Lectures in Applied Mathematics
Volume 18, 1980

AN APPROACH TO DETECTABILITY AND OBSERVERS

M. L. J. Hautus* and Eduardo D. Sontag**

ABSTRACT. This paper proposes an approach to the problem of establishing the existence of observers for deterministic dynamical systems. This approach differs from the standard one based on Luenberger observers in that the observation error is not required to be Markovian given the past input and output data. A general abstract result is given, which specializes to new results for parametrized families of linear systems, delay systems and other classes of systems. Related problems of feedback control and regulation are also studied.

1. INTRODUCTION. An *observer* for a given dynamical system Σ is, roughly, a system $\hat{\Sigma}$ which accepts as input the inputs $u(\cdot)$ and outputs $y(\cdot)$ of Σ, and whose output $\hat{x}(t)$ at time t asymptotically approaches the internal state $x(t)$ of the original system, whatever $x(0)$ was. The definition may require that this convergence occur for any initial state $z(0)$ of $\hat{\Sigma}$, or just for a fixed state, say $z(0) = 0$. A reasonable design requirement is that $\hat{\Sigma}$ be somehow stable itself. For a linear observer $\hat{\Sigma}$ this will imply that Σ will remain an observer for any initial state $z(0)$. The definitions of "asymptotically," "stable," etc. will, of course, have to be

*This research was completed while the author was occupying the Kranzberg Chair in Electronics at the Department of Electrical Engineering, Technion, Haifa.

**Supported by U.S. Air Force Grant AFOSR - F49620-79-C-0117.

made precise in the different contexts. A stronger design objective usually adds that observers be obtained with given rates of convergence of $\hat{x}(t) - x(t)$ to zero; more generally, one may ask for specific dynamics for the observation error $e(t) = \hat{x}(t) - x(t)$. When Σ and $\hat{\Sigma}$ are linear systems, the error convergence rates are independent of $u(\cdot)$; in a nonlinear context the situation is more delicate, and when rates are independent of the particular input in a fixed set, one speaks of "uniform" observers with respect to that set. It is also of interest to determine when an observer $\hat{\Sigma}$ exists which is of the same "type" as $\hat{\Sigma}$; for example, if Σ is a delay-differential system with all delays multiples of, say, $\tau_1,\ldots,\tau_r$, one would like to study the existence of observers $\hat{\Sigma}$ which are also delay-differential with all delays multiples of the τ_i.

One may also consider observers for a parametrized family of systems Σ_λ. Indeed, assume that each Σ_λ is a system of a given kind and that the Σ_λ are finitely parametrized by a (vector) parameter λ; for example the coefficients defining Σ_λ may be polynomial in λ or rational functions (with no poles). It is natural then to ask for a corresponding family of observers $\hat{\Sigma}_\lambda$, with the $\hat{\Sigma}_\lambda$ also finitely parametrized and if possible using the same kind of functions (polynomials, rational functions). This situation appears when a design is desired previous to the identification of certain parameters, or when these parameters are subject to change.

The standard (textbook) approach to observer design for linear systems is based on the "Luenberger observer" or "deterministic Kalman filter"; in this case one searches for observers in which the error $e(t)$ given the past input/output data is Markovian. In other words, attention is restricted to observers $\hat{\Sigma}$ whose measurement function is the identity: $\hat{x}(t) = z(t)$. Because of the success of this approach, the study of more general linear situations (delay and other distributed linear systems, family of systems, and multidimensional filters), as well as the study of nonlinear observers, has been directed towards such Luenberger-like observers. This has run into a number of highly technical difficulties, for example, those dealing with the extensions of the pole-shifting theorem to systems over rings, as well as necessitating strong algebraic notions of observability. Recent counter-examples [Bumby et al., 1979] show that this approach may in general fail to work.

This paper de-emphasizes "Luenberger" observers through a direct construction methodology for more general observers. It appears that the approach presented here is new even in the "classical" linear finite dimensional case. *The central fact is that while the standard observer construction is an inherently nonlinear problem, the construction of more general observers can be posed in many cases as a linear problem once "linearity"* is properly interpreted. We shall present a series of abstract results which characterize the existence of observers, (with arbitrary or with fixed rates of convergence), in the context of linear systems over rings. Necessary and sufficient conditions are given in terms of corresponding notions of detectability. These results will be then specialized to delay systems and to families of systems. In the first case we shall obtain observers that have a delay structure similar to that of the original system and, it appears from examples, a simpler structure than that obtained when other methods in the literature are applicable. In the case of families of systems, the results will be basically that a polynomial or rational family, when each system is observable, admits a similarly parametrized family of observers. Although the results are in principle linear, we shall point out potential applications to bilinear and other state-affine systems.

Some of the above results admit dualizations into statements about (dynamic) state-feedback controllers. We shall present a result along these lines, as well as a partial result on the solution of the regulator problem. Applied for examples to delay-differential systems, the latter permits the input/output regulation of a wide class of transfer functions, using delay systems of the same type, under conditions very much weaker than those obtained previously ("split realization" construction).

The next section will present the basic definitions and some preliminary results, while section 3 describes and proves the main ("detectability is equivalent to existence of observers") result. After that, we specialize to families of systems and delay systems, and briefly treat the regulator problem. Various open problems are posed through the paper.

2. DEFINITIONS AND PRELIMINARIES. Our approach in this paper will rely heavily upon the theory of linear systems over rings, as developed for example in Rouchaleau [1972], Rouchaleau et al. [1974], Kamen [1975], Sontag [1976], Kamen [1978], Rouchaleau and Sontag [1979], and others. Since we want our results to apply in general, including to both discrete- and continuous-time systems, we shall work formally with transfer functions being rational in a symbol "s", which will correspond to either the Laplace or the z-transform variable, depending on the applications. Similarly, systems will be just formal objects identified with their defining matrices. The various manipulations with transfer matrices can be made rigorous with respect to the different applications, in terms of Laplace or z-transforms, or in terms of operators in function spaces.

Unless otherwise stated, R will denote an arbitrary but fixed integral domain with an identity element; $R[s]$ will be the ring of polynomials in the indeterminate $s \cdot$ A (causal) *transfer function* $w(s)$ is just a rational function $p(s)q(s)^{-1}$ with p,q in $R[s]$, q monic in s, and $\deg q \geq \deg p$; $w(s)$ is called *strictly causal* when the latter inequality is strict. A [strictly] causal transfer matrix $W = (w_{ij})$ is a matrix for which each entry is a [strictly] causal transfer function. A *system* (over R, of dimension n, with m inputs and p outputs) is a quadruple $\Sigma = (F,G,H,J)$, where F is an $n \times n$, G an $n \times m$, H a $p \times n$, and J a $p \times m$ matrix.

The intuitive interpretation of a system is given by the equations

$$\begin{aligned} sx &= Fx + Gu \\ y &= Hx + Ju \end{aligned} \tag{2.1}$$

where the x's play the role of "state variables," u of inputs and y of outputs. Sometimes we refer to the equations (2.1) rather than to the quadruple (F,G,H,J) as the system Σ. When G, H, or J are irrelevant we shall refer to the "system" (F,H), or (F,G), etc., the meaning being clear from the context. A system is called *strictly causal* if $J = 0$.

It is well known from the theory of linear systems over rings (see e.g., Eilenberg [1974, Chapter XVI]) that a causal W admits a *realization*, i.e., a system $\Sigma = (F,G,H,J)$ such that $W = H(sI - F)^{-1}G + J$. (All operations are performed for matrices over the ring of rational functions $R(s)$). In fact, and this will be of importance later, if q is a common denominator for the entries of W, then a realization can be obtained for which the characteristic polynomial of F is a power of q. (Thus in applications, q having "no unstable zeroes" will insure a suitable "internal stability" for at least one realization.) Of course, many realizations are possible for a given W, but this need not concern us here.

Intuitively, if a system Σ given by (2.1) realizes W, then elimination of x in (2.1) yields $y = Wu$, i.e., W is the transfer matrix from u to y of Σ, and hence characterizes the input/output behavior of Σ. Conversely, given a Σ as in (2.1), there is a well-defined causal transfer matrix $W_\Sigma := H(sI - f)^{-1}G + J$. Strict causality of (2.1) corresponds to W_Σ being strictly causal.

Recall that a linear finite-dimensional continuous-time system (F,G,H) is *detectable* when all its unstable modes are observable or (see Hautus [1969, 1970]) equivalently when

$$\operatorname{rank} \begin{pmatrix} sI - F \\ H \end{pmatrix} = n \tag{2.2}$$

for all complex s with $\mathrm{Re}\, s \geq 0$. For discrete-time systems this is replaced by (2.2) for $|s| \geq 1$. It is well known that for such systems (2.2) is equivalent to the existence of observers. More generally, one may call (F,G,H) *Ω-detectable*, where Ω is a subset of $\mathbb{C}$, if the span of the generalized s-eigenvectors, for s in Ω, contains no state indistinguishable from zero, or equivalently if (2.2) holds for all s in Ω. When Ω is the set of s with $\mathrm{Re}\, s \geq -a$ ($|s| \geq a$ for discrete time), Ω-detectability is equivalent to the existence of observers with convergence rates better than a (as e^{-bt} in continuous time, or b^t in discrete time, for any $b > a$). *Observability* is equivalent to detectability with arbitrary rates, i.e., (2.2) for all $s \in \mathbb{C}$.

We now want to generalize the above concepts. Although a definition of detectability is possible for arbitrary commutative rings R, we shall state it only for rings R which are rings of rational functions over a field k. This restriction will simplify matters considerably and will suffice for the applications to follow. Assume then that R is a subring of $k(\sigma_1,\ldots,\sigma_r)$, with k a field and the σ_i algebraically independent over k. Let K be an algebraically closed extension field of k. Let Ω be a subset of K^{r+1}. We shall say that a system $\Sigma = (F,G,H,J)$ over R, or just the pair (F,H), is *Ω-detectable* if and only if

$$\mathrm{rank}\begin{pmatrix} \bar{s} - F(\bar{\sigma}) \\ \\ H(\bar{\sigma}) \end{pmatrix} = n \qquad (2.3)$$

for all $(\bar{s},\bar{\sigma}) \in \Omega$, where $\bar{\sigma} = (\sigma_1,\ldots,\sigma_r)$. The definition implicitly assumes that Ω is *admissible*, i.e., that these evaluations are always defined on R. (Note that we use bars to denote particular elements $(\bar{s},\bar{\sigma})$, *not* complex conjugation.) For the more standard finite-dimensional linear system case, k above is either $\mathbb{R}$ or $\mathbb{C}$, $r = 0$ and $K = \mathbb{C}$, so that Ω is a subset of $\mathbb{C}$ itself. In most examples of interest it will turn out that we may choose $K = k_c$, the algebraic closure of k.

We shall say that Σ is (*strongly* or *algebraically*) *observable* if

$$P' := [H', F'H', \ldots, F'^{n-1}H']$$

has a right inverse over R. (Prime indicates transpose.) This concept of observability is not to be confused with the much weaker condition of observability that appears in realization theory over rings, namely that P' have rank n over the quotient field of R. Rather, this condition of (strong) observability is equivalent to (F', H') being (ring-)reachable.

Observability and detectability are related according to the following:

(2.4) PROPOSITION. (i) *If* Σ *is observable then* Σ *is* Ω*-detectable for all admissible* Ω *(and all* K*).* (ii) *If* K *is a universal field over* k *(i.e., an algebraically closed infinite transcendental extension of* k*), and* Σ *is* Ω*-detectable for all admissible* Ω*, then* Σ *is observable.*

Proof. (i) Let $(\bar{s}, \bar{\sigma})$ be in some Ω. Then $P'(\bar{\sigma})$ is right invertible over the field K and hence (see Hautus [1969]) (2.3) is true for all $\bar{s} \in K$.

(ii) If Σ is not observable then there is a maximal ideal M such that P' reduced mod M is not right invertible over R/M. But R being a k-algebra implies that R/M is a (finitely generated) field extension of k and hence that it can be embedded in K. Reducing mod M becomes then evaluation at some point of K^r. Thus for some $\bar{\sigma}$ in K^r, $P'(\bar{\sigma})$ is not right invertible, and, again by the results over fields, there is an $\bar{s}$ such that (2.3) fails. So, (F,H) is not $\{(\bar{s},\bar{\sigma})\}$-detectable, contrary to the assumption. ⌑

Turning now towards the definition of observer, we first need to define a property corresponding to convergence, or equivalently for our purposes, to stability. We do this by choosing a family of "stable polynomials." A *stability set* S will be any multiplicative subset of R[s] which consists entirely of monic polynomials, contains at least one linear monomial $s + \alpha$,

α in R, and which is saturated, i.e. such that $pq \in S$ and p and q monic implies that both p and q are in s. [An example to keep in mind is S = set of monic polynomials over $\mathbb{R}$, or over $\mathbb{C}$, with no roots with real part greater than -a, for some fixed real number a.] With respect to such a fixed S, a *stable transfer function* is one that admits a representation pq^{-1} with q in S; a stable transfer *matrix* has each entry stable. Thus W is stable if and only if it admits some realization (F,G,H,J) which is *stable*, i.e., such that the characteristic polynomial of F is in S.

Let $\Sigma = (F,G,H,J)$ be an n-dimensional system with m inputs and p outputs. An observer $\hat{\Sigma} = (A,(B_1,B_2), C, (D_1,D_2))$ for Σ (relative to a given stability set S) will be a stable system (over the same ring R) with m + p inputs and n outputs such that, solving formally the equations

$$
\begin{aligned}
sx &= Fx + Gu + v \\
y &= Hx + Ju \\
sz &= Az + B_1u + B_2y \\
\hat{x} &= Cz + D_1u + D_2y
\end{aligned}
\tag{2.5}
$$

leads to an equation

$$\hat{x} - x = M(s)v , \tag{2.6}$$

for some stable transfer matrix M. (The phrase "solving formally" may be rigorously interpreted by assuming that x,u,v, etc., are vectors of independent variables over R[s], and operating in a suitable extension of this ring; we shall be doing this implicitly.)

Several comments are in order. The definition of observer sketched in the introduction involved initial states and convergence of $e = \hat{x} - x$ to zero. It is not reasonable, however, to introduce initial states in our context, since our setup will specialize into areas where values of x *do not* correspond to true states, in that knowledge of such values at a given time does not determine a unique solution of the equation for future times. (For example, in the case of delay systems the natural "initial states" are in appropriate function spaces; more generally, when R is a ring of operators states may belong to sets quite unrelated to R itself.) "Solving formally" translates in such applications into assuming that all variables are zero for sufficiently negative times. The concept of initial state can be replaced by the addition of a "disturbance input" v whose effect on Σ is what the observer tries to determine. (An independent "disturbance input" w could be added to the observer (z) equation in (2.5) to take into account the effect of initial states of $\hat{\Sigma}$, but stability of $\hat{\Sigma}$ insures that the effect of w on e will be only through a stable transfer matrix, and hence will not effect observer performance.) Finally, we shall restrict attention to strictly causal Σ $(J = 0)$ and observers with $D_1 = 0$: if $\hat{\Sigma}$ observes $(F,G,H,0)$ then adding a "$+Ju$" term to $\hat{x}$ gives an observer for (F,G,H,J).

We shall be interested also in questions of feedback control. A *reachable* system has the matrix $[G,FG,\ldots,F^{n-1}G]$ right invertible over R. This is again equivalent to a dual condition to (2.3), since reachability corresponds to observability of (F',G'). Again with reference to a stability set S, a *feedback controller* for a system Σ will be a system $\Sigma_1 = (A,B,C,D)$ with n inputs and m outputs such that solving formally the equations

$$\begin{aligned} sx &= Fx + Gu + v \\ sz &= Az + Bx \\ y &= Cz + Dx \\ u &= -y \end{aligned} \tag{2.7}$$

results in $x = W(s)v$ and $z = (sI - A)^{-1}BW(s)v$, where $W(s)$ and $(sI - A)^{-1}BW(s)$ are both stable transfer matrices. Thus a feedback controller Σ_1 will, in applications, when started with $z = 0$, force the state of Σ to zero while keeping its own state-variable "small." The problem of finding feedback controllers will be dual to that of finding observers with a special structure.

Finally, we define what we mean by a *regulator* (with respect to a given S) for a transfer matrix W, or for a realization Σ of W: this is a stable Σ_2 which accepts inputs and *outputs* of Σ as inputs, and such that, as in (2.7) the state of Σ becomes for the closed-loop system, a stable function of disturbances. A regulator for W is by definition a regulator for *some* realization of W.

3. GENERAL RESULTS. We shall first investigate the structure of the ring $Tr(S)$ of transfer functions stable with respect to a fixed stability set S. Recall that there is an α in R with $s + \alpha$ in S. Working in the quotient field $Q(s)$ of $R[s]$, consider the field automorphism π induced by the evaluation

$$\pi : s \mapsto s^{-1} - \alpha.$$

For each

$$q(s) = s^n + q_{n-1}s^{n-1} + \dots + q_0$$

in S, we denote

$$\tilde{q}(s) := s^n \cdot q(s^{-1} - \alpha)$$

and let T be the subset of $R[s]$ consisting of all such $\tilde{q}$. We claim that

$$\pi(Tr(S)) = T^{-1}(R[s]) . \qquad (3.1)$$

(The right-hand term denotes the ring of fractions with respect to T.) It is clear that $\pi(Tr(S))$ is included in $T^{-1}(R[S])$; to prove the converse it will be enough to see that the monomial s and every $\tilde{q}^{-1}$ are in the image of π. But $s = \pi((s+\alpha)^{-1})$, and for q as above,

$$\pi((s + \alpha)^n q^{-1}) = \tilde{q}^{-1} ,$$

as wanted.

Thus $Tr(S)$ is isomorphic to a ring of fractions of a polynomial ring. This representation helps in characterizing the maximal ideals of the former. Indeed, by standard results in ring theory, the set of maximal ideals M of $T^{-1}(R[s])$ is in a one-to-one correspondence with the set of those prime ideals P in $R[s]$ which are maximal with respect to the property of not intersecting T. Moreover, the corresponding quotient fields satisfy

$$T^{-1}(R[s])/M \simeq Q(R[s]/P).$$

and the respective residue maps coincide on $R[s]$. In other words, reduction modulo maximal ideals of $T^{-1}(R[s])$ corresponds to homomorphisms ("evaluations")

$$\gamma : R[s] \to E$$

into fields E such that $\ker \gamma$ is maximal with respect to not intersecting T. Such maps can be extended canonically to $T^{-1}(R[s])$. The corresponding evaluations for the original ring $Tr(S)$ are then obtained by composition through π. (The geometric interpretation of all this for R a polynomial ring will be discussed in the next section.)

A residue map on $Tr(S)$ is then obtained as $\gamma \cdot \pi$, with γ as above. We would like to be able to express such a composite map in terms of R and s directly. But a map like this may not admit an extension which includes $R[s]$, since $Tr(S)$ does not contain s (the latter is not a causal transfer function). An extension will exist precisely when

$$\gamma(s) = (\gamma \cdot \pi)(s+\alpha)^{-1} =: a \neq 0 , \tag{3.3}$$

This is an obviously necessary condition, and we prove it is also sufficient. Indeed, assume that $\gamma \cdot \pi$ maps $Tr(S)$ into a field E and (3.3) holds. Let ε be the restriction of γ to R; this is the same as the restriction of $\gamma \cdot \pi$ to R, because π leaves R invariant. Since s is independent over R, we may extend ε to:

$$\varepsilon : R[s] \to Q, \quad \varepsilon(s) := a^{-1} - \varepsilon(\alpha)$$

By the theorem on extension of places, (see e.g., Bourbaki [1972 VI. 2.4]) there is a common extension ε' of ε and $\gamma \cdot \pi$ to a subring of $Q(s)$ containing both $R[s]$ and $Tr(S)$. Thus $(\gamma \cdot \pi)(s)$ has a well-defined meaning, since ε' is uniquely determined by $\gamma \cdot \pi$.

Thus the evaluations at maximal ideals of $Tr(S)$ can be classified into two types: those that are zero at $(s+\alpha)^{-1}$ and those that are a fortiori defined on s. We let $Max(S)$ be the *set of maximal ideals corresponding to the latter kind of map*. When, as in the previous section, R is a subring of $k(\sigma_1,\ldots,\sigma_r)$, a maximal ideal of $T^{-1}(R[s])$ corresponds to an evaluation at a point $(\bar{s},\bar{\sigma})$ in K^{r+1} (K universal). The point $(\bar{s},\bar{\sigma})$ is uniquely determined from M modulo the Galois group of K over k. We denote by Ω_T a representative set of points $(\bar{s},\bar{\sigma}) \in K^{r+1}$ corresponding to maximal ideals M in $Max(S)$, and define (for this choice):

$$\Omega(S) := \{((\bar{s}+\alpha(\bar{\sigma}))^{-1},\bar{\sigma}) \mid (\bar{s},\bar{\sigma}) \in \Omega_T\}. \tag{3.4}$$

(*Representative* meaning that each maximal M appears as an evaluation at some such point.)

For instance, in the classical case, say for discrete-time, S is the set of real polynomials with no zeros s with $|s| \geq 1$, α can be taken to be zero, and T is the set of polynomials with no roots with $|s| \leq 1$. The residue maps for $T^{-1}(R[s])$ are thus the evaluations at such s, and Ω_T can be naturally taken as corresponding to evaluations at the s with $0 < |s| < 1$ (assuming that $K = \mathbb{C}$). So $\Omega(S)$ corresponds to evaluations at s with $|s| \geq 1$. Note that the (only) maximal ideal missing from $\Omega(S)$ is precisely the kernel of "evaluations at s = infinity."

Let S be a fixed stability set, and R a ring of rational functions as above. We then have the

(3.5) THEOREM. *The system Σ admits an observer with respect to S if and only if Σ is $\Omega(S)$-detectable.*

This result will be a consequence of the following result, valid for any domain R:

(3.6) LEMMA. *The system* Σ *admits an observer with respect to* S *if and only if the matrix*

$$\begin{pmatrix} (s+\alpha)^{-1}(sI-F) \\ H \end{pmatrix} \tag{3.7}$$

has rank n *when reduced modulo every maximal ideal of* Tr(S).

We note first that (3.6) indeed implies (3.5) if R is a ring of rational functions. By the previous remarks, maximal ideals of Tr(S) are kernels of γ's which satisfy $\gamma(s) = 0$ or which evaluate at points of $\Omega(S)$. Consider first those of type $\gamma(s) = 0$. Applying π to each entry of the matrix in (3.7) results in

$$\begin{pmatrix} I - sF_1 \\ H \end{pmatrix} \tag{3.8}$$

when $F_1 := \alpha I + F$. But (3.8) always has rank n when evaluated at a point with $s = 0$. Thus, checking (3.7) at such ideals is redundant. For the ideals corresponding to points $(\bar{s},\bar{\sigma})$ of $\Omega(S)$, evaluations give

$$\begin{pmatrix} (\bar{s} + \alpha(\bar{\sigma}))^{-1}(\bar{s}I - F(\bar{\sigma})) \\ H(\bar{\sigma}) \end{pmatrix} \tag{3.9}$$

and this matrix has rank n if and only if

$$\text{rank} \begin{pmatrix} \bar{s}F - F(\bar{\sigma}) \\ H(\bar{\sigma}) \end{pmatrix} = n ,$$

as wanted. Thus (3.5) and (3.6) are equivalent.

The validity of (3.6) will in turn follow from:

(3.10) LEMMA. *The system* Σ *admits an observer if and only if there exist stable causal transfer matrices* $M(s)$, $N(s)$ *with*

$$[M(s)\ N(s)]\begin{bmatrix} sI - F \\ H \end{bmatrix} = I \qquad (3.11)$$

If such M,N *exist,* M *is necessarily strictly causal.*

Assume that (3.11) holds. Then

$$[(s+\alpha)M(s) \quad N(s)] \qquad (3.12)$$

is a left inverse for the matrix (3.7) over the ring $Tr(S)$. Conversely, if (3.7) admits a left inverse then there are M,N as in (3.11). Over any ring, left invertibility can be checked at each maximal ideal (see e.g. Bourbaki [1972, Chapter II]). In other words, lemma (3.10) becomes just a restatement of lemma (3.6).

We now prove (3.10). Assume first that (3.11) holds. Consider the transfer matrix $W(s)$ with $m+p$ inputs and n outputs corresponding to

$$\hat{x} = M(s)Gu + N(s)y \qquad (3.13)$$

This transfer matrix admits a stable realization $\hat{\Sigma}$, since M,N are causal and stable. Assume that x satisfies

$$sx = Fx + Gu + v, \quad y = Hx \qquad (3.14)$$

as in (2.5). Note that $M(s)(sI-F) + N(s)H = I$ and $(sI-F)x = Gu + v$. Then

$$\hat{x} - x = M(s)Gu + N(s)y - M(s)(sI-F)x - N(s)Hx = M(s)v,$$
$$\text{with } M \text{ stable.} \qquad (3.15)$$

Thus $\hat{\Sigma}$ is an observer for Σ.

Conversely, assume that such a $\hat{\Sigma}$ exists, and consider its transfer matrix $W(s)$, written as

$$\hat{x} = R(s)u + N(s)y\ . \qquad (3.16)$$

Solving formally,

$$\hat{x} - x = R(s)u + N(s)Hx - x$$

$$= R(s)u + (N(s)H - I)((sI - F)^{-1}Gu + (sI - F)^{-1}v)$$

$$= [R(s) + (N(s)H - I)(sI - F)^{-1}G]u$$

$$+ (N(s)H - I)(sI - F)^{-1}v. \qquad (3.17)$$

This must equal $M(s)v$, so

$$R(s) = -(N(s)H - I)(sI - F)^{-1}G \qquad (3.18)$$

and

$$(N(s)H - I)(sI - F)^{-1} = M(s) . \qquad (3.19)$$

The last equation implies that

$$N(s)H - I = M(s)(sI - F) , \qquad (3.20)$$

i.e., that $[M \ \ N]$ provides a left inverse as in (3.11). This completes the proof of (3.10), and hence of theorem (3.5). ⌑

Over any domain R observers will exist if and only if the conditions in (3.6) or (3.10) are satisfied. Since an n-column matrix A over R has rank n over every residue field precisely when the ideal generated by the n-minors $\Delta_i(s)$ of A is trivial, the above conditions amount to requiring the existence of stable rational functions $a_i(s)$ such that $\Sigma a_i(s)\Delta_i(s)$ is a stable rational function. It is again easier to work with $T^{-1}(R[s])$. Thus applying π and letting Δ_i' be the minors of (3.8), observers will exist if and only if there exist $b_i(s)$ in $R[s]$ such that

$$\Sigma b_i(s)\Delta_i'(s) \in T . \qquad (3.21)$$

The actual computation of such observers, and even the verification of (3.21), will of course depend on the ring in question. Although such questions are usually "decidable" in the sense of computer science for effectively presented R and suitable finiteness conditions (Noetherian, etc.), no general reasonable algorithm can be expected at this level of abstraction, and our result is purely existential. When R is a ring of rational functions, on the other hand, methods from elimination theory can be used; examples of such rings will appear in the next section.

We turn now to questions of feedback control. The most general type of statement thatone would like to have is that $\Sigma = (F,G,H)$ admits an (S-)feedback controller if and only if (F',H',G') is $\Omega(S)$-detectable. Unfortunately such a statement is too general, since feedback controllers are not the precise dual notion to observers. However, the following somewhat less general theorem is valid, for any R:

(3.22) THEOREM. *The following statements are equivalent for* $\Sigma = (F,G)$:

(a) *For each* α *in* R, *there is a feedback controller with respect to the stability set*

$$S_\alpha = \{(s+\alpha)^t, \quad t \geq 0\}.$$

(b) *For each* α *in* R *and* S_α *as above, there exist stable causal transfer matrices* V,W *such that*

$$[sI - F \quad G] \begin{pmatrix} W \\ V \end{pmatrix} = I \tag{3.23}$$

and W *causally divides* V, *i.e.* $V = KW$ *for some causal* K.

(c) (F,G) *is reachable.*

<u>Proof</u>. We shall prove that (a) ⇒ (b) ⇒ (c) ⇒ (a). If (a) holds, then we may write $u = K(s)x$, with a suitable $K(s)$, and $x = W(s)v$, with

$$W(s) = (sI - F + GK(s))^{-1} . \tag{3.24}$$

Further, $W(s)$ and $V(s) = K(s)W(s)$ are both stable, by the definition of feedback controller. Writing (3.24) as $(sI - F + GK)W = I$, we conclude that

$$(sI - F)W + G(KW) = I ,$$

as wanted.

We now prove that (b) implies (c). Let M be a maximal ideal of R and let $(F(M),G(M))$ denote the reduction mod M of (F,G). Working on $(R/M)[s]$, we have that for each α in R there are polynomial matrices W_1, V_1 over $(R/M)[s]$ such that, for some j,

$$(sI - F(M))W_1 + GV_1 = (s + \alpha(M))^j I \ . \qquad (3.25)$$

But R/M being a field, this implies that

$$[sI - F(M),\ G(M)]$$

has rank n for each s in the algebraic closure of R/M (given M,s, take any α with $\alpha(M) \neq -s$). Thus $(F(M),G(M))$ is reachable for each M, so the original system is reachable.

Assume finally that (c) holds, and pick α in R. Let $L := F + \alpha I$. Since (L,G) is again reachable, there is a right-inverse to $G, LG, \ldots, L^{n-1}G$, i.e. there exist m by n matrices $b_1, \ldots, b_n$ over R with

$$L^{n-1}Gb_1 + \ldots + LGb_{n-1} + Gb_n = I \ . \qquad (3.26)$$

For notational convenience, we let $b_0 := b_{n+1} := 0$. We define:

$$K_1(s) := (s+\alpha)^n I_m - \sum_{j=1}^{n} (s+\alpha)^{n-j} b_j (F+\alpha I)^n G, \qquad (3.27a)$$

$$K_2(s) := \sum_{i=1}^{n} (s+\alpha)^{n-j} (b_{j+1}(F+\alpha I)^n - b_j (F+\alpha I)^{n+1}), \qquad (3.27b)$$

$$K(s) := K_1(s)^{-1} K_2(s) \ . \qquad (3.27c)$$

By the result in the appendix, $K(s)$ is a causal transfer matrix admitting a realization $\Sigma_2 = (A,B,C,D)$ such that both $W(s) := (sI - F + GK(s))^{-1}$ and $(sI - A)^{-1}BW(s)$ are causal transfer matrices, in fact, polynomial matrices in $(s+\alpha)^{-1}$, so in particular S_α-stable. So Σ_2 is a feedback controller for (F,G). ⌑

The above feedback construction admits a dualization in terms of a generalized notion of "Luenberger" observer. The dual statement is that an observable system Σ admits an observer whose "error" $e = \hat{x} - x$ satisfies an equation

$$se = (F - GK(s))e + v \, . \tag{3.28}$$

This form of the observer equation is, of course, more particular than the original definition, and it corresponds to the condition that M causally divide N.

4. PARAMETRIC FAMILIES OF SYSTEMS. As an application of the results in the previous sections we consider problems related to observation and control of families of linear systems, parametrized either polynomially or rationally. For the types of rings and stability sets that appear in these applications stronger results than obtained in the previous section can be given. This will be considered now.

Let R be a ring of polynomials $k[\sigma_1,\ldots,\sigma_r]$, and assume that S is a stability set of the type

$$S_X = \{q \in R[s] \mid q \text{ monic in } s \text{ and } q(s,\sigma) \neq 0 \text{ for all } (s,\sigma) \text{ in } X\}, \tag{4.1}$$

where X is a subset of k_c^{r+1} and $k_c \subseteq K$ is the algebraic closure of k. For any subset X, S_X is a saturated multiplicative set, and S_X will contain a monomial $s + \alpha$ if some hyperplane $s = -\alpha$ fails to intersect X. We shall assume that there is such a hyperplane, so that S is a stability set as before. For the rest of this section, R and S will have this form, unless otherwise stated.

In the main examples we wish to consider here, k is the real or complex field, $K = k_c$ is the complex field, and, for any fixed nonpositive real number a,

$$X = \{(s,\sigma) \mid \mathrm{Re}\, s \geq -a, \ \sigma \text{ in } k\} \, . \tag{4.2}$$

Thus a stable transfer function is here a family of transfer functions, parametrized by polynomials in σ, each of which has all poles in a suitable left-hand semiplane. For any $\alpha > a$ the monomial $s+\alpha$ is in S_X. The development proceeds for general X, but this will be our main illustration.

The detectability condition implies checking evaluations at $\Omega(S)$. It is always true that X can be taken as a subset of $\Omega(S_X)$; when these two sets can be taken equal we shall say that X is *perfect*. In general, however, it may not be enough to check full rank at points of X. We wish to see in some more detail what else is required.

It is again more convenient to work with $T^{-1}(R[s])$. Note that if $S = S_X$ we may introduce

$$X^{-1} = \{((s+\alpha)^{-1},\sigma)|(s,\sigma) \in X\} \cup \{(0,\sigma)|\sigma \in k^r\}$$

so that T becomes $S_{X^{-1}}$. The ring of stable transfer functions becomes then isomorphic to the ring of those rational functions which have no zeroes in X^{-1}.

Recall from (3.21) that the condition for existence of observers is that there be some element of T in the ideal of $R[s]$ generated by the minors $\Delta_i'(s)$. In the present case, this means that there must be some b_i in $R[s]$ such that

$$\Sigma b_i\Delta_i' = f \tag{4.3}$$

and f has no zeroes in X^{-1}. Equivalently, *the set of common zeroes (over* k_c*) of the* Δ_i' *must be contained in the set of zeroes of some stable* f. Indeed, any common zero of the Δ_i' is clearly a zero of an f as in (4.3) and, conversely, if all common zeroes of the Δ_i' are contained in the zeroes of some g in T, then by Hilbert's Nullstellensatz (see Bourbaki [1972, V. 3.3]) there is some t with $f = g^t$ satisfying (4.3). Since T is multiplicative, this f is in T, as desired.

Thus the problem of deciding if an observer exists becomes one of "interpolating" the common zeroes of the Δ_i' by a stable polynomial. When $r = 1$ the situation is somewhat simpler. Assume that the rank condition is satisfied for all points of X^{-1}, i.e. that at each point of X^{-1} there is some Δ_i' which does not vanish there. Since $r = 1$, the Δ_i' have either a *finite* set of common zeroes or they have a common divisor. In the latter case, this common divisor is already stable and interpolates common zeroes, so we shall assume the Δ_i' have finitely many zeroes $\{x_i\}$ in common. If for each x_i there is a polynomial f_i in T which has $f_i(x_i) = 0$, then the product f of the f_i is again in T and has all the common zeroes of the Δ_i' as its zeroes. We then have the

(4.4) LEMMA. *Let* $r = 1$, $S = S_X$. *The system* (F,G,H) *admits an observer if and only if for each common root* $(\bar{s},\bar{\sigma})$ *(in* k_c^2*) of the minors of*

$$\begin{pmatrix} sI - F(\sigma) \\ H(\sigma) \end{pmatrix}$$

there is a $p(s,\sigma) \in S$ with $p(\bar{s},\bar{\sigma}) = 0$.

The above will be especially useful in the context of delay systems. For families of systems, the situation will be even better. We have:

(4.5) PROPOSITION. *Assume that* X *is of the form*

$$X_1 \times k_c^r \ ,$$

where X_1 *is an infinite subset of* k_c *which is conjugate-closed, i.e. such that* X_1 *is invariant under the Galois group of* k_c *over* k. *Then* X *is perfect. In particular, the detectability condition in (3.5) needs only be checked for points in* X.

Proof. We work with X^{-1}. This set again satisfies the hypothesis of the lemma; say $X^{-1} = X_2 \times k_c^r$. We must prove that the prime ideals of $k[\sigma_2,\ldots,\sigma_r,s]$ which are maximal with respect to not intersecting T are precisely the kernels of the evaluations into $(\bar{s},\bar{\sigma})$, with $(\bar{s},\bar{\sigma})$ in X^{-1}. Let P be such an ideal, and let $V(P)$ be the zero set of P, i.e. the irreducible algebraic subset of k_c^{r+1} at which all polynomials in P vanish. We prove that $V(P)$ must contain a point $(\bar{s},\bar{\sigma})$

with $\bar{s}$ in X_2. If $V(P)$ does *not* contain such a point, then the projection of $V(P)$ on the first coordinate fails to intersect X_2. Since X_2 is infinite, this means (see Dieudonné [1974, p. 109]) that the projection has only finitely many points, and hence by irreducibility just one. So $V(P)$ is included in some hyperplane $s = \beta$. Then the product of all $(s - \beta_i)$ where β_i are the conjugates of β in k_c, is in $V(P) \cap T$, contradicting the fact that $T \cap P$ was supposed empty. We conclude that $V(P)$ contains a point $(\bar{s},\bar{\sigma})$ with $\bar{s}$ in X_2. But P is then contained in the maximal ideal M which is the kernel of evaluation at $(\bar{s},\bar{\sigma})$, and $M \cap T = \emptyset$. By maximality, $P = M$ as wanted. ⌑

In fact, in this case elements of T, (and thus also those of S), *must be* constant polynomials (i.e., in $k[s]$), since the set of zeroes of an irreducible polynomial $p(s,\sigma)$ cannot have a projection that fails to intersect X_2 except if it is contained in a hyperplane, in which case it is by dimensionality and irreducibility the whole hyperplane. The numerators of transfer functions, however, may still be arbitrary polynomials, so that the calculations involved are not all trivial, and in general rely on methods from elimination theory. With low-dimensional specific examples it is best to try to solve directly for transfer functions as in (3.11) rather than making effective the various steps of the theoretical results; the latter insures us that solutions will exist.

Note that a family of polynomially parametrized systems $\Sigma_\lambda = (F(\lambda), G(\lambda), H(\lambda))$, λ in k^r, is detectable (with X as in (4.5)) precisely when the systems Σ_λ are each one detectable, for the corresponding X_1. For example, a polynomial continuous-time family of linear systems over $\mathbb{C}$ each of which satisfies a detectability condition, say,

$$\operatorname{rank} \begin{pmatrix} sI - F \\ H \end{pmatrix} = n$$

for all $\operatorname{Re} \bar{s} \geq -a$, will admit a polynomial parametrized family of observers such that for example for each fixed λ, $\hat{\Sigma}_\lambda$ will observe the state of the corresponding Σ_λ with error of the order e^{-bt}, for some $b > a$. Similarly, a polynomial family over $\mathbb{R}[\sigma_1,\ldots,\sigma_r]$ will admit a *real*-polynomial family of observers *provided that* the detectability condition be satisfied for all real *and* complex parameters, as follows from (4.5). (The intuitive reason why complex parameter values are relevant is evidenced by examples of one-dimensional systems with $y = (\lambda^2 + 1)x$: the inversion of this map cannot be carried out over the ring of real (or complex) polynomials, even though for each *real* λ *the system* will be observable!) A polynomially (or even rationally) parametrized family of observable real systems, however, will admit a rational family of observers (with no real poles, i.e. well-defined for each real λ); this follows by considering the ring of rational functions $p(\sigma_1,\ldots,\sigma_r)q(\sigma_1,\ldots,\sigma_r)^{-1}$ over $\mathbb{R}$ with no real poles, and applying the dual version of (3.22). As a simple illustration, consider a two-parameter family of linear systems over $\mathbb{R}$:

$$\begin{aligned} \dot{x}_1(t) &= -x_2(t), \quad \dot{x}_2(t) = -x_1(t) \\ y_1(t) &= (\lambda^2+\mu)x_2(t), \quad y_2(t) = (\mu-1)x_1(t), \end{aligned} \tag{4.7}$$

where λ,μ are allowed to take arbitrary real values. (As shown in Bumby, et al. [1979], pole-shifting arguments cannot be in general applied to real 2-parameter families). The family is observable as a system over the ring of rational functions defined on $\mathbb{R}^2$, since the rank of

$$\begin{pmatrix} s & -1 \\ -1 & s \\ 0 & \lambda^2+\mu \\ \mu-1 & 0 \end{pmatrix} \tag{4.8}$$

is 2 for all complex s and real λ,μ. Thus there exists a family of observers with any given convergence rate, say $s = 1$. Indeed, a left inverse of (4.8) is for example given by (a_{ij}), where

$$a_{11} = [(\lambda^2+1)s^2 + (3\lambda^2-1+4\mu)s + 4(\lambda^2+\mu)][(\lambda^2+1)(s+1)^3]^{-1}$$

$$a_{12} = a_{21} = (s-1)(s+1)^{-3}$$

$$a_{13} = -a_{24} = 4[(\lambda^2+1)(s+1)^2]^{-1} \tag{4.9}$$

$$a_{14} = -a_{23} = -4s[(\lambda^3+1)(s+1)^2]^{-1}$$

$$a_{22} = [(\lambda^2+1)s^2 + (3+\lambda^2-4\mu)s + 4(\mu-1)][(\lambda^2+1)(s+1)^3]^{-1}.$$

From this matrix (a_{ij}) an observer over this ring, i.e. a family of observers for the original family, is obtained as in (3.13).

(4.10) REMARK. A further area of application of the present setup is yet to be explored in detail. This concerns certain types of nonlinear systems. Specifically, consider *state-linear* systems Σ of the following type:

$$\begin{aligned} x(t+1)[\text{or } \dot{x}(t)] &= F(u(t))x(t) , \\ y(t) &= H(u(t))x(t) , \end{aligned} \tag{4.11}$$

where u is a (vector) control. Realization and other properties of such systems were studied in some detail in Sontag [1979]. Recently, Grasselli and Isidori [1979] have studied the construction of uniform observers for (continuous-time) *internally-bilinear* systems, i.e. the particular case where F is affine in u and H is constant. They have shown that if the system is observable for each *constant* u, then there are observers for "slowly varying" (and bounded) controls u. Their proof relies implicitly in constructing observers for each *fixed* value of u, viewing essentially (4.11) as a family of systems $(F(\lambda), H(\lambda))$, showing that such an observer construction can be done smoothly in u (i.e., λ), and proving that the resulting "observer" remains such for slowly varying u. Now, even in the case of bilinear systems their proof insures only that the observer has F,H differentiable. It would be, of course, more desirable to obtain a finite structure for F,H. The methods developed in this section provide in principle this finite structure, since observers for each (constant) u are defined rationally in u. Since the results of Grasselli and Isidori are expressed for Lueberger-like observers, however, the technical details of this extension are not quite trivial, and we leave this topic as a suggestion for further research. ⌑

5. DELAY SYSTEMS. A *delay-differential* system is defined by equations of the type

$$\begin{aligned} \dot{x}(t) &= \Sigma\, F_i x(t-\alpha_i) + \Sigma\, G_j u(t-\beta_j) \\ y(t) &= \Sigma\, H_\mu x(t-\gamma_\mu) + \Sigma J_\nu\, u(t-\theta_\nu) \end{aligned} \tag{5.1}$$

where the α_i, β_i, γ_μ, θ_ν are all nonnegative real numbers. If all delays are integral multiples of certain noncommensurable delays $\tau_1,\ldots,\tau_r$, it is natural to model (5.1) as a system over the polynomial ring $\mathbb{R}[\sigma_1,\ldots,\sigma_r]$, (or $\mathbb{C}[\sigma_1,\ldots,\sigma_r]$) as suggested by Kamen [1975, 1978]. Thus, the results of sections 3 and 4 can be applied in this context. The proper notion of stability is now different from that in section 4. Here the natural stability set to consider (see Bellman and Cooke [1962] or El'sgol'Ts [1966]) is S_X with (for fixed $a \geq 0$):

$$X = \{(s, e^{-\tau_1 s}, \ldots, e^{-\tau_t s}) \mid \mathrm{Re}\ s \geq -a\}\ . \tag{5.2}$$

Observers and regulators with respect to such a stability set will have exponential convergence rates of order e^{-bt} for some $b > a$.

Problems of control and regulation for (5.1) have been studied by various authors. Two approaches in particular have been dominant. One is the functional-analytic approach, which regards (5.1) as an ordinary differential system in a suitable function space and generalizes the classical results to this context. Along these lines, Pandolfi [1975] and Bhat and Koivo [1976] have proved that Luenberger observers will exist under the natural detectability condition (i.e., that (2.3) hold for every $(\bar{s},\bar{\sigma})$ in the above X) if one allows for much more general systems than (5.1) as observers. The latter may now contain in the right-hand side terms with "distributed" delays of the form

$$\int_{t-\tau}^{t} \lambda(\alpha)x(\alpha)\ dx\ , \tag{5.3}$$

for various functions λ and numbers τ. A different, more algebraic, approach was used by Morse [1976] who proved that observers will exist (using only "point delays") with arbitrary convergence rates, if (1) the system is observable and (2) $r = 1$, i.e. all delays are multiples of some fixed unit. Each of these assumptions is, of course, rather restrictive. Somewhat more general conditions were given by Sontag [1976], but this involved allowing for somewhat more general observer configurations. A wide extension of this line of argument was obtained by Kamen [1978a], who considered even more general observer configurations. (Strictly speaking, all of the above concentrate on feedback controllers and treat observes by duality.) Again in the work of Kamen, however, one encounters distributed delays as in (5.3) when designing observers for (5.1).

The above restrictions to observability and to $r = 1$ can be overcome using the present setup. If one has observability then *observers* (with arbitrary convergence rates) *will exist whether* $r = 1$ *or not*; this is clear from (3.22), and the dual statement holds for feedback controllers. And the observers (controllers) are again of type (5.1). But even if one does not have observability, however, the results in the previous sections do insure that observers will exist under the respective detectability assumptions. For example, if X is as in (5.2), with, say, $a = 0$, observers will exist for any system that is $\Omega(S_X)$-detectable. If one could prove that X is perfect in the sense of section 4, this statement would be directly comparable to those of Pandolfi and of Bhat and Koivo, with the greatly strengthened conclusion that the observer has the same structure as the original system. Unfortunately, it appears that X is *not* perfect and that the determination of $\Omega(S_X)$ is nontrivial.

Take for simplicity $r = 1$. In view of (4.4), a point is *not* in $\Omega(S_X)$ if it is a zero of a stable polynomial. For example, the points (s_0, σ_0) with $\mathrm{Re}\, s_0 < 0$ are not in $\Omega(S_X)$ (because $s-s_0$ over $\mathbb{C}$, or $(s-s_0)(s-\hat{s}_0)$ over $\mathbb{R}$, where $\hat{s}_0$ is the conjugate of s_0, are stable polynomials zero at such a point). Other examples can be obtained by taking the zeroes of any given stable polynomial. For instance, we have the following result (for $r = 1$).

(5.4) PROPOSITION. *If* $|\sigma_0| > 1$, *then* (s_0, σ_0) *is not in* $\Omega(S_X)$.

Proof. We note that for $a > 0$, $b \in \mathbb{C}$, $|b| < a$, the polynomial $p(s,\sigma) := s + a - b\sigma$ is stable. Indeed

$$|p(s,e^{-\ s})| \geq |s+a| - |b| \geq a - |b| > 0 \quad \text{for} \quad \tau \geq 0,$$
$$\mathrm{Re}\, s \geq 0.$$

Therefore, if $|\sigma\ | > 1$, we may choose $b := (a+s_0)/\sigma_0$ and a so large that

$$|b| = \frac{|a+s_0|}{|\sigma_0|} = \frac{a}{|\sigma_0|}\left(1 + \frac{|s_0|}{a}\right) < a \qquad \Pi$$

Also (see El'Sgol'Ts [1966]) $(0,0)$ is not in $\Omega(S_X)$ for any $\tau > 0$, since the polynomial $s + \mu\sigma$ has no roots of the form $(s, e^{-\tau s})$ with $\mathrm{Re}\, s \geq 0$ whenever

$$0 < \mu < \frac{\mu}{2\tau} \quad .$$

Other points not in $\Omega(S_X)$ can be found by analysing the stability of the polynomials $s + a + b\sigma$ for various a,b (see again the book by El'Sgol'Ts). On the other hand, it appears (E. Kamen [personal communication]) that the point $(1,0)$ is in $\Omega(S_X)$ (but not in X), and thus that X is not perfect. We leave as an open problem the characterization of $\Omega(S_X)$.

Thus, restricting for simplicity to $r = 1$, we can insure the existence of observers precisely when each point at which

$$\operatorname{rank}\begin{pmatrix} sI - F(\sigma) \\ \\ H(\sigma) \end{pmatrix} < n \tag{5.5}$$

is the root of at least one stable polynomial p, i.e. a $p(s,\sigma)$ such that $p(s,e^{-\tau s}) \neq 0$ whenever $\operatorname{Re} s \geq 0$. If a real such p can be found for every point at which (5.5) holds, the observer will have real coefficients; if p is complex the resulting observer will be a priori complex, and then the method described in section 7 can be applied to obtain from this a real-coefficient observer.

(5.6) EXAMPLE. We work in some detail (the dual of) the main example given by Kamen [1978a] as an illustration of his method for pole-shifting. The equations here are:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t-\tau) + b_1 u(t) \\ \dot{x}_2(t) &= x_1(t-\tau) + x_2(t) + b_2 u(t) \\ y(t) &= x_2(t). \end{aligned} \tag{5.7}$$

(We added an indeterminate input term (b_1,b_2) since the dualization of Kamen's example gives us just F and H.) Note that this system is unstable for any (positive) τ, since

$$x^2 - x - e^{-2\tau x}$$

has a positive real root, being negative at zero and positive for large x. The problem of finding an observer is therefore nontrivial. We must find then a left inverse to

$$\begin{pmatrix} s & -\sigma \\ -\sigma & s-1 \\ 0 & 1 \end{pmatrix} \tag{5.8}$$

over the ring of stable transfer functions. The minors of (5.8) are $(s^2 - s - \sigma^2)$, s, $-\sigma$, so the only common root is $(0,0)$, which as remarked before is not in $\Omega(S_X)$. Thus the rank of (5.8) is 2 for all points in $\Omega(S_X)$, and the system is detectable. By inspection we can find a left inverse:

$$\begin{bmatrix} \frac{1}{s+\mu\sigma} & \frac{-\mu}{s+\mu\sigma} & \frac{\sigma+\mu s-\mu}{s+\mu\sigma} \\ 0 & 0 & 1 \end{bmatrix}$$

where μ is any real number such that $0 < \mu < \pi/2\tau$. Obtaining a realization as in the proof of (3.5), we conclude that an observer $\hat{\Sigma}$ is given by equations

$$\dot{z}(t) = -\mu z(t-\tau) + (b_1-\mu b_2)u(t) + (1-\mu^2)y(t-2) - \mu y(t)$$

$$\hat{x}_1(t) = z(t) + y(t), \qquad \hat{x}_2(t) = y(t) . \tag{5.9}$$

It is interesting to contrast this observer with the one obtained by Kamen's method, using his feedback matrix to define a Luenberger observer:

$$\dot{z}_1(t) = z_2(t-\tau) - 2z_2(t) - z_2(t-2\tau) + 2y(t) + y(t-\tau) + b_1u(t)$$

$$\begin{aligned} \dot{z}_2(t) = {} & z_1(t-\tau) + z_2(t) - 2\int_{t-\tau}^{t} z_2(\alpha)d\alpha - \int_{t-3\tau}^{t-2\tau} z_2(\alpha)d\alpha \\ & -4z_2(t) + 2\int_{t-\tau}^{t} y(\alpha)d\alpha + \int_{t-3\tau}^{t-2\tau} y(\alpha)d\alpha \\ & +4y(t) + b_2u(t) , \end{aligned} \tag{5.10}$$

and $\hat{x}_i = z_i$. Close inspection of (5.10) reveals that in this particular case it so happens that this system can be enlarged to one containing only point delays, since the distributed delays that appear are just finite-time integrators. In more general examples, however, distributed delays may become more involved. For instance, if the original system is perturbed by adding a term "$\varepsilon \dot{x}_1(t)$" to the first equation, the new system is still detectable in our sense, but (5.10) includes now distributed delays of more complicated form. We leave as an open problem the comparison between these two approaches; each seems to have a different domain of applicability and to lead to very different configurations when both can be applied.

6. REGULATION. We saw in section 3 how to construct a state-feedback controller for a reachable system, and how to obtain a state-observer for a detectable system. We do not know as yet how to complete the general input/output regulator design combining these two results. A regulator may be shown to exist, however, under the stronger hypothesis that a *nondynamic* feedback controller exists. We let R be any integral domain and S any stability set.

(6.1) PROPOSITION. *Assume that* Σ *is described by equations*

$$\begin{aligned} sx &= Fx + Gu + v \\ y &= Hx , \end{aligned} \tag{6.2}$$

that

$$\hat{x} = M(s)Gu + N(s)y$$

defines the transfer matrix of an observer for Σ, *and that* $K(s)$ *is a stable matrix such that*

$$W(s) := (sI - F + GK(s))^{-1} \tag{6.4}$$

is stable (which is in particular the case if $K(s)$ *is a nondynamic solution of the feedback controller problem). Then, defining*

$$L(s) := K(s)(I + M(s)GK(s))^{-1}N(s) , \tag{6.5}$$

and considering equation (6.2) *together with*

$$x = -L(s)y , \tag{6.6}$$

results in a stable closed-loop system, i.e.

$$(sI - F + GL(s)H)^{-1} \tag{6.7}$$

is stable.

Proof. This is just an algebraic manipulation re-expressing (6.7) in a suitable way. Since

$$x = M(sI - F)x + Ny = \hat{x} + Mv \ , \tag{6.8}$$

we have that

$$\begin{aligned} sx &= Fx - GLy + v \\ &= Fx - GK\hat{x} + v \\ &= Fx - G(Kx - KMv) + v \\ &= Fx - GKx + GKMv + v. \end{aligned} \tag{6.9}$$

Thus

$$(sI - F + GLH)^{-1} = W(GKM + I) \ , \tag{6.10}$$

and the right-hand term is a product of two stable transfer matrices; the left-hand term is therefore stable as wanted. π

Thus, at least one can solve the regulation problem as defined in section 2 when the given transfer matrix has a realization which is detectable and which admits a pole-shifting theorem. The latter happens for example when the realization is reachable and R is a polynomial ring in one variable [Morse, 1976] and for some other rings [Bumby et al., 1979]. It is thus of interest to know when such realizations exist. We answer this question now for the case $m = p = 1$; the resulting condition turns out to be a generalization of the *split* condition in Sontag [1978] (see also Byrnes [1978]) which studies the stronger requirement of reachable and *observable* (not just detectable) realizations. We assume that the ring R is completely integrally closed; this is a relatively mild technical condition, which is satisfied for all our examples. Further, we state the result for the case where R is a ring of rational functions, i.e., $R \subseteq k(\sigma_1,\ldots,\sigma_r)$.

(6.11) THEOREM. *Let* $w(s)$ *be a transfer function. The following properties are equivalent, for fixed* Ω*:*

(a) $w(s)$ *has a reachable and detectable realization;*

(b) *the canonical realization of* $w(s)$ *is reachable and detectable;*

(c) *if* $w(s)$ *is written as* $p(s)q(s)^{-1} + w_0$ *with* w_0 *in* R *and* $q(s)$ *of minimal degree, then* $p(s,\sigma)$ *and* $q(s,\sigma)$ *have no common roots in* Ω.

Proof. Since (b) trivially implies (a), it will be sufficient to prove that (a) implies (b) and that (b), (c) are equivalent. Assume that (a) holds for a realization $\Sigma = (F,G,H,J)$. Since Σ is reachable, there is a system epiomorphism which maps Σ onto the canonical realization Σ_w (see Eilenberg [1974, Ch. 16]). Since Σ_w is free (because R is completely integrally closed, (see Rouchaleau and Sontag [1979])), this morphism splits, i.e. the matrices (F,H) of Σ can be written as

$$F = \begin{pmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{pmatrix}, \quad H = [H_1 \quad 0]$$

in a suitable basis, and (F_{11}, H_1) are matrices defining (F,H) for the canonical realization, which is a priori reachable and *weakly* observable, but not necessarily detectable. If (F,H) is Ω-detectable, (6.12) implies that (F_{11}, H_1) is also Ω-detectable, as wanted.

To prove that (b) and (c) are equivalent, we write the canonical realization with F in the "flat" (i.e., controllable canonical) form:

$$F = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \\ a_0 & a_1 & \cdots & & a_{n-1} \end{pmatrix} \quad H = [b_0,\ldots,b_{n-1}] \qquad (6.13)$$

where $w = pq^{-1} + w_0$, $p = \sum_{j=0}^{n-1} b_j(\sigma)\, s^j$, $q = s^n + \sum_{i=0}^{n-1} a_i(\sigma)s^i$.

Assume that (b) does not hold, so that for some $(\bar{s},\sigma)$ in Ω the matrix

$$A(\bar{s},\bar{\sigma}) = \begin{pmatrix} \bar{s}I - F(\bar{\sigma}) \\ \\ H(\bar{\sigma}) \end{pmatrix}$$

admits a nonzero kernel element $v = [v_1,\ldots,v_n]'$ in K^n. Because of the form of (6.13), this implies that $v_2 = \bar{s}v_1$, $v_3 = \bar{s}v_2,\ldots,v_n = \bar{s}v_{n-1}$. Thus

$$v = v_1 \begin{pmatrix} 1 \\ \bar{s} \\ \vdots \\ \bar{s}^{n-1} \end{pmatrix} , \tag{6.15}$$

and we may take $v_1 = 1$. From the last two entries of $Av = 0$ we conclude that p,q have a common root, i.e. (c) does not hold. Reversing the above argument yields the converse implication: a common root $(\bar{s},\bar{\sigma})$ of p and q gives rise to a nonzero vector in the kernel of $A(\bar{s},\bar{\sigma})$. ⌑

The above condition (c) is of course much weaker than the "split" condition which requires that p,q have no common roots at all in k_c^{r+1}. In fact, for $k = \mathbb{R}$ or $\mathbb{C}$, this condition is "generic" when Ω is a "thin" set. In the context of delay systems with $r = 1$, the latter would happen if X is perfect, since then Ω would be an analytic set of codimension one. Even if X fails to be perfect, the condition is rather weak. As an illustration, take the transfer function $w = \sigma/s$, corresponding to the input/output map

$$\dot{y}(t) = u(t-1) . \tag{6.16}$$

This is not a "split" i/o map, but the only common root $\bar{s} = \bar{\sigma} = 0$ is not in the Ω considered in section 4, so a regulator can be built for this transfer function. Specifically, the canonical realization is given by

$$\dot{x}(t) = u(t), \quad y(t) = x(t-1) \; ; \tag{6.17}$$

then the feedback law $u(t) := -x(t)$ in conjunction with the observer

$$\hat{x}(t) = -\hat{x}(t-1) + y(t) + u(t) \tag{6.18}$$

serves as a regulator.

7. REMARKS. In some applications requiring the existence of observers and/or controllers with real coefficients, we may be able to construct directly only an observer (or controller) with complex coefficients. It is, however, possible in general to modify such a construction in order to obtain also a real controller or observer. For example, let

$$sz = Az + B_1u + B_2y, \quad \hat{x} = Cz + Dy \tag{7.1}$$

be an observer having matrices in $\mathbb{C}[\sigma_1,\ldots,\sigma_r]$, for a system whose defining matrices are in $\mathbb{R}[\sigma_1,\ldots,\sigma_r]$ and for which $u(t)$, $v(t)$ are real-valued. Separating into real and complex parts: $z = z^{(1)} + iz^{(2)}$, $A = A^{(1)} + iA^{(2)}$, etc., we may consider the new observer (defined over the reals):

$$\begin{aligned} sz_1 &= A^{(1)}z_1 - A^{(2)}z_2 + B_1^{(1)}u + B_2^{(1)}y \\ sz_2 &= A^{(2)}z_1 + A^{(1)}z_2 + B_1^{(2)}u + B_2^{(2)}y \\ \hat{x} &= C^{(2)}z_1 - C^{(2)}z_2 + D^{(1)}y \end{aligned} \tag{7.2}$$

Thus the error $\hat{x} - x$ for the new observer will be the real part of the error for the original one, and, for most reasonable definitions of "convergence" (stability sets), this error will become "small" if the original does.

In the case of feedback controllers and of i/o regulators the situation is slightly more delicate due to the fact that inputs u for the original system are now produced by the controller and hence a priori may be complex valued. A similar reduction can be performed, however, essentially by making the (ideal) "complex part" of the original system part of the controller. We omit the (straightforward) details.

APPENDIX

We prove here the technical result needed for theorem (3.22). Letting $z := s + \alpha$ and $L := F + \alpha I$, $L^{n-1}Gb_1 + \dots + Gb_n = I$, it must be proved that if (with $b_o = b_{n+1} = 0$):

$$K_1(z) := z^n I_m - \sum_{j=1}^{n} z^{n-j} b_j L^n G \ ,$$

$$K_2(z) := \sum_{j=0}^{n} z^{n-j} (b_{j+1} L^n - b_j L^{n+1}) \ ,$$

$$K(z) := K_1(z)^{-1} K_2(z) \ ,$$

and

$$W(z) = (zI - L + GK(z))^{-1} \ ,$$

then $W(z)$ is polynomial in z^{-1} and there is a factorization $K(z) = C(zI-A)^{-1}B + D$ such that $\hat{K}(z) = (zI - A)^{-1}BW(z)$ is also a polynomial in z^{-1}. The statement about $W(z)$ will be a consequence of (a) the fact that $W(z)$ is the transfer function from u to x for the discrete-time system given by

$$x(t+1) = Lx(t) - G\xi(t) + u(t)$$

$$\xi(t+n) = \sum_{j=0}^{n} (b_{j+1} L^n - b_j L^{n+1}) x(t+n-j) \qquad (A.1)$$

$$+ \sum_{j=1}^{n} b_j L^n G \xi(t+n-j)$$

when starting with $x(t) = 0$, $\xi(t) = 0$ for $t < 0$, and of (b) the fact that $x(t) = 0$ when $t \geq n$, whenever x, ξ are as above and u is such that $u(t) = 0$ for $t \neq -1$. The statement about K will similarly follow from the fact that $\xi(t) = 0$ for $t \geq n$ when x, ξ and u are as above, so that a choice of $\xi(t-1), \ldots, \xi(t-n)$ at time t will give a realization of the transfer function K from v to ξ which has the above stability property.

We prove then that x, ξ both become eventually zero. Equations (A.1) are equivalent to

$$x(t) = Lx(t-1) - G\xi(t-1) + u(t-1)$$

$$\xi(t) = b_2 L^n x(t-1) + \sum_{j=2}^{n} (b_{j+1} L^n - b_j L^{n+1}) x(t-j) \qquad (A.2)$$

$$+ \sum_{j=2}^{n} b_j L^n G \xi(t-j) + b_1 L^n u(t-1) .$$

For notational convenience, introduce $b_{n+2}, b_{n+3}, \ldots$ all zero, so that the above summations have upper limit infinity. An easy induction argument shows from (A.2) that if $x(t)$ and $\xi(t)$ are zero for $t < 0$ and if $u(-1) = u_o$ is the only nonzero value of u, then

$$x(t) = L^t u_o - \sum_{i=1}^{t} L^{t-i} G b_i L^n u_o$$

and

$$\xi(t) = b_{t+1} L^n u_o$$

for all $t \geq 0$. Thus $\xi(t)$ is indeed zero for large t, and also, for $t \geq n$:

$$x(t) = L^t u_o - \sum_{i=1}^{n} L^{t-i} G b_i L^n u_o$$

$$= L^t u_o - L^{t-n} \left(\sum_{i=1}^{n} L^{n-i} G b_i\right) L^n u_o ,$$

$$= L^t u_o - L^{t-n} L^n u_o$$

$$= 0 \qquad \text{as wanted.}$$

⌑

REFERENCES

Bellman, R. and K. L. Cooke, *Differential-Difference Equations*, Academic Press, 1963.

Bhat, K. P. M. and H. N. Koivo, *Modal characterization of controllability and observability for time-delay systems*, IEEE Trans. Aut. Contr., Vol. AC-21, 1976, pp. 232-233.

Bourbaki, N., *Commutative Algebra*, Addison-Wesley, 1972.

Bumby, R., E. D. Sontag, H. Sussmann, and W. Vasconcelos, *Some remarks on pole shifting problems over rings*, to appear in J. Pure Appl. Algebra, 1979.

Byrnes, C., *On the control of certain infinite dimensional systems by algebra-geometric techniques*, Amer. J. of Math., December 1978.

Dieudonne, *Cours de Geometrie Algêbrique*, Univ. France Press, 1974.

Eilenberg, S., *Automata, Languages, and Machines, Vol. A*, Academic Press, 1974.

El'Sgol'Ts, L. E., *Introduction to the Theory of Differential Equations with Deviating Argument*, Holden-Day, 1966.

Grasselli, O. M. and A. Isidori, *Uniform observers for bilinear systems*, submitted for publication, 1979.

Hautus, M. L. J., *Controllability and observability conditions of linear autonomous systems*, Nederl. Akad. Wetensch., Proc., Ser. A72, 1969, pp. 443-448.

Hautus, M. L. J., *Stabilization, controllability and observability of linear autonomous systems*, Nederl. Akad. Wetensch., Proc., Ser. A73, 1970, pp. 448-455.

Kamen, E. W., *On the algebraic theory of systems defined by convolution operators*, Math. Syst. Theory, Vol. 9, 1975, pp. 57-74.

Kamen, E. W., *Lectures on algebraic system theory: Linear systems over rings*, NASA Contractor Report 3016, 1978.

Kamen, E. W., *An operator theory of linear functional differential equations*, J. of Diff. Equations, Vol. 27, 1978, pp. 274-297.

Morse, A. S., *Ring models for delay-differential systems*, Automatica, Vol. 12, 1976, pp. 529-531.

Pandolfi, L., *On feedback stabilization of functional differential equations*, Bull. Un. Mat. Ital., Vol. 11, 1975, pp. 626-635.

Rouchaleau, Y., *Linear, discrete-time, finite-dimensional dynamical systems over some classes of commutative rings*, Ph.D. dissertation, Stanford University, Stanford, California, 1972.

Rouchleau, Y. and E. D. Sontag, *On the existence of minimal realizations of linear dynamical systems over Noetherian integral domains*, J. Comp. Syst. Sciences, Vol. 18, 1979, pp. 65-75.

Rouchleau, Y., B. F. Wyman, and R. E. Kalman, *Algebraic structure of linear dynamical systems, III. Realization theory over a commutative ring*, Proc. Nat. Acad. Sci. U.S.A., Vol. 69, 1978, pp. 3404-3406.

Sontag, E. D., *Linear systems over commutative rings: A survey*, Ric. di Automatica, Vol. 7, 1976, pp. 1-34.

Sontag, E. D., *On Split realizations of response maps over rings*, Information and Control, Vol. 37, 1978, pp. 23-33.

Sontag, E. D., *Polynomial Response Maps*, Lecture Notes in Control and Info. Sciences, Springer, 1979.

*DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TECHNOLOGY
EINDHOVEN, NETHERLANDS

**DEPARTMENT OF MATHEMATICS
RUTGERS UNIVERSITY
NEW BRUNSWICK, NEW JERSEY
U.S.A. 08903