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Analysis of Nonlinear Tridiagonal Cooperative Systems using Totally Positive Linear Differential Systems

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***Abstract*— Cooperative tridiagonal dynamical systems appear often in biological and engineering applications. The most important theorem for such systems was arguably one proved by Smillie in 1984, and subsequently refined by other authors. Smillie showed that–under mild technical assumptions–precompact trajectories always converge to equilibria. The key to his proof was the construction of an integer-valued Lyapunov function that certifies that the number of sign variations in the vector of derivatives of states eventually stabilizes.**

This paper shows how to re-derive Smillie's theorem by appealing to results from Binyamin Schwarz, who analyzed the sign variations in solutions of linear systems whose flows are totally nonnegative or totally positive (meaning that all minors are nonnegative or positive, respectively). The connection is through the variational equation associated to the original system. In addition to connecting two seemingly disparate areas of research, the connection allows one to both simplify proofs and extend the validity of Smillie's Theorem.

I. Introduction

Monotone (and particularly cooperative) dynamical systems appear often in biological and engineering applications. They are characterized by the existence of a partial order induced by proper cone K (cooperative systems are the special case $K = \mathbb{R}^n_+$) which is preserved by the solution operator. One of the most celebrated results in this field was shown by Moe Hirsch in the early 1980s, and guarantees convergence to equilibria from an open dense or full measure set of initial states, assuming a slightly stronger property called "strong" monotonicity. A particularly good presentation of Hirsch's Quasi-Convergence Theorem, and more generally of monotone systems, is Hal Smith's book [1]. Hirsch's result leaves open the possibility that there may be periodic orbits as well as more exotic behaviors in low-dimensional manifolds, and this can indeed happen for general strongly monotone systems. Thus, one may ask whether additional structural constraints might provide global convergence, not necessarily just from a "large" set of initial states. Along these lines, Smillie [2] considered time-invariant strongly cooperative *tridiagonal* systems and showed that, for such systems, every precompact trajectory converges to an equilibrium point. This result has found many applications in systems biology (see, e.g. [3], [4], [5]).

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Smillie's proof relies upon the following idea. Consider a system

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is $n-1$ times continuously differentiable, and consider the variational equation

$$\dot{z}(t) = J(x(t))z(t), \tag{2}$$

where $z := \dot{x}$ and $J(x) := \frac{\partial}{\partial x} f(x)$ is the Jacobian of the vector field f. Smillie showed that the number of sign variations in the vector $z(t)$, denoted $\sigma(z(t))$, is a non-increasing function of t. Recall that for a vector $y \in \mathbb{R}^n$ with no zero entries the number of sign variations in y is $\sigma(y) := |\{i \in \{1, \dots, n-1\} : y_i y_{i+1} < 0\}|$. The domain of σ can be extended, via continuity, to the open set $\mathcal{V} := \{y \in \mathbb{R}^n : y_1 \neq 0,\ y_n \neq 0, \text{ and if } y_i = 0 \text{ for some } i \in \{2, \dots, n-1\} \text{ then } y_{i-1}y_{i+1} < 0\}$. For example, for $n = 3$ the vector $r := \begin{bmatrix} -1 & \varepsilon & 2 \end{bmatrix}' \in \mathcal{V}$ and $\sigma(r) = 1$ for all $\varepsilon \in \mathbb{R}$.

To explain the intuition behind Smillie's proof, consider the case $n = 3$. Seeking a contradiction, assume that the sign pattern of $z(t)$ near some time $t = t_0$ is as follows:

	$t = t_0^-$	$t = t_0$	$t = t_0^+$
$z_1(t)$	$-$	$-$	$-$
$z_2(t)$	$-$	$-$	$-$
$z_3(t)$	$-$	0	$+$

In this case $\sigma(t) := \sigma(z(t))$ *increases* from $\sigma(t_0^-) = 0$ to $\sigma(t_0^+) = 1$. However, since the system is tridiagonal and strongly cooperative, (2) yields $\dot{z}(t_0) = \begin{bmatrix} * & + & 0 \\ + & * & + \\ 0 & + & * \end{bmatrix} \begin{bmatrix} - \\ - \\ 0 \end{bmatrix}$, where $+$ $[*]$ means a positive [arbitrary] value, and thus $\dot{z}_3(t_0) < 0$, and the case described in the table above cannot take place. Smillie's analysis shows rigorously that $\sigma(t)$ is well-defined, except perhaps at up to $n-1$ discrete times τ_i, $i = 1, \dots, n-1$, and that at these times $\sigma(\tau_i^+) < \sigma(\tau_i^-)$. This is based on direct analysis of the ODEs and is non trivial. Smillie then used the behavior of $\sigma(z(t))$ to show that for any a in the state-space of (1) the omega limit set $\omega(a)$ can include no more than a single point, and thus every trajectory that remains in a compact set converges to an equilibrium.

Smith [6] has generalized Smillie's approach to periodically time-varying strongly cooperative and tridiagonal systems. He showed that every trajectory that remains in some compact set converges to a periodic trajectory with the same period as the vector field. This entrainment property is important in numerous natural and artificial systems. For example, proper functioning of various processes in biological organisms requires entrainment to the 24h solar day.

Here, we show that the results of Smillie and Smith can be generalized, and their proofs simplified, by relating them to a classical topic from linear algebra: the sign variation diminishing property (SVDP) of totally nonnegative (TN) matrices [7], [8], and more precisely to the notion of *totally positive differential systems* (TPDSs) and *totally nonnegative differential systems* (TNDSs) introduced by Schwarz in 1970 [9].

To explain this, we first recall another definition for the number of sign variations in a vector [7]. For $y \in \mathbb{R}^n$, let $s^-(y)$ denote the number of sign variations in the vector y after deleting all its zero entries, and let $s^+(y)$ denote the maximal possible number of sign variations in y after each zero entry is replaced by either $+1$ or -1. Note that $s^-(y) \leq s^+(y)$ for all $y \in \mathbb{R}^n$. An immediate yet important observation is that $\mathcal{V} = \{y \in \mathbb{R}^n : s^-(y) = s^+(y)\}$.

A classical result from the theory of TN matrices [7] states that if A is totally positive (TP) and $x \neq 0$ then $s^+(Ax) \leq s^-(x)$, whereas if A is TN then $s^-(Ax) \leq s^-(x)$.

To apply this SVDP to the stability analysis of (1) note that if the *transition matrix* corresponding to the LTV system (2) is TP for all time t then we may expect the number of sign variations in $z(t)$ to be a nonincreasing function of time.

In 1970, Schwarz [9] considered the linear matrix differential equation $\dot{Y}(t) = A(t)Y(t)$, $Y(t_0) = I$, with $A(t) \in \mathbb{R}^{n\times n}$ a continuous matrix function of t. Schwarz gave a formula for the induced dynamics of the minors of $Y(t)$. He defined the system as a TPDS [TNDS] if for every t_0 and every $t > t_0$ the matrix $Y(t)$ is TP [TN].[1] His analysis is based on what is now known as the theory of cooperative dynamical systems: the system is a totally nonnegative dynamical system if the dynamics maps any set of nonnegative minors to a set of nonnegative minors (note that any minor of $Y(t_0) = I$ is either zero or one and is thus nonnegative). However, the work of Schwarz seems to have been largely forgotten and its potential for the analysis of *nonlinear* dynamical systems has been overlooked.[2]

We provide here a review of TPDSs and show how they can be immediately linked to recent results on the stability of nonlinear cooperative systems. We also generalize the results of Schwarz to systems in the form $\dot{Y}(t) = A(t)Y(t)$, with $A(t)$ a measurable, rather than continuous, matrix function of time. We also show how this can be used to derive the interesting results of Smillie [2] and Smith [6] under milder technical conditions and with simpler proofs.

We use small [capital] letters to denote vectors [matrices]. I is the identity matrix, with dimension that should be clear from context. For a (column) vector $z \in \mathbb{R}^n$, z_i is the ith entry of z, and z' is the transpose of z. A square matrix B is called Metzler if every off-diagonal entry of B is nonnegative.

The next section reviews relevant definitions and results from the theory of TN matrices. Section III reviews TPDSs. Section IV shows how these results can be applied to analyze nonlinear time-varying tridiagonal strongly cooperative systems. Due to space limitations, many of the proofs and other details are omitted. These may be found in an extended version of this paper [10].

We believe that highlighting the deep and unknown connections between the work of Schwarz and more recent work on nonlinear tridiagonal cooperative systems opens the door to many new research directions.

II. TOTALLY NONNEGATIVE MATRICES

We briefly review known results from the rich and beautiful theory of TN and TP matrices that will be used later on. We consider only square and real matrices, as this is the case that is relevant for our applications. For more information and proofs we refer to the two excellent monographs [7], [8]. Unfortunately, this field suffers from nonuniform terminology. We follow the more modern terminology in [7].

Definition 1 *A matrix $A \in \mathbb{R}^{n\times n}$ is called totally nonnegative [totally positive] if the determinant of* every *square submatrix is nonnegative [positive].*

In particular, if a matrix is TN [TP] then every entry of the matrix is nonnegative [positive].

The product of two TN [TP] matrices is a TN [TP] matrix. This follows from the Cauchy-Binet formula for the minors of the product AB in terms of the minors of A and the minors of B [11, Ch. 0].

Certain matrices with a special structure are known to be TN. We review one such example.

Example 1 *A sufficient condition for the tridiagonal matrix*

$$\begin{bmatrix} a_1 & b_1 & 0 & \dots & 0 \\ c_1 & a_2 & \ddots & \dots & \vdots \\ 0 & \ddots & \ddots & \dots & \vdots \\ \vdots & \ddots & \ddots & \dots & b_{n-1} \\ 0 & \dots & \dots & c_{n-1} & a_n \end{bmatrix} \tag{3}$$

with $b_i, c_i \geq 0$ to be TN is the dominance condition

$$a_i \geq b_i + c_{i-1} \quad \text{for all } i \in \{1, \dots, n\}, \tag{4}$$

where we define $c_0 := 0$ and $b_n := 0$ [7, Ch. 0].

An important subclass of TN matrices are the *oscillatory matrices* studied in the pioneering work of Gantmacher and Krein [12] who analyzed the properties of an elastic segmental continuum under small transverse oscillations. A matrix $A \in \mathbb{R}^{n\times n}$ is called *oscillatory* if A is TN and there exists an integer $k > 0$ such that A^k is TP. A TN matrix A is oscillatory if and only if it is non-singular and irreducible [7, Ch. 2], and in this case A^{n-1} is TP. For

[1] We use a slightly different terminology than that used in [9].

[2] According to Goggle Scholar the paper [9] has been cited 22 times since its publication in 1970.

example, the matrix (3) with $b_i, c_i > 0$ and the dominance condition (4) is irreducible and TN. If it is also non-singular then it is oscillatory.

At this point we can already provide an intuitive explanation revealing the so far unknown connection between Smillie's results and oscillatory matrices. To do this, consider for simplicity the system $\dot{z} = Jz$, where J is a *constant* tridiagonal matrix with positive entries on the super- and sub-diagonals. Then

$$\begin{aligned} z(t) &= \exp(Jt)z(0) \\ &= \lim_{k\to\infty} (I + (Jt/k))^k z(0). \end{aligned}$$

Fix $t > 0$. Then for any k sufficiently large the matrix $(I + (Jt/k))$ is TN (by Example 1), irreducible, and nonsingular, so it is oscillatory.

A. Spectral properties of TN matrices

TN matrices have a strong spectral structure: all their eigenvalues are real and nonnegative, and the corresponding eigenvectors have special sign patterns. This spectral structure is particularly evident in the case of oscillatory matrices [13].

If $A \in \mathbb{R}^{n\times n}$ is an oscillatory matrix then its eigenvalues are all real, positive, and distinct. and thus can be ordered as $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$. If $u^k \in \mathbb{R}^n$ is the eigenvector corresponding to λ_k then for any $1 \le i \le j \le n$ and any $c_i, \dots, c_j \in \mathbb{R}$, that are not all zero,

$$i - 1 \le s^-(\sum_{k=i}^{j} c_k u^k) \le s^+(\sum_{k=i}^{j} c_k u^k) \le j - 1.$$

In particular, $s^-(u^i) = s^+(u^i) = i - 1$, $i = 1, \dots, n$.

B. Sign variation diminishing property

TN matrices enjoy a remarkable variety of mathematical properties. For our purposes, the most relevant property is that multiplication by a TN matrix cannot increase the sign variation of a vector.

Theorem 1 *[7, Ch. 4] If $A \in \mathbb{R}^{n\times n}$ is TN then*

$$s^-(Ax) \le s^-(x) \text{ for all } x \in \mathbb{R}^n. \tag{5}$$

In general, this cannot be strengthened to

$$s^+(Ax) \le s^+(x), \text{ for all } x \in \mathbb{R}^n. \tag{6}$$

For example, if J_3 is the 3×3 matrix with all entries equal to oneł and $x = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix}'$ then J_3 is TN, $s^+(x) = 1$ and $s^+(J_3 x) = s^+(\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}') = 2$. However, (6) does hold if A is an *invertible* TN matrix [7, Ch. 4].

TP matrices satisfy a stronger SVDP.

Theorem 2 *[7, Ch. 4] If $A \in \mathbb{R}^{n\times n}$ is TP then*

$$s^+(Ax) \le s^-(x), \text{ for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{7}$$

If A is TN and nonsingular then $s^+(Ax) \le s^-(x)$ for all $x \in \mathbb{R}^n$ such that either x has no zero entries or Ax has no zero entries.

A natural question is whether the property in (7) characterizes TP matrices. Recall that a matrix is called *strictly sign-regular* (SSR) if all minors of a given size are nonzero and share a common sign (that may vary from size to size). For example, $A = \begin{bmatrix} 1 & 2 \\ 3 & 1/2 \end{bmatrix}$ is SSR because all 1×1 minors are positive, and the single 2×2 minor is non-zero. Obviously, TP matrices are SSR.

Theorem 3 *[14] A nonsingular matrix $A \in \mathbb{R}^{n\times n}$ satisfies* (7) *if and only if it is SSR.*

C. Dynamics of compound matrices

Schwarz [9] studied the following question. Consider the matrix differential equation $\dot{Y}(t) = A(t)Y(t)$. What is the dynamics of some minor of Y? It turns out that we can express the dynamics of every $p \times p$ minor in terms of all the $p \times p$ minors of Y and the n^2 entries of A. To explain this, we review multiplicative and additive compound matrices and their role in analyzing differential equations [15].

Given a matrix $A \in \mathbb{R}^{n\times n}$ and an integer p, with $1 \le p \le n$, consider the $\binom{n}{p}^2$ minors of A of size $p \times p$. Each minor is defined by a set of row indexes $1 \le i_1 < i_2 < \cdots < i_p \le n$ and column indexes $1 \le j_1 < j_2 < \cdots < j_p \le n$. This minor is denoted by $A(\alpha|\beta)$ where $\alpha := \{i_1, \dots, i_p\}$ and $\beta := \{j_1, \dots, j_p\}$.

The pth *multiplicative compound matrix* $A^{(p)}$ is the $\binom{n}{p} \times \binom{n}{p}$ matrix that includes all these minors ordered lexicographically. For example, for $n = 3$ and $p = 2$ there are nine minors. The $(1, 1)$ entry of $A^{(2)}$ is $A(\{1, 2\}, \{1, 2\})$, entry $(1, 2)$ of $A^{(2)}$ is $A(\{1, 2\}, \{1, 3\})$, and so on.

An important property that follows from the Cauchy-Binet formula is $(AB)^{(p)} = A^{(p)}B^{(p)}$. This justifies the term multiplicative compound.

The pth *additive compound matrix* of A is defined by $A^{[p]} := \frac{d}{dh}(I + hA)^{(p)}|_{h=0}$. Applying Cauchy-Binet again gives $(I + hA)^{(p)}(I + hB)^{(p)} = (I + hA + hB + o(h))^{(p)}$, and this yields $(A + B)^{[p]} = A^{[p]} + B^{[p]}$, justifying the term additive compound.

The entry of $A^{[p]}$ corresponding to $(\alpha|\beta) = (i_1, \dots, i_p|j_1, \dots, j_p)$ is

$$\begin{cases} \sum_{k=1}^{p} a_{i_k j_k}, & |\alpha \cap \beta| = p, \\ (-1)^{\ell+m} a_{i_\ell j_m}, & |\alpha \cap \beta| = p - 1, i_\ell \ne j_m, \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

(see, e.g. [16]). The first line in (8) corresponds to the case where $i_\ell = j_\ell$ for all $\ell = 1, \dots, p$, and this corresponds to diagonal entries of $A^{[p]}$. The second line describes the case where all the indexes in α and β coincide except for a single index $i_\ell \ne j_m$.

The additive compound is important when studying the dynamics of the multiplicative compound. To see this, suppose that $\dot{Y} = AY$. Then

$$\begin{aligned} Y^{(p)}(t + h) &= (Y(t) + hA(t)Y(t) + o(h))^{(p)} \\ &= (I + hA(t) + o(h))^{(p)} Y^{(p)}(t), \end{aligned}$$

and this gives

$$\frac{d}{dt}Y^{(p)}(t) = A^{[p]}(t)Y^{(p)}(t). \tag{9}$$

Example 2 *For $p = 1$ we have $Y^{(1)} = Y$ and* (8) *yields $A^{[1]} = A$, so we obtain $\dot{Y} = AY$. For $p = n$, $Y^{(n)} = \det(Y)$, and* (8) *yields $A^{[n]} = \operatorname{trace}(A)$, so* (9) *yields $\frac{d}{dt}\det(Y(t)) = \operatorname{trace}(A(t))\det(Y(t))$, i.e. the Abel-Jacobi-Liouville identity.*

In view of the powerful structure of TN matrices, a natural question is: when will the transition matrix of a dynamical system be TN or TP and what will be the consequences of this?

III. Totally positive differential systems

Consider the system

$$\dot{Y}(s) = A(s)Y(s), \quad Y(t_0) = I, \tag{10}$$

with $A(s)$ a continuous function of t. Let $Y(t, t_0)$ denote the solution of (10) at time t. Schwarz [9] called (10) a TNDS if for every t_0 the solution $Y(t, t_0)$ is TN for all $t \geq t_0$, and a TPDS if for every t_0 the solution $Y(t, t_0)$ is TP for all $t > t_0$.

Schwarz combined the Peano-Baker representation for the solution of (9), and (8) to derive necessary and sufficient conditions for a system to be TNDS [TPDS]. Stated in modern terms, his analysis is based on the fact that (10) is TNDS [TPDS] iff (9) is a cooperative [strongly cooperative] dynamical system. Indeed, note that $Y(t_0) = I$ implies that $Y^{(p)}(t_0) = I$ for all p, so in particular all the minors at time t_0 are nonnegative.

Schwarz also studied the implications of TNDS/TPDS of (10) on the number of sign variations in a solution of the vector differential equation $\dot{z}(t) = A(t)z(t)$.

Fix a time interval (a, b) with $-\infty \leq a < b \leq \infty$. Consider the system

$$\frac{d}{dt}\Phi(t, t_0) = A(t)\Phi(t, t_0), \quad \Phi(t_0, t_0) = I, \tag{11}$$

with $a < t_0 \leq t < b$. We assume throughout a more general case than in [9], namely, that

$$A : (a, b) \to \mathbb{R}^{n\times n} \text{ is a matrix of locally (essentially) bounded measurable functions.} \tag{12}$$

Recall that (12) implies that (11) admits a unique, locally absolutely continuous, invertible solution for all $t \in (a, b)$ (see, e.g., [17, Appendix C]).

Recall that if x is the solution of $\dot{x} = f(x)$ then $z := \dot{x}$ satisfies the variational equation $\dot{z}(t) = J(x(t))z(t)$. This suggests that in order to study the evolution of the number of sign changes in $z(t)$ one needs to consider the LTV system $\dot{z}(t) = A(t)z(t)$ or, equivalently, the associated matrix differential equation (11). The formula $z(t) = \Phi(t, t_0)z(t_0)$ suggests that if $\Phi(t, t_0)$ is TP then $\sigma(z(t))$ will be no larger than $\sigma(z(t_0))$. The next result formulates this idea.

Theorem 4 *Consider the time-varying linear system:*

$$\dot{z}(t) = A(t)z(t), \tag{13}$$

with $A(t)$ satisfying (12) *and suppose that*

$$\Phi(t, t_0) \text{ is TP for all } a < t_0 < t < b. \tag{14}$$

If $z(t)$ is not the trivial solution $z(t) \equiv 0$ then
(1) the functions $s^-(z(t)), s^+(z(t))$ are non-increasing functions of time on (a, b);
(2) $z(t) \in \mathcal{V}$ for all $t \in (a, b)$, except perhaps for up to $n-1$ discrete values of t.

As we will see in Section IV, these properties are useful in the analysis of nonlinear ODEs.

Proof of Thm. 4. Pick t_0, t such that $a < t_0 < t < b$. Since $z(t) = \Phi(t, t_0)z(t_0)$ and $\Phi(t, t_0)$ is TP, the SVDP (7) yields

$$s^+(z(t)) \leq s^-(z(t_0)). \tag{15}$$

If $z(t_0) \in \mathcal{V}$ [$z(t_0) \notin \mathcal{V}$] then $s^-(z(t_0)) = s^+(z(t_0))$ [$s^-(z(t_0)) < s^+(z(t_0))$], so (15) yields $s^+(z(t)) \leq s^+(z(t_0))$ [$s^+(z(t)) < s^+(z(t_0))$]. Thus, $s^+(z(t))$ never increases, and it strictly decreases as $z(t)$ goes through a point that is not in $\mathcal{V}$. Since s^+ takes values in the finite set $\{0, 1, \ldots, n-1\}$, this implies that $z(t) \in \mathcal{V}$ for all t, except perhaps for up to $n-1$ discrete points. ∎

Thm. 4 implies that we may view $s^+(z(t))$ as an integer-valued Lyapunov function of the time-varying linear system (13). Since $z(t) \in \mathcal{V}$ for all t except perhaps for up to $n-1$ discrete time points, and $s^-(z(t)) = s^+(z(t)) = \sigma(z(t))$ when $z(t) \in \mathcal{V}$, we conclude that $s^-(z(t))$ and $\sigma(z(t))$ are also integer-valued Lyapunov functions.

The TP perspective allows to derive a converse to Thm. 4. Indeed, suppose that the solution of $\dot{z} = Az$ satisfies (15) for all $a < t_0 < t < b$ and all $z(t_0) \in \mathbb{R}^n \setminus \{0\}$. Since $z(t) = \Phi(t, t_0)z(t_0)$, Thm. 3 implies that $\Phi(s, r)$ is SSR for all $a < r < s < b$. Since the matrix $\Phi(t_0, t_0) = I$ has a minor of every size that is one, we conclude by continuity that $\Phi(t, t_0)$ is TP for all $t_0 < t$.

We now formally state the definitions of a TNDS and a TPDS for the system defined by (11) and (12).

Definition 2 *We say that* (11) *is a TNDS on (a, b) if for all $a < t_0 \leq t < b$ the matrix $\Phi(t, t_0)$ is TN. We say that* (11) *is a TPDS on (a, b) if for all $a < t_0 < t < b$ the matrix $\Phi(t, t_0)$ is TP.*

Example 3 *For $A(t) = \begin{bmatrix} 0 & t^2 \\ t^2 & 0 \end{bmatrix}$ the solution of* (11) *is $\Phi(t, t_0) = \begin{bmatrix} \cosh((t^3 - t_0^3)/3) & \sinh((t^3 - t_0^3)/3) \\ \sinh((t^3 - t_0^3)/3) & \cosh((t^3 - t_0^3)/3) \end{bmatrix}$. Every entry here is positive for all $t > t_0$ and $\det(\Phi(t, t_0)) \equiv 1$, so $\Phi(t, t_0)$ is TP on any interval (a, b). Thus the system is a TPDS on such an interval.*

For $A(t) = \begin{bmatrix} 0 & t^2 \\ 0 & 0 \end{bmatrix}$ the solution of (11) *is $\Phi(t, t_0) = \begin{bmatrix} 1 & (t^3 - t_0^3)/3 \\ 0 & 1 \end{bmatrix}$. This matrix is TN (but not TP) for all $t \geq t_0$, so the system is a TNDS on any interval (a, b).*

Since the product of TN [TP] matrices is a TN [TP] matrix, a sufficient condition for TNDS [TPDS] on (a, b) is that there exists $\varepsilon > 0$ such that $\Phi(t_0 + \varepsilon, t_0)$ is TN [TP] for all $a < t_0 < b - \varepsilon$.

The next step is to state conditions on $A(t)$ guaranteeing that (11) is a TNDS or a TPDS. A closely related question has already been addressed in 1955 by Loewner [18] who studied the infinitesimal generators of the group of TN matrices. It is useful to first consider the case of a constant matrix. Let $\mathbb{M} \subset \mathbb{R}^{n\times n}$ denote the set of tridiagonal matrices with nonnegative entries on the sub- and super-diagonal. Let $\mathbb{M}^+ \subset \mathbb{M}$ denote the set of tridiagonal matrices with positive entries on the sub- and super-diagonal.

The next result provides a necessary and sufficient condition for (11), with A a *constant* matrix, to be TNDS/TPDS.

Theorem 5 *[9] The system $\dot{U}(t) = AU(t)$ is TNDS [TPDS] on any interval (a, b) if and only if $A \in \mathbb{M}$ [$A \in \mathbb{M}^+$].*

The proof in [9] is based on what is now known as the theory of cooperative systems. To explain the basic idea, pick $1 \le p \le n$ and consider the induced dynamics for the $p \times p$ minors of $U(t)$. Recall that this is given by

$$\dot{U}^{(p)} = A^{[p]}U^{(p)}, \quad U^{(p)}(0) = I, \tag{16}$$

where $\dot{U}^{(p)} := \frac{d}{dt}(U^{(p)}(t))$, and $A^{[p]}$ is given in (8). Schwarz [9] showed that if $A \in \mathbb{M}$ then $A^{[p]}$ is Metzler, so (16) is a cooperative system. Since all the entries of $U^{(p)}(0) = I$ are non-negative, this implies that all the $p\times p$ minors remain non-negative for all $t \ge 0$. If $A \in \mathbb{M}^+$ then $A^{[p]}$ is Metzler and irreducible and it follows that every entry of $U^{(p)}$ is positive for all $t > 0$.

We now turn to consider the time-varying case.

Theorem 6 *The system* (11) *with* $A(t)$ *satisfying* (12) *is TNDS on (a, b) iff $A(t) \in \mathbb{M}$ for almost all $t \in (a, b)$.*

The proof of this follows from arguing as in the proof of Thm. 5 and using a known necessary and sufficient condition for positivity of the system $\dot{x}(t) = B(t)x(t)$, with $B(t)$ Metzler for almost all t (see, e.g., [19]).

The next result provides a sufficient condition for TPDS.

Theorem 7 *Suppose that $A(t) \in \mathbb{M}^+$ for almost all $t \in (a, b)$ and, furthermore, that $a_{ij}(t) \ge \delta > 0$ for all $|i-j| = 1$ and almost all $t \in (a, b)$. Then* (11) *is TPDS on (a, b).*

In the special case where $A(t)$ is continuous, we conclude that a necessary and sufficient condition for TPDS is that $A(t) \in \mathbb{M}$ for all t and every off-diagonal entry of $A(t)$ does not vanish on an interval of time.

The next step is to explain how the notion of a linear TPDS can be applied to analyze nonlinear dynamical systems.

IV. Applications to stability analysis

Consider the nonlinear time-varying dynamical system

$$\dot{x}(t) = f(t, x(t)), \tag{17}$$

whose trajectories evolve on an invariant, compact, and convex set $\Omega \subset \mathbb{R}^n$, that is, for any $x_0 \in \Omega$ and any $t_0 \ge 0$ a unique solution $x(t, t_0, x_0)$ exists and satisfies $x(t, t_0, x_0) \in \Omega$ for all $t \ge t_0$. From here on we take $t_0 = 0$. We also assume that the Jacobian $J(t, x) = \frac{\partial}{\partial x} f(t, x)$ exists for all $t \ge 0$ and $x \in \Omega$.

We consider the case where f is T-periodic, that is, $f(t, z) = f(t + T, z)$ for all $t \ge 0$ and all $z \in \Omega$. Note that in the particular case where f is time-invariant this property holds for all T.

We also assume that for any $t \ge 0$ and along any line $\gamma : [0, 1] \to \Omega$ the matrix

$$A(t) := \int_0^1 J(t, \gamma(r))\, \mathrm{d}r \tag{18}$$

is well-defined, locally (essentially) bounded, measurable, and satisfies the conditions for TPDS. Note that since f is T-periodic, so is $A(t)$.

Under these assumptions, we now state the main stability result that generalizes a result of Smith [6].

Theorem 8 *Every solution of* (17) *converges to a periodic trajectory with period T.*

If the vector field satisfies $f(t, x) = h(x, u)$, with $u(t)$ a T-periodic input, one may view u as a periodic excitation. Thm. 8 implies that every solution converges to a periodic solution with the same period as the excitation, that is, the system *entrains* to the excitation. Entrainment is important in many natural and artificial systems from biology [20], [21], [22], physics and chemistry [23], and more.

To prove Thm. 8 we require the following "eventual monotonicity" result.

Lemma 1 *Pick $a, b \in \Omega$, with $a \ne b$, and consider the solutions $x(t, a)$, $x(t, b)$ of* (17)*. There exists a time $s \ge 0$ such that for all $t \ge s$ either $x_1(t, a) > x_1(t, b)$ or $x_1(t, a) < x_1(t, b)$.*

Proof. Let $z(t) := x(t, a) - x(t, b)$. Then $\dot{z}(t) = A(t)z(t)$, with $A(t)$ defined in (18), and $\gamma(r) := rx(t, a) + (1 - r)x(t, b)$, with $r \in [0, 1]$. By assumption, this LTV system is TPDS, so Thm. 4 yields $z(t) \in \mathcal{V}$ for all t except for up to $n - 1$ discrete time points τ_i. Pick $s > \max_i \tau_i$. Then the definition of $\mathcal{V}$ implies that $z_1(t) \ne 0$ for all $t \ge s$. ■

We can now prove Thm. 8. Pick $a \in \Omega$. If the solution $x(t, a)$ of (17) is T-periodic then there is nothing to prove. Thus, suppose that $x(t, a)$ is not T-periodic. Then $x(t + T, a)$ is another solution of (17) that is different from $x(t, a)$. Lemma 1 implies that without loss of generality we may conclude that there exists an integer $m \ge 0$ such that

$$x_1(kT, a) - x_1((k + 1)T, a) > 0 \text{ for all } k \ge m. \tag{19}$$

Define the Poincaré map $P_T : \Omega \to \Omega$ by $P_T(y) := x(T, y)$. Then P_T is continuous, and for any integer $k \ge 1$ the k-times composition of P_T satisfies $P_T^k(y) = x(kT, y)$.

Define the omega limit set $\omega_T : \Omega \to \Omega$ by $\omega_T(y) := \{z \in \Omega :$ there exists a sequence $n_1, n_2, \dots$ with $n_k \to \infty$ and $\lim_{k\to\infty} P_T^{n_k}(y) = z\}$. This set is not empty (recall that we assume that Ω is compact), and invariant under P_T, that is, $P_T(\omega_T(y)) = \omega_T(y)$. In particular, if $\omega_T(y) = \{q\}$ then $P_T(q) = q$, that is, the solution emanating from q is T-periodic. Thus, to prove Thm. 8 we need to show that $\omega_T(a)$ is a singleton. Seeking a contradiction, assume that this is not the case, i.e. there exist $p, q \in \omega_T(a)$ with $p \neq q$. This means that there exist integer sequences $n_k \to \infty$ and $m_k \to \infty$ such that $\lim_{k\to\infty} x(n_kT, a) = p$, $\lim_{k\to\infty} x(m_kT, a) = q$. Combining this with (19) yields $p_1 = q_1$. We conclude that any two points in $\omega_T(a)$ have the same first entry. Consider the solutions emanating from p and from q at time zero, that is, $x(t, p)$ and $x(t, q)$. We know that there exists an integer $\ell \geq 0$ such that, say, $x_1(kT, p) - x_1(kT, q) > 0$ for all $k \geq \ell$. However since $p, q \in \omega_T(a)$, we have $x(kT, p), x(kT, q) \in \omega_T(a)$ for all k, so $x_1(kT, p) = x_1(kT, q)$ for all k. This contradiction completes the proof of Thm. 8. ■

The *time-invariant* nonlinear dynamical system:

$$\dot{x}(t) = f(x(t)) \tag{20}$$

is T-periodic for all $T > 0$, so Thm. 8 yields the following result that generalizes Smillie's theorem [2].

Corollary 1 *Suppose that the solutions of* (20) *evolve on an invariant compact and convex set* $\Omega \subset \mathbb{R}^n$*; that* $f \in C^1$*; and that the matrix* $J(x) := \frac{\partial}{\partial x} f(x) \in \mathbb{M}^+$ *for all* $x \in \Omega$*. Then for every* $x_0 \in \Omega$ *the solution* $x(t, x_0)$ *converges to an equilibrium point.*

V. Conclusions

TN and TP matrices enjoy a rich set of powerful properties and have found applications in numerous fields. A natural question is when is the transition matrix of the linear dynamical system $\dot{z} = Az$ TN or TP and what are the implications of this for $z(t)$? This question has been solved by Schwarz [9] yielding the notion of TNDS and TPDS. One important property of such systems is that for any vector solution $z(t)$ the number of sign variations $\sigma(z(t))$ is non-increasing with time. His approach is based on what is now known as cooperative systems theory: a system is TNDS [TPDS] if all the minors of the transition matrix, that are all either zero or one at the initial time t_0, are non-negative [positive] for all $t > t_0$. However, the seminal work of Schwarz has been largely forgotten, perhaps because he did not show how to apply these results to analyze *nonlinear* dynamical systems.

More recently, $\sigma(z(t))$, where $z(t) = \dot{x}(t)$, has been used by several authors as an integer-valued Lyapunov function for a nonlinear dynamical system. In these works, the fact that $\sigma(z(t))$ is non-increasing with time has been proved by a direct and tedious analysis.

Here, we reviewed these seemingly different lines of research and showed that the system describing the evolution of z (i.e., the variational system) is in fact TPDS. This allows to generalize several known results, while greatly simplifying the proofs. The new connection between these research fields may lead to many new and interesting research directions.

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References

[1] H. L. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, ser. Mathematical Surveys and Monographs. Providence, RI: Amer. Math. Soc., 1995, vol. 41.

[2] J. Smillie, "Competitive and cooperative tridiagonal systems of differential equations," *SIAM J. Mathematical Analysis*, vol. 15, pp. 530–534, 1984.

[3] M. Margaliot and T. Tuller, "Stability analysis of the ribosome flow model," *IEEE/ACM Trans. Comput. Biol. Bioinf.*, vol. 9, pp. 1545–1552, 2012.

[4] L. O. Chua and T. Roska, "Stability of a class of nonreciprocal cellular neural networks," *IEEE Trans. Circuits and Systems*, vol. 37, no. 12, pp. 1520–1527, 1990.

[5] P. Donnell, S. A. Baigent, and M. Banaji, "Monotone dynamics of two cells dynamically coupled by a voltage-dependent gap junction," *J. Theoretical Biology*, vol. 261, no. 1, pp. 120–125, 2009.

[6] H. L. Smith, "Periodic tridiagonal competitive and cooperative systems of differential equations," *SIAM J. Math. Anal.*, vol. 22, no. 4, pp. 1102–1109, 1991.

[7] S. M. Fallat and C. R. Johnson, *Totally Nonnegative Matrices*. Princeton, NJ: Princeton University Press, 2011.

[8] A. Pinkus, *Totally Positive Matrices*. Cambridge, UK: Cambridge University Press, 2010.

[9] B. Schwarz, "Totally positive differential systems," *Pacific J. Math.*, vol. 32, no. 1, pp. 203–229, 1970.

[10] M. Margaliot and E. D. Sontag, "Revisiting totally positive differential systems: A tutorial and new results," *Automatica*, 2018, to appear. [Online]. Available: https://arxiv.org/abs/1802.09590

[11] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed. Cambridge University Press, 2013.

[12] F. R. Gantmacher and M. G. Krein, *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems*. Providence, RI: American Mathematical Society, 2002, translation based on the 1941 Russian original.

[13] A. Pinkus, "Spectral properties of totally positive kernels and matrices," in *Total Positivity and its Applications*, M. Gasca and C. A. Micchelli, Eds. Dordrecht: Springer Netherlands, 1996, pp. 477–511.

[14] T. Ando, "Totally positive matrices," *Linear Algebra Appl.*, vol. 90, pp. 165–219, 1987.

[15] J. S. Muldowney, "Compound matrices and ordinary differential equations," *The Rocky Mountain J. Math.*, vol. 20, no. 4, pp. 857–872, 1990.

[16] M. Fiedler, *Special Matrices and Their Applications in Numerical Mathematics*, 2nd ed. Mineola, NY: Dover Publications, 2008.

[17] E. D. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, 2nd ed. New York: Springer, 1998.

[18] C. Loewner, "On totally positive matrices," *Mathematische Zeitschrift*, vol. 63, no. 1, pp. 338–340, 1955.

[19] D. Angeli and E. D. Sontag, "Monotone control systems," *IEEE Trans. Automat. Control*, vol. 48, pp. 1684–1698, 2003.

[20] G. Russo, M. di Bernardo, and E. D. Sontag, "Global entrainment of transcriptional systems to periodic inputs," *PLOS Computational Biology*, vol. 6, p. e1000739, 2010.

[21] M. Margaliot, E. D. Sontag, and T. Tuller, "Entrainment to periodic initiation and transition rates in a computational model for gene translation," *PLoS ONE*, vol. 9, no. 5, p. e96039, 2014.

[22] N. C. Grassly and C. Fraser, "Seasonal infectious disease epidemiology," *Proc. Royal Society B: Biological Sciences*, vol. 273, pp. 2541–2550, 2006.

[23] M. Margaliot, L. Grüne, and T. Kriecherbauer, "Entrainment in the master equation," *Royal Society Open Science*, vol. 5, no. 4, p. 171582, 2018.