Omega)$, then the system is *state-independent uniformly output stable* with $w \in \mathcal{M}(\Omega)$.

The proof of Lemma A.2 follows the same idea as in [24], which depends on the following result on uOS (see Theorem 3.2 of [24]).

Lemma A.5. *Let Ω be a compact metric space, and suppose that a system* (57) *is uniformly output stable with respect to $w \in \mathcal{M}(\Omega)$. Then the system admits a smooth Lyapunov function V satisfying the following properties*:

- *There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that*

$$\alpha_1(|h(\xi)|) \leqslant V(\xi) \leqslant \alpha_2(|\xi|), \quad \forall \xi \in \mathbb{R}^n, \tag{61}$$

- *there exists $\alpha_3 \in \mathcal{KL}$ such that*

$$DV(\xi) f(\xi, v) \leqslant -\alpha_3(V(\xi), |\xi|), \quad \forall \xi \in \mathbb{R}^n, \ \forall v \in \Omega. \tag{62}$$

Moreover, if the system is output-Lagrange uniformly output stable with respect to $w \in \mathcal{M}(\Omega)$, then (61) *can be strengthened to*

$$\alpha_1(|h(\xi)|) \leqslant V(\xi) \leqslant \alpha_2(|h(\xi)|), \quad \forall \xi \in \mathbb{R}^n, \tag{63}$$

for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. Finally, if the system is state-independent uniformly output stable with respect to $w \in \mathcal{M}(\Omega)$, then (61) *can be strengthened to* (63) *and also* (62) *can be strengthened to*:

$$DV(\xi) f(\xi, v) \leqslant -\alpha_4(V(\xi)), \quad \forall \xi \in \mathbb{R}^n, \ \forall v \in \Omega. \tag{64}$$

for some $\alpha_4 \in \mathcal{K}$.

A.2. Proof of Lemma A.2

As in the work [24], to prove Lemma A.2, we need first to explore some relations among several output stability properties.

A.2.1. Relations among the output stability properties

Lemma 4 in [23] can be generalized to the following:

Lemma A.6. *Suppose system* (47) *is uIOS. Then there exists a smooth $\mathcal{K}_\infty$-function λ such that the system*

$$\dot{x}(t) = f(x(t), d(t)\lambda(|h(x(t))|), w(t)), \quad y(t) = h(x(t)), \tag{65}$$

where $d \in \mathcal{M}(\mathcal{B}_m)$ is uOS (*with* (d, w) *as disturbances*).

Lemma A.6 will be proved in Section A.2.3.

As defined in [23], we say that a system (47) is uOLIOS *under output redefinition* if there exist a locally Lipschitz map $h_0 : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ with $h_0(0) = 0$, and a $\chi \in \mathcal{K}_\infty$, such that

$$h_0(\xi) \geqslant \chi(|h(\xi)|), \quad \forall \xi,$$

and that the system

$$\dot{x} = f(x, u, w), \quad y = h_0(x) \tag{66}$$

is uOLIOS. One of the main objectives of this appendix is to generalize Theorem 6 in [23] to the following result:

Lemma A.7. *The following are equivalent for a system* (47):

(i) *The system is uIOS.*
(ii) *The system is uOLIOS under output redefinition*.

A.2.2. Proof of Lemma A.7

The implication (ii) ⇒ (i) should be clear. Below we prove the converse.

Assume that system (47) with $y=h(x)$ is uIOS with an estimate as (51) for some $\beta \in \mathcal{KL}$ and some $\gamma \in \mathcal{K}$. Without loss of generality, we assume that $\gamma \in \mathcal{K}_\infty$. Let $h_0 : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ be defined by

$$h_0(\xi) = \sup_{t \geqslant 0, u, w} \{\max\{|y(t, \xi, u, w)| - \gamma(\|u\|), 0\}\}. \tag{67}$$

Observe that forward completeness is being used in this definition. Since $|y(0, \xi, u, w)| - \gamma(\|u\|) = |h(\xi)| \geqslant 0$ for $u \equiv 0$, the above is equivalent to

$$h_0(\xi) = \sup_{t \geqslant 0, u, w} \{|y(t, \xi, u, w)| - \gamma(\|u\|)\}.$$

It is clear that

$$|h(\xi)| \leqslant h_0(\xi) \leqslant \beta_0(|\xi|), \quad \forall \xi \in \mathbb{R}^n,$$

where $\beta_0(s) = \beta(s, 0)$. Since, for any u_0 with $\gamma(\|u_0\|) \geqslant \beta_0(s)$,

$$\max\{\beta(|\xi|, t), \gamma(\|u_0\|)\} - \gamma(\|u_0\|) \leqslant \max\{\beta_0(|\xi|), \gamma(\|u_0\|)\} - \gamma(\|u_0\|) = 0,$$

it follows that

$$h_0(\xi) = \sup\{|y(t, \xi, u, w)| - \gamma(\|u\|) : t \geqslant 0,\ \|u\| \leqslant \gamma^{-1}(\beta_0(|\xi|)),\ w \in \mathcal{M}(\Omega)\}. \tag{68}$$

Also note that for any $\tau \geqslant 0$, any control u_0 and any disturbance w_0,

$$\begin{aligned} h_0(x(\tau,\xi,u_0,w_0)) &= \sup_{t\geqslant 0,u,w}\{|y(t,x(\tau,\xi,u_0,w_0),u,w)| - \gamma(\|u\|)\} \\ &= \sup_{t\geqslant 0,u,w}\{|y(t+\tau,\xi,u_0\sharp_\tau u,w_0\sharp_\tau w)| - \gamma(\|u_0\sharp_\tau u\|_{[\tau,\infty)})\} \\ &\leqslant \sup_{t\geqslant 0,u}\{\max\{\beta(|\xi|,t+\tau),\gamma(\|u_0\sharp_\tau u\|)\} - \gamma(\|u_0\sharp_\tau u\|_{[\tau,\infty)})\} \\ &\leqslant \sup_{u}\{\max\{\beta(|\xi|,\tau),\gamma(\|u_0\sharp_\tau u\|)\} - \gamma(\|u_0\sharp_\tau u\|_{[\tau,\infty)})\} \\ &\leqslant \sup_{u}\{\max\{\beta(|\xi|,\tau),\ \gamma(\|u_0\sharp_\tau u\|) - \gamma(\|u_0\sharp_\tau u\|_{[\tau,\infty)})\}\} \\ &\leqslant \max\{\beta(|\xi|,\tau),\ \gamma(\|u_0\|_{[0,\tau)})\}, \end{aligned} \tag{69}$$

where for any two functions v_1 and v_2 defined on $\mathbb{R}_{\geqslant 0}$, $v_1\sharp_\tau v_2$ is the concatenation of v_1 and v_2 defined by

$$v_1\sharp_\tau v_2(t) = \begin{cases} v_1(t), & \text{if } 0\leqslant t<\tau, \\ v_2(t-\tau), & \text{if } t\geqslant\tau. \end{cases}$$

This shows that the system (47) with the output map $y = h_0(x)$ satisfies an uIOS-type estimate (51) with the same functions β and γ as the original system.

Next, let us show that (47) with $y = h_0(x)$ also satisfies an output Lagrange estimate (49), with $\sigma_1(r) = 2r$ and $\sigma_2(r) = 2\gamma(r)$. Indeed, for any input u_0, any disturbance w_0 and any $\tau \geqslant 0$, we have

$$\begin{aligned} h_0(x(\tau,\xi,u_0,w_0)) &= \sup_{t\geqslant 0,u,w}\{|y(t,x(\tau,\xi,u_0,w_0),u,w)| - \gamma(\|u\|)\} \\ &= \sup_{t\geqslant 0,u,w}\{|y(t+\tau,\xi,u_0\sharp_\tau u,w_0\sharp_\tau w)| - \gamma(\|u_0\sharp_\tau u\|_{[\tau,\infty)})\} \\ &\leqslant \sup_{s\geqslant 0,u,w}\{|y(s,\xi,u_0\sharp_\tau u,w_0\sharp_\tau w)| - \gamma(\|u_0\sharp_\tau u\|) + \gamma(\|u_0\|)\} \\ &\leqslant \sup_{s\geqslant 0,\tilde{u},\tilde{w}}\{|y(s,\xi,\tilde{u},\tilde{w})| - \gamma(\|\tilde{u}\|) + \gamma(\|u_0\|)\} \\ &= h_0(\xi) + \gamma(\|u_0\|) \leqslant \max\{2h_0(\xi), 2\gamma(\|u_0\|)\}, \end{aligned} \tag{70}$$

as desired.

Define $C := \{\xi : h_0(\xi) = 0\}$. Then for any $\xi \notin C$, it holds that

$$h_0(\xi) = \sup_{0\leqslant t\leqslant t_\xi,\|u\|\leqslant\gamma^{-1}(\beta_0(|\xi|)),w\in\mathscr{M}(\Omega)}\{|y(t,\xi,u,w)| - \gamma(\|u\|)\},$$

where $t_\xi = T_{|\xi|}(h_0(\xi)/2)$, and $T_r(s)$ is associated with β as in Lemma A.9.

Lemma A.8. *The function h_0 is locally Lipschitz on the set where $h_0(\xi) \neq 0$ and continuous everywhere.*

Proof. We first remark that

$$\varliminf_{\xi\to\xi_0} h_0(\xi) \geqslant h_0(\xi_0), \quad \forall \xi_0 \in \mathbb{R}^n, \tag{71}$$

that is, $h_0(\xi)$ is lower semi-continuous on $\mathbb{R}^n$. Indeed, pick ξ_0 and let $c := h(\xi_0)$. Take any $\varepsilon > 0$. Then there are some u_0, w_0 and t_0 so that $|y(t_0, \xi_0, u_0, w_0)| - \gamma(\|u_0\|) \geqslant c - \varepsilon/2$. By continuity of $y(t_0, \cdot, u_0, w_0)$, there is some neighborhood $\tilde{U}_0$ of ξ_0 so that $|y(t_0, \xi, u_0, w_0)| - \gamma(\|u_0\|) \geqslant c - \varepsilon$ for all $\xi \in \tilde{U}_0$. Thus, $h_0(\xi) \geqslant c - \varepsilon$ for all $\xi \in \tilde{U}_0$, and this establishes (71).

Fix any $\xi_0 \notin C$, and let $c_0 = h_0(\xi_0)/2$. Then there exists a neighborhood U_0 of ξ_0 with compact closure such that

$$h_0(\xi) \geqslant c_0, \quad \forall \xi \in U_0.$$

Let s_0 be such that $|\xi| \leqslant s_0$ for all $\xi \in U_0$. Then

$$h_0(\xi) = \sup\{|y(t, \xi, u, w)| - \gamma(\|u\|) : t \in [0, t_1],\ \|u\| \leqslant b,\ w \in \mathscr{M}(\Omega)\}, \quad \forall \xi \in U_0,$$

where $t_1 = T_{s_0}(c_0/2)$, and $b = \gamma^{-1}(\beta_0(s_0))$. By [13, Proposition 5.5], one knows that $x(t, \xi, u, w)$ is Lipschitz in $\xi \in U_0$ uniformly on the set $\|u\| \leqslant b$, $w \in \mathscr{M}(\Omega)$ and $t \in [0, t_1]$, and therefore, so is $y(t, \xi, u, w)$. Let L_1 be a constant such that

$$\begin{aligned} &|y(t, \xi, u, w) - y(t, \eta, u, w)| \\ &\quad \leqslant L_1 |\xi - \eta| \forall \xi, \eta \in U_0, \quad \forall 0 \leqslant t \leqslant t_1,\ \forall \|u\| \leqslant b,\ \forall w \in \mathscr{M}(\Omega). \end{aligned}$$

For any $\varepsilon > 0$ and any $\xi \in U_0$, there exist some $t_{\xi,\varepsilon} \in [0, t_1]$, some $u_{\xi,\varepsilon}$ and some $w_{\xi,\varepsilon}$ such that

$$h_0(\xi) \leqslant |y(t_{\xi,\varepsilon},\ \xi, u_{\xi,\varepsilon},\ w_{\xi,\varepsilon})| - \gamma(\|u_{\xi,\varepsilon}\|) + \varepsilon.$$

Then it follows that, for any $\xi, \eta \in U_0$, for any $\varepsilon > 0$,

$$\begin{aligned} h_0(\xi) - h_0(\eta) &\leqslant |y(t_{\xi,\varepsilon}, \xi, u_{\xi,\varepsilon}, w_{\xi,\varepsilon})| - \gamma(\|u_{\xi,\varepsilon}\|) \\ &\qquad + \varepsilon - (|y(t_{\xi,\varepsilon}, \eta, u_{\xi,\varepsilon}, w_{\xi,\varepsilon})| - \gamma(\|u_{\xi,\varepsilon}\|)) \\ &\leqslant L_1 |\xi - \eta| + \varepsilon. \end{aligned}$$

Consequently,

$$h_0(\xi) - h_0(\eta) \leqslant L_1 |\xi - \eta|, \quad \forall \xi, \eta \in U_0.$$

By symmetry,

$$h_0(\eta) - h_0(\xi) \leqslant L_1 |\xi - \eta|, \quad \forall \xi, \eta \in U_0.$$

This proves that h_0 is locally Lipschitz on $\mathbb{R}^n \backslash C$.

We now show that h_0 is continuous on C. Fix $\xi_0 \in C$. One would like to show that

$$\lim_{\xi \to \xi_0} h_0(\xi) = 0. \tag{72}$$

Assume that this does not hold. Then there exists a sequence $\{\xi_k\}$ with $\xi_k \to \xi_0$ and some $\varepsilon_0 > 0$ such that $h_0(\xi_k) > \varepsilon_0$ for all k. Without loss of generality, one may assume that, for some $s_1 \geqslant 0$, $|\xi_k| \leqslant s_1$ for all k. Then it follows that

$$h_0(\xi_k) = \sup\{|y(t, \xi_k, u, w)| - \gamma(\|u\|) : t \in [0, t_2], \|u\| \leqslant b_1, w \in \mathscr{M}(\Omega)\},$$

where $t_2 = T_{s_1}(\varepsilon_0/2)$, and $b_1 = \gamma^{-1}(\beta_0(s_1))$. Hence, for each k, there exists some u_k with $\|u_k\| \leqslant b_1$, some $w_k \in \mathscr{M}(\Omega)$, and some $\tau_k \in [0, t_2]$ such that

$$|y(\tau_k, \xi_k, u_k, w_k)| - \gamma(\|u_k\|) \geqslant h_0(\xi_k) - \varepsilon_0/2 \geqslant \varepsilon_0/2. \tag{73}$$

Again, by the locally Lipschitz continuity of the trajectories, one knows that there is some $L_2 > 0$ such that

$$|y(t, \xi_k, u, w) - y(t, \xi_0, u, w)| \leqslant L_2 |\xi_k - \xi_0|$$
$$\forall k \geqslant 0, \quad \forall 0 \leqslant t \leqslant t_2, \ \forall \ \|u\| \leqslant b_1, \ \forall w \in \mathscr{M}(\Omega).$$

Hence,

$$|y(\tau_k, \xi_0, u_k, w_k)| - \gamma(\|u_k\|) \geqslant |y(\tau_k, \xi_k, u_k, w_k)| - \varepsilon_0/4 - \gamma(\|u_k\|) \geqslant \varepsilon_0/4$$

for k large enough, contradicting the fact that $h_0(\xi_0) = 0$. This shows that (72) holds on C. □

Below we follow the same proof as in [23] to modify h_0 to get an output function $\tilde{h}$ that is locally Lipschitz everywhere so that system (66) with $\tilde{h}$ is uOLIOS.

We first pick a function $\bar{h}(\xi)$ that is smooth on $\mathbb{R}^n \backslash C$ with the property

$$\frac{h_0(\xi)}{2} \leqslant \bar{h}(\xi) \leqslant 2h_0(\xi), \quad \forall \xi \in \mathbb{R}^n.$$

This can be done according to, e.g., Theorem B.1 in [13]. By Lemma 4.3 in [13], there exists a $\mathscr{K}_\infty$-function ρ such that $\rho \circ \bar{h}$ is smooth everywhere. Let $\tilde{h} = \rho \circ \bar{h}$. Note then that

$$\rho(h_0(\xi)/2) \leqslant \tilde{h}(\xi) \leqslant \rho(2h_0(\xi)). \tag{74}$$

Combining this with the fact that $h_0(\xi) \geqslant |h(\xi)|$, one sees that

$$\tilde{h}(\xi) \geqslant \chi(|h(\xi)|), \quad \forall \xi,$$

where $\chi(s) = \rho(s/2)$. Because of (69), one has

$$\tilde{h}(x(t, \xi, u, w)) \leqslant \max\{\tilde{\beta}(|\xi|, t), \tilde{\gamma}(\|u\|)\}, \quad \forall t \geqslant 0,$$

where $\tilde{\beta}(s, t) = \rho(2\beta(s, t))$, and $\tilde{\gamma}(s) = \rho(2\gamma(s))$, and because of (74) and (70), one has

$$\begin{aligned} \tilde{h}(x(t, \xi, u, w)) &\leqslant \rho(2h_0(x(t, \xi, u, w))) \leqslant \max\{\rho(4h_0(\xi)), \rho(4\gamma(\|u\|))\} \\ &\leqslant \max\{\rho(8\rho^{-1}(\tilde{h}(\xi))), \rho(4\gamma(\|u\|))\}, \quad \forall t \geqslant 0, \end{aligned}$$

that is,

$$\tilde{h}(x(t, \xi, u, w)) \leqslant \max\{\tilde{\sigma}_1(\tilde{h}(\xi)), \tilde{\sigma}_2(\|u\|)\}, \quad \forall t \geqslant 0,$$

for all ξ, all u and all w, where $\tilde{\sigma}_1(s) = \rho(8\rho^{-1}(s))$ and $\tilde{\sigma}_2(s) = \rho(4\gamma(s))$. We conclude that system (66) with the output function $y = \tilde{h}(x)$ is uOLIOS.

In the above proof, we used the following result (see Lemma A.1 in [24]) regarding $\mathcal{KL}$-functions:

Lemma A.9. *For any $\mathcal{KL}$-function β, there exists a family of mappings $\{T_r\}_{r \geqslant 0}$ such that*

- *for each fixed $r > 0$, $T_r : \mathbb{R}_{>0} \xrightarrow{\text{onto}} \mathbb{R}_{>0}$ is continuous and strictly decreasing, and $T_0(s) \equiv 0$;*
- *for each fixed $s > 0$, $T_r(s)$ is strictly increasing as r increases, and is such that $\beta(r, T_r(s)) < s$, and consequently, $\beta(r, t) < s$ for all $t \geqslant T_r(s)$.*

A.2.3. Proof of Lemma A.6

We will follow the same idea as in [23] to prove Lemma A.6. We first establish the following.

Lemma A.10. *Assume that the system* (47) *is forward complete and admits an output-Lagrange estimate as in* (49). *Then, there is a smooth $\mathcal{K}_\infty$ function λ such that the system*

$$\dot{x} = g(x, d, w) := f(x, d\lambda(|y|), w), \quad y = h(x), \tag{75}$$

where $d \in \mathcal{M}(\mathcal{B}_m)$, is forward complete and

$$\sigma_2(|d(t)|\lambda(|y_\lambda(t, \xi, d, w)|)) \leqslant \tfrac{1}{2}|h(\xi)| \tag{76}$$

holds a.e. on $[0, \infty)$, *where we have used $y_\lambda(t, \xi, d, w)$ to denote the output function of* (75) *with the initial state ξ and the disturbance functions d and w.*

Proof. Let σ_1, σ_2 be $\mathcal{K}$-functions such that (49) holds. Without loss of generality, we assume that both are in $\mathcal{K}_\infty$ and that $\sigma_1(s) \geqslant s$ for all $s \geqslant 0$. Hence, $\sigma_1^{-1}(s) \leqslant s$ for all $s \geqslant 0$. Let λ be any smooth $\mathcal{K}_\infty$-function such that

$$\sigma_2(\lambda(s)) < \tfrac{1}{4}\, \sigma_1^{-1}(s), \quad \forall s > 0.$$

Below we show that with such a choice of λ, the resulting system (75) satisfies the desired properties.

To show that system (75) is complete, we first prove that (76) holds a.e. on the maximal interval of definition $[0, T_{x_\lambda})$ of the solution x_λ.

Pick any ξ, any d and any w, and use simply $x_\lambda(t)$ and $y_\lambda(t)$ to denote the corresponding trajectory and the output function respectively. To prove (76) on $[0, T_{x_\lambda})$, it is enough to show that

$$\sigma_2(\lambda(|y_\lambda(t)|)) \leqslant \tfrac{1}{2}\, |h(\xi)|, \tag{77}$$

for such t. □

Case 1: $h(\xi) \neq 0$. Since $\sigma_2(\lambda(|y_\lambda(0)|)) = \sigma_2(\lambda(|h(\xi)|)) < \frac{1}{4}\,\sigma_1^{-1}(|h(\xi)|) \leqslant \frac{1}{4}\,|h(\xi)|$, it follows that $\sigma_2(\lambda(|y_\lambda(t)|)) \leqslant \frac{1}{4}\,|h(\xi)|$ for t small enough. Let

$$t_1 = \inf\left\{t \in (0, T_{x_\lambda}) : \sigma_2(\lambda(|y_\lambda(t)|)) > \tfrac{1}{2}\,|h(\xi)|\right\},$$

with $t_1 = T_{x_\lambda}$ if the set is empty. Suppose by way of contradiction that $t_1 < T_{x_\lambda}$. Then (77) holds on $[0, t_1)$, and hence, (76) holds a.e. on $[0, t_1)$. Note that on $[0, T_{x_\lambda})$, $y_\lambda(t) = y(t, \xi, u, w)$ with $u(t) = d(t)\lambda(|y_\lambda(t)|)$. With (49), one sees that $|y_\lambda(t)| \leqslant \sigma_1(|h(\xi)|)$ for all $0 \leqslant t \leqslant t_1$, and in particular, $|y_\lambda(t_1)| \leqslant \sigma_1(|h(\xi)|)$. Consequently,

$$\sigma_2(\lambda(|y_\lambda(t_1)|)) \leqslant \tfrac{1}{4}\,\sigma_1^{-1}(|y_\lambda(t_1)|) \leqslant \tfrac{1}{4}\,|h(\xi)|,$$

contradicting the definition of t_1. Thus, (77) holds for all $t \in [0, T_{x_\lambda})$.

Case 2: $h(\xi) = 0$. In this case, it is enough to show that $y_\lambda(t) = 0$ for all $t \in [0, T_{x_\lambda})$. Suppose this is not true. Then there exists some $\varepsilon > 0$ and some $t_2 \in (0, T_{x_\lambda})$ such that $|y_\lambda(t_2)| \geqslant \varepsilon$. Let $0 < \varepsilon_0 < \varepsilon$ be such that $\lambda^{-1}(\sigma_2^{-1}(\varepsilon_0)) < \varepsilon/2$. Then there is some $\tau \in (0, t_2)$ such that $|y_\lambda(\tau)| = \varepsilon_0$. Applying (77) proved for case 1 to the new initial state $\xi_1 := x_\lambda(\tau)$, one sees that

$$|y_\lambda(t)| \leqslant \lambda^{-1}\left(\sigma_2^{-1}\left(\tfrac{1}{2}\,|y_\lambda(\tau)|\right)\right) \leqslant \lambda^{-1}(\sigma_2^{-1}(\varepsilon_0)) < \varepsilon/2$$

for all $t \in [\tau, T_{x_\lambda})$, and in particular, $|y_\lambda(t_2)| < \varepsilon/2$, a contradiction. This shows that $y_\lambda(t) = 0$ for all $t \in [0, T_{x_\lambda})$.

We have shown that in both cases, (77) holds for all $t \in [0, T_{x_\lambda})$, which implies that, for any ξ, any d and any w, the function $u(t) := d(t)\lambda(|y_\lambda(t, \xi, d, w)|)$ remains essentially bounded on $[0, T_{x_\lambda})$. Suppose $T_{x_\lambda} < \infty$. Then, by the forward completeness property of system (47), the trajectory $x_\lambda(t, \xi, d, w)$ (which is in fact $x(t, \xi, u, w)$ with $u(t) = d(t)\lambda(|y_\lambda(t, \xi, d, w)|)$) is bounded on $[0, T_{x_\lambda})$. This contradicts the maximality of T_{x_λ}. Therefore, $T_{x_\lambda} = \infty$ for every ξ, every d and every w. Consequently, (76) holds for all $t \in [0, \infty)$. □

Lemma A.11. *Suppose a system* (47) *is uOLIOS. Then there exists some smooth* $\mathcal{K}_\infty$ *function* λ *such that the resulting system as in* (75) *is output-Lagrange uniformly output stable with* (d, w) *as disturbances.*

Proof. Suppose estimates as in (51) and (49) hold for some $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$, $\sigma_1 \in \mathcal{K}$ and $\sigma_2 \in \mathcal{K}$. Without loss of generality, we assume that $\sigma_1(r) \geqslant r$ for all r and $\sigma_2 \in \mathcal{K}_\infty$.

Let the function λ be as in Lemma A.10 so that system (75) is forward complete and (76) holds for almost all $t \geqslant 0$. By [1, Corollary 2.3], there exist some $\mathcal{K}$-functions ϱ_1, ϱ_2 and some $c \geqslant 0$ such that

$$|x_\lambda(t, \xi, d, w)| \leqslant \varrho_1(t) + \varrho_2(|\xi|) + c, \tag{78}$$

for all ξ, all d, all w and all $t \geqslant 0$.

Below we will show that system (75) satisfies the two properties listed in Remark A.12. First, by (76) and (49), one has:

$$|y_\lambda(t, \xi, d, w)| \leqslant \max\left\{\sigma_1(|h(\xi)|), \tfrac{1}{2}\,|h(\xi)|\right\} = \sigma_1(|h(\xi)|) \leqslant \tilde{\sigma}(|\xi|), \quad \forall t \geqslant 0, \tag{79}$$

where $\tilde{\sigma}$ is any $\mathcal{K}$-function such that $\sigma_1(|h(\xi)|) \leqslant \tilde{\sigma}(|\xi|)$ for all ξ. Property 1 follows readily. To prove Property 2, we first show the following:

Claim. For each $r > 0$, $s > 0$, there is some $T_{r,s} > 0$ such that

$$t \geqslant T_{r,s}, \ |\xi| \leqslant r, \ |h(\xi)| \leqslant s \ \Rightarrow \ |y_\lambda(t, \xi, d, w)| \leqslant s/2, \quad \forall d, \ \forall w. \tag{80}$$

To prove the claim, note that by (51) and (76), one has, for all ξ as in (80),

$$|y_\lambda(t, \xi, d, w)| \leqslant \max\left\{\beta(|\xi|, t), \frac{|h(\xi)|}{2}\right\} \leqslant \max\left\{\beta(r, t), \frac{s}{2}\right\}, \quad \forall\, t \geqslant 0.$$

Since $\beta \in \mathcal{KL}$, there is some $T_{r,s} > 0$ such that $\beta(r, t) \leqslant s/2$ for all $t \geqslant T_{r,s}$, and consequently,

$$|y_\lambda(t, \xi, u, w)| \leqslant \frac{s}{2}, \quad \forall t \geqslant T_{r,s}.$$

This $T_{r,s}$ satisfies the requirements of the claim.

Let κ be a $\mathcal{K}$-function such that $|h(\xi)| \leqslant \kappa(|\xi|)$ for all ξ. Let $\varepsilon > 0$ be given. Pick any $\xi \neq 0$ and let $r = |\xi|$. Then $|h(\xi)| \leqslant \kappa(r)$. Let $l > 0$ be such that $2^{-l}\kappa(r) < \varepsilon$. Let $s_1 = \kappa(r)$ and $s_i = s_{i-1}/2$ for $i \geqslant 2$. By (80), there is some $T_{r,s_1} > 0$ such that

$$|y_\lambda(t, \xi, d, w)| \leqslant s_1/2, \quad \forall t \geqslant T_{r,s_1}, \ \forall d, \ \forall w.$$

By (78), one has

$$|x_\lambda(T_{r,s_1}, \xi, d, w)| \leqslant \varrho_1(T_{r,s_1}) + \varrho_2(r) + c := r_2, \quad \forall d, \ \forall w.$$

Applying (80) to r_2 and s_2, one sees that there is some T_{r_2,s_2} such that the following holds:

$$|y_\lambda(t + T_{r,s_1}, \xi, d, w)| \leqslant s_2/2, \quad \forall t \geqslant T_{r_2,s_2}, \ \forall d, \ \forall w.$$

Inductively, letting $\tilde{T}_k = \sum_{i=1}^{k} T_{r_i,s_i}$ (where $r_1 = r$) and applying (80) to s_{k+1} and

$$r_{k+1} := \varrho_1(\tilde{T}_k) + \varrho_2(r) + c,$$

one sees that there is some $T_{r_{k+1},s_{k+1}}$ such that

$$|y_\lambda(t + \tilde{T}_k, \xi, d, w)| \leqslant \frac{s_k}{2}, \quad \forall t \geqslant T_{r_{k+1},s_{k+1}}, \ \forall d, \ \forall w.$$

Finally, we let $T = \tilde{T}_{l+1}$. Then, for any $t \geqslant T$, any d and any w,

$$|y_\lambda(t, \xi, d, w)| \leqslant \frac{s_1}{2^l} < \varepsilon.$$

Observe that in the above argument, T only depends on $|\xi|$ and ε. Thus, the system satisfies Property 2 of Remark A.12. Consequently, system (75) admits an estimate as in (58) with (d, w) as disturbances. The output Lagrange condition required as in (59) follows from (79). □

In the above proof, we have used the following (c.f. [13]):

Remark A.12. Suppose system (75) is forward complete. Then, the existence of a β as in (58) is equivalent to the following:

1. There is a $\mathcal{K}_\infty$-function $\delta(\cdot)$ such that for any $\varepsilon > 0$, it holds that

$$|y(t, \xi, w)| \leqslant \varepsilon, \quad \forall t \geqslant 0, \ \forall w,$$

   whenever $|\xi| \leqslant \delta(\varepsilon)$; and
2. for any $r \geqslant 0$ and any $\varepsilon > 0$, there exists some $T_{r,\varepsilon} > 0$ such that

$$|y(t, \xi, w)| \leqslant \varepsilon,$$

   for all $t \geqslant T_{r,\varepsilon}$, all w, and all $|\xi| \leqslant r$.

The proof of Lemma A.6 then follows from Lemmas A.7 and A.11.

A.2.4. Sketch of the proof of Lemma A.2

Finally, we sketch the idea of the proof of Lemma A.2. The sufficiency parts can be done with some comparison principle (and in fact can be done by exactly the same proofs as in [24]).

To prove the necessity part of Statement 2 of Lemma A.2, first note that if a system is uOLIOS, then by Lemma A.11, there exists some smooth $\mathcal{K}_\infty$-function λ such that the corresponding system (65) is output-Lagrange uniformly output stable with $(d, w) \in \mathcal{M}(\mathcal{B}_m) \times \mathcal{M}(\Omega)$. By Lemma A.5, there exists a Lyapunov function V satisfying (63) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and

$$DV(\xi) f(\xi, \mu\lambda(|h(\xi)|), v) \leqslant -\alpha_3(V(\xi), |\xi|), \quad \forall \xi \in \mathbb{R}^n, \ \forall |\mu| \leqslant 1, \ \forall v \in \Omega, \tag{81}$$

for some $\alpha_3 \in \mathcal{KL}$. Note then that (81) implies that

$$V(\xi) \geqslant \chi(|\mu|) \ \Rightarrow \ DV(\xi) f(\xi, \mu, v) \leqslant -\alpha_3(V(\xi), |\xi|),$$

for all $\xi \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$ and $v \in \Omega$, where $\chi(r) = \alpha_2(\lambda^{-1}(r))$.

The proof of the necessity of Statement 3 of Lemma A.2 follows the same idea as the proof of Statement 2. The necessity of Statement 1 of Lemma A.2 follows from Lemma A.7 and the necessity of Statement 2.

Remark A.13. In the works [23] and [24], it was assumed that $f(0, 0) = 0$. This assumption was in fact not necessary. Indeed, in many interesting applications, it is not reasonable to assume that 0 is an equilibrium for the zero-input system.

Appendix B. An approximation lemma

Lemma B.1. *Let $h : \mathbb{R}^n \to \mathbb{R}^p$ be a continuous map. Then there exists a smooth function $\hat{h} : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ so that for some $\mathscr{K}_\infty$-functions ρ_1, ρ_2,*

$$\rho_1(|h(\xi)|) \leqslant |\hat{h}(\xi)| \leqslant \rho_2(|h(\xi)|), \quad \forall \xi \in \mathbb{R}^n.$$

Proof. Let h be a continuous function, and let $\mathcal{O} = \{\xi : h(\xi) \neq 0\}$. Applying [3, Theorem 4.8] to the continuous function $|h(\cdot)|$, there is a $\mathscr{C}^\infty$ function $\tilde{h} : \mathcal{O} \to \mathbb{R}_{\geqslant 0}$ such that

$$||h(\xi)| - \tilde{h}(\xi)| < \frac{|h(\xi)|}{2}, \quad \forall \xi \in \mathcal{O}.$$

Extend $\tilde{h}$ to $\mathbb{R}^n$ by letting $\tilde{h}(\xi) = 0$ if $\xi \notin \mathcal{O}$. Then $\tilde{h}$ is $\mathscr{C}^\infty$ on $\mathcal{O}$, continuous on $\mathbb{R}^n$, and $\tilde{h}(\xi) = 0$ on $\mathbb{R}^n \setminus \mathcal{O}$. Note also that

$$\tfrac{1}{2}\,|h(\xi)| \leqslant \tilde{h}(\xi) \leqslant 2|h(\xi)|, \quad \forall \xi \in \mathbb{R}^n.$$

Applying [13, Lemma 4.3] to the function $\tilde{h}$, one sees that there exists some $\mathscr{K}_\infty$-function χ such that $\chi(\tilde{h}(\cdot))$ is $\mathscr{C}^\infty$. The lemma is proved by letting $\hat{h}(\xi) = \chi(\tilde{h}(\xi))$, $\rho_1(s) = \chi(s/2)$ and $\rho_2(s) = \chi(2s)$. □

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