

Universal formulas for feedback stabilization with respect to Minkowski balls[☆]

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Received 29 August 1999; received in revised form 9 March 2000

Abstract

This note provides explicit algebraic stabilizing formulas for clf's when controls are restricted to certain Minkowski balls in Euclidean space. Feedbacks of this kind are known to exist by a theorem of Artstein, but the proof of Artstein's theorem is nonconstructive. The formulas are obtained from a general feedback stabilization technique and are used to construct approximation solutions to some stabilization problems. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Constrained controls; Control-Lyapunov functions

1. Introduction

Smooth control-Lyapunov functions (clf's) provide foundations for much current feedback control design. See for instance the appropriate sections in the textbooks [3–6,11]. The theory of smooth clf's had its origins in Artstein's paper [1]. See also [9] for *nonsmooth* clf's, and the recent work [2] for applications of the latter. A very useful characteristic of clf's is the existence of 'universal formulas' for stabilization (cf. [10] and the above textbooks).

This note continues the search, started in [7] (see also [8]), for universal clf formulas for *constrained* controls. By a universal formula one means, informally (with precise definitions given later), an expression for a feedback law in terms of the directional (or 'Lie') derivatives of the given clf (which is assumed known) in the directions of the vector fields that define the system which renders the plant globally asymptotically stable. Paper [7] provided one such formula in the case of unit balls $\{x \in \mathbb{R}^m: \|x\|_2 < 1\}$ with respect to Euclidean norms. The formula is a very simple algebraic function of the Lie derivatives. This paper will present analogous formulas for the case of control sets

$$B_{m,p} := \left\{ x = (x_1, \dots, x_m)' \in \mathbb{R}^m: \|x\|_p := \left[\sum_{j=1}^m |x_j|^p \right]^{1/p} < 1 \right\}$$

[☆] This research was supported in part by US Air Force Grant F49620-98-1-0242.

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(which we call *unit balls in p -norms*) for certain values of $p > 1$ other than two. These new formulas are then used to approximately solve the stabilization problem for the control set $(-1, +1)^2$ (which we will sometimes denote by $B_{2,\infty}$) and the control set $B_{m,1}$ and to exactly stabilize with respect to these latter two sets when a ‘strong clf’ (cf. Section 6) is known.

Consider the following control system on \mathbb{R}^n :

$$\dot{x} = f(x) + G(x)u. \quad (1)$$

The entries of f and of the $n \times m$ matrix G are smooth functions on \mathbb{R}^n and $f(0) = 0$. Controls take values in $B_{m,p}$, where p is a positive number we further specify shortly. This note will consider the following problem.

Problem. Find a control law $k : \mathbb{R}^n \rightarrow B_{m,p}$ with the property that the closed-loop system

$$\dot{x} = f(x) + G(x)k(x) \quad (2)$$

has a continuous right side for $x \neq 0$ and is globally asymptotically stable about $x = 0$.

For example, find a feedback for a multi-input plant with two independent saturating inputs which renders the plant globally asymptotically stable with respect to $x = 0$. This is a significant saturation problem which frequently arises in aerospace engineering, robotics, chemical process control, etc. Since $B_{2,2} \subseteq (-1, +1)^2$, a natural approach to solving this problem is to find a clf for the plant with controls in $(-1, +1)^2$ and then to use that clf in the stabilizing feedback formula of Lin and Sontag [7] for plants with controls in $B_{2,2}$. This procedure gives a feedback which is valued in $(-1, +1)^2$, but the feedback that results may not render the plant globally asymptotically stable with respect to $x = 0$. As an example, set

$$f((x_1, x_2)') := (x_1, x_2)' \quad \text{and} \quad G((x_1, x_2)') := -\frac{2}{3} \begin{pmatrix} x_1 & \delta((x_1, x_2)')x_1 \\ x_2 & \delta((x_1, x_2)')x_2 \end{pmatrix}, \quad (3)$$

where

$$\delta((x_1, x_2)') := \frac{1}{2} \left[1 + \frac{x_1^2 + x_2^2}{1 + x_1^2 + x_2^2} \right].$$

The function $\frac{1}{2}(x_1^2 + x_2^2)$ is a clf for the system defined by (3) with controls in $(-1, +1)^2$, but if we use the feedback law of Lin and Sontag [7] with this clf, then we get a feedback that does not stabilize the plant. We will study this example in greater detail in Section 2, and Section 6 gives a general technique for stabilizing systems with saturating inputs.

Let us now suppose that some feedback $k : \mathbb{R}^n \rightarrow B_{m,p}$ is such that (2) has a continuous right side for $x \neq 0$ and is globally asymptotically stable about $x = 0$. Converse Lyapunov theorems then guarantee the existence of a positive definite (meaning, $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$), proper (i.e., $V(x) \rightarrow \infty$ as $\|x\|_2 \rightarrow \infty$), smooth mapping $V = V_k : \mathbb{R}^n \rightarrow \mathbb{R}$ such that the Lyapunov condition

$$\inf_{u \in B_{m,p}} \{a(x) + b(x)u\} < 0 \quad (4)$$

holds for each $x \neq 0$, where $a(x) := \nabla V_k(x)f(x)$ and $b(x) := \nabla V_k(x)G(x)$. To show that V_k exists, simply find a Lyapunov function for (2) and put $u = k(x)$ in (4). Any positive definite, proper, smooth function V_k satisfying (4) for each $x \neq 0$ is called a *control-Lyapunov function (clf)* with respect to (1) with controls in $B_{m,p}$. If k is continuous at the origin, then V_k has an additional *small control property (scp)*: For each $\varepsilon > 0$, there is a $\delta > 0$ such that if $0 \neq \|x\|_2 < \delta$, then there is some u with $\|u\|_2 < \varepsilon$ such that $a(x) + b(x)u < 0$. Observe, for future use, that if $B_{m,p} \subseteq W$, then any clf with respect to (1) and controls in $B_{m,p}$ having the scp is also a clf with respect to (1) and controls in W having the scp.

In [1], the following elegant converse to these facts is shown: If there is a clf V with respect to (1) with controls in $B_{m,p}$, then there is a feedback $k : \mathbb{R}^n \rightarrow B_{m,p}$ which is smooth on $\mathbb{R}^n \setminus \{0\}$ and which globally stabilizes the system, and this k can be taken to be continuous on \mathbb{R}^n if V has the scp. The result holds for rather arbitrary control-value sets, including all Minkowski unit balls. The proof is nonconstructive, being based on partitions of unity.

In [7], an explicit formula for k which is algebraic in a and b is given for (1) for the case of controls valued in $B_{m,2}$. The formula is a simple algebraic function of the Lie derivatives $a(x)$ and $b(x)$. However, as we saw above, this formula is not appropriate for controls in other Minkowski balls (cf. Section 2).

In this paper, we give the following formulas for feedback stabilizers for controls valued in $B_{m,p}$ when $p = 2r/(2r - 1)$ and r is any positive integer:

$$k_p(x) = (k_{p,1}(x), \dots, k_{p,m}(x))', \quad (5)$$

where

$$k_{p,j}(x) = \lambda_p(a(x), \|b(x)\|_{2r}^{2r})(b_j(x))^{2r-1} \quad \text{for } j = 1, \dots, m, \quad (6)$$

and

$$\lambda_p(a, b) = - \begin{cases} \frac{a + \sqrt[2r]{a^{2r} + b^{2r}}}{(1 + \sqrt[2r]{1 + b^{2r-1}})b} & \text{if } b > 0, \\ 0 & \text{if } b = 0. \end{cases} \quad (7)$$

Call a function on \mathbb{R}^n *almost smooth* if it is smooth on $\mathbb{R}^n \setminus \{0\}$ and continuous. We generalize the construction in [7] by proving the following.

Theorem 1. *Let $p = 2r/(2r - 1)$ for some $r \in \mathbb{N}$. If V is a control-Lyapunov function with respect to (1) with controls in $B_{m,p}$, then (5)–(7) is smooth on $\mathbb{R}^n \setminus \{0\}$, takes values in $B_{m,p}$, and globally stabilizes the system with respect to $x = 0$. Moreover, if the right-hand side of (1) is real analytic in x and V is real analytic, then k is real analytic on $\mathbb{R}^n \setminus \{0\}$. Furthermore, if V satisfies the small control property, then the feedback k_p is almost smooth on \mathbb{R}^n .*

Note that (6) with $r = 1$ gives the feedback stabilizer of Lin and Sontag [7]. Also note that the case of controls valued in the *closure* of the relevant Minkowski ball offers no difficulty, since if (4) holds with such controls, then by continuity it also holds with $u \in B_{m,p}$, so that if the Lyapunov condition holds with the closed constraint set, then there is a feedback taking values in the ball itself (and hence in particular in its closure).

As an application of Theorem 1, we deduce the following approximation theorem. For each $S \subset \mathbb{R}^m$ and $\varepsilon > 0$, we put

$$S^\varepsilon := \left\{ x \in \mathbb{R}^m : \inf_{y \in S} \|x - y\|_2 < \varepsilon \right\},$$

the ε -enlargement of S , and we set

$$B_{m,\infty} := \left\{ x = (x_1, \dots, x_m)' \in \mathbb{R}^m : \max_{i=1, \dots, m} |x_i| < 1 \right\}$$

for each $m \in \mathbb{N}$.

Theorem 2. *Let $j = 1$ or ∞ (in the second case, assuming $m = 2$). If V is a control-Lyapunov function satisfying the small control property for (1) with controls in $B_{m,j}$ and $\varepsilon > 0$ is given, then there is an almost smooth feedback ϕ_j , taking values in $B_{m,j}^\varepsilon$ and algebraic in the corresponding Lie derivatives, which globally stabilizes the system with respect to the equilibrium $x = 0$. This feedback is real analytic on $\mathbb{R}^n \setminus \{0\}$ if the right side of (1) is real analytic in x and V is real analytic.*

In this way, algebraic feedback stabilization is also ‘almost’ possible for $B_{m,1}$ and $B_{2,\infty}$, modulo vanishingly small overflows of the feedbacks’ values.

This paper is organized as follows. In Section 2, we show that the use of the earlier formula of Lin and Sontag [7] can lead to wrong results when the control set is $B_{m,p}$ for $p \neq 2$. This is followed in Section 3 by the precise definitions of universal stabilizing formulas (usf’s) and a lemma which produces usf’s for

rotated Minkowski balls. In Section 4 we introduce the notion of universal stability maps (usm's) and their associated control sets, and we give a general technique useful for finding stabilizing feedback laws for control sets associated with invertible usm's. In Section 5, the general technique is used to prove Theorem 1 and the main results of Lin and Sontag [7] as well as Theorem 2, which is a consequence of the rotation lemma of Section 3. We also give a procedure, illustrated in Section 6, for *exact* almost smooth stabilization for $(-1, +1)^2$ and $B_{m,1}$ for cases where the known clf has extra regularity (cf. Theorem 4 below).

2. Examples

We illustrate, by means of simple counterexamples, how the use of the previously known formula for the feedback of Artstein's theorem for controls constrained to $B_{m,2}$ can lead to wrong results if the constraint set is $B_{m,p}$ for some p other than 2. First, we consider the case where the inputs of the plant saturate and the known clf for the plant with controls in $(-1, +1)^2$ is not a clf for the plant with controls in $B_{2,2}$. In this case, the feedback law from Lin and Sontag [7] is valued in the right control set, but it does not stabilize the system. We also consider the case where the stabilizing formula for controls in $B_{m,2}$ cannot be used for systems with controls in $B_{m,p}$ for $p \neq 2$ because it takes some of its values outside $B_{m,p}$. In this case, the previously known formula stabilizes the plant, but the input restrictions on the plant prevent the stabilizer from being used.

2.1. Clf's for $(-1, +1)^m$ which are not clf's for $B_{m,2}$

Since $B_{m,2} \subseteq (-1, +1)^m$, one natural approach to stabilizing a plant with two independent saturating inputs is to find a clf for the plant with controls in $(-1, +1)^2$ and to then use this clf in the stabilizing formula in [7] for controls in $B_{2,2}$. As noted above, any clf for our system with respect to controls in a set U is also a clf with respect to \tilde{U} whenever $\tilde{U} \supseteq U$. However, it is not always the case that V is a clf for (1) and controls in \tilde{U} when $\tilde{U} \subset U$ and V is a clf for (1) and controls in U . Thus, use of the $B_{m,2}$ feedback law of Lin and Sontag [7] with a clf for $B_{m,\infty}$ can lead to a feedback which is valued in the correct control set but which does not globally asymptotically stabilize the system.

For example, put $n = m = 2$, with

$$q := \begin{pmatrix} x \\ y \end{pmatrix}, \quad f(q) := q, \quad \delta(q) := \frac{1}{2} \left[1 + \frac{x^2 + y^2}{1 + x^2 + y^2} \right] \quad \text{and} \quad G(q) := -\frac{2}{3} \begin{pmatrix} x & \delta(q)x \\ y & \delta(q)y \end{pmatrix}.$$

Set $V(q) = \frac{1}{2} \|q\|_2^2$. Then, for $q \neq 0$, we get

$$x^2 + y^2 = \nabla V(q) f(q) > \|\nabla V(q) G(q)\|_2 = \frac{2}{3} \sqrt{[x^2 + y^2]^2 (1 + \delta(q)^2)} = \frac{2}{3} (x^2 + y^2) \sqrt{1 + \delta(q)^2}$$

(since $\delta < 1$), so V cannot be a clf with respect to $B_{2,p}$ for any $p \leq 2$. In fact, if k is a feedback law for the system which is valued in $B_{2,p}$ for some $p \in [1, 2]$, and ζ is a trajectory for the corresponding closed loop system with $\zeta(0) \neq 0$, then we have

$$\begin{aligned} \frac{d}{dt} V(\zeta(t)) &= \nabla V(\zeta(t)) f(\zeta(t)) + \nabla V(\zeta(t)) G(\zeta(t)) k(\zeta(t)) \\ &\geq \nabla V(\zeta(t)) f(\zeta(t)) - \|\nabla V(\zeta(t)) G(\zeta(t))\|_2 \|k(\zeta(t))\|_2 \\ &\geq \nabla V(\zeta(t)) f(\zeta(t)) - \|\nabla V(\zeta(t)) G(\zeta(t))\|_2 \|k(\zeta(t))\|_p \\ &\geq \nabla V(\zeta(t)) f(\zeta(t)) - \|\nabla V(\zeta(t)) G(\zeta(t))\|_2 \\ &> 0 \end{aligned}$$

for $t > 0$, so the system is not g.a.s. for any feedback law valued in $B_{2,p}$ for $p \leq 2$.

On the other hand, if

$$r^2 := x^2 + y^2 \neq 0,$$

then

$$\nabla V(q) \left[f(q) + G(q) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] = r^2 - \frac{2}{3}(r^2, \delta(q)r^2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = r^2 \{1 - \frac{2}{3}[1 + \delta(q)]\} < 0,$$

since $\delta > \frac{1}{2}$. By continuity, we conclude that V is a clf with respect to $B_{2,\infty}$, even though it is not a clf with respect to any of the Minkowski balls $B_{2,p}$ for $p \leq 2$.

2.2. Cases where the $B_{2,2}$ formula gives values outside $B_{2,p}$ for $p < 2$

Even in the presence of a clf V with the scp for a given Minkowski ball $B_{m,p}$ for $p < 2$, use of the $B_{m,2}$ law, evaluated along the Lie derivatives for V , can give feedback values outside $B_{m,p}$. Therefore, use of the formula from Lin and Sontag [7] for plants with controls in $B_{2,p}$ for $p \neq 2$ can give excessive feedback values which overwhelm the system. We show this to be the case in a rather strong sense by exhibiting an analytic control-affine system with equilibrium value zero which has

$$V(\cdot) := \frac{1}{2} \|\cdot\|_2^2$$

as a clf with the scp for all the Minkowski balls and which has the property that for any $p \in (1, 2)$, the $B_{m,2}$ law given above, evaluated along the Lie derivatives $a(\cdot) = \nabla V(\cdot)f(\cdot)$ and $b(\cdot) = \nabla V(\cdot)G(\cdot)$, takes some values outside $B_{m,p}$.

Take $m = n = 2$, with

$$f \equiv 0 \quad \text{and} \quad G(x, y) := \begin{pmatrix} x & x^2 \\ y & y \end{pmatrix}.$$

Then,

$$\nabla V(x, y)G(x, y) \begin{pmatrix} u \\ v \end{pmatrix} = u(x^2 + y^2) + v(x^3 + y^2) < 0$$

except at the origin for suitable (u, v) 's in any Minkowski ball (and, in fact, in any neighborhood of the origin, by merely picking $v = 0$ and $u = -\varepsilon/(x^2 + y^2)$ with $\varepsilon > 0$ small). So V is a clf for the system with respect to all the $B_{2,p}$'s and the origin. Moreover, these clf's have the scp. The $B_{m,2}$ feedback law in this case is

$$-\frac{1}{1 + \sqrt{1 + (x^2 + y^2)^2 + (x^3 + y^2)^2}} \begin{pmatrix} x^2 + y^2 \\ x^3 + y^2 \end{pmatrix}.$$

When $x = 0$, this has p norm

$$\left[\frac{2^{1/2}y^2}{1 + \sqrt{1 + 2y^4}} \right] 2^{(2-p)/2p}.$$

When $p \in [1, 2)$, this tends to a limit above 1 as $|y|$ increases. Thus, the $B_{m,2}$ feedback law may be invalid for all the control sets $B_{m,p}$ with $p \in [1, 2)$, even when the feedback laws are constructed using a clf which is valid for all of these Minkowski balls.

3. Universal stabilizing formulas

Following Lin and Sontag [8], we reduce the search for regular feedbacks to a search for universal formulas. For any $U \subset \mathbb{R}^m$, we set $\mathcal{D}(U) := \{(a, b) \in \mathbb{R} \times \mathbb{R}^m : a + bu < 0 \text{ for some } u \in U\}$ and recall the following definition.

Definition 3.1. Let $U \subseteq \mathbb{R}^m$. A *universal stabilizing formula* relative to U is a real-analytic function

$$\alpha : \mathcal{D}(U) \subseteq \mathbb{R} \times \mathbb{R}^m \rightarrow U \subseteq \mathbb{R}^m$$

such that the following two conditions hold:

1. For any (a, b) in $\mathcal{D}(U)$, $a + b\alpha(a, b) < 0$.
2. For any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\{(a, b) \in \mathcal{D}(U)\} \wedge \{a < \delta \|b\|_2\} \wedge \{|a| < \delta\} \wedge \{\|b\|_2 < \delta\} \Rightarrow \|\alpha(a, b)\|_2 < \varepsilon. \quad (8)$$

We require analyticity to disallow ‘tricks’ such as partitions of unity. Observe that condition 2 can be replaced by the following: 2'. For any $\tilde{\varepsilon} > 0$ and any norms $\|\cdot\|_s, \|\cdot\|_t$, and $\|\cdot\|_u$ which are equivalent to $\|\cdot\|_2$, there is a $\tilde{\delta} > 0$ such that

$$\{(a, b) \in \mathcal{D}(U)\} \wedge \{a < \tilde{\delta} \|b\|_s\} \wedge \{|a| < \tilde{\delta}\} \wedge \{\|b\|_t < \tilde{\delta}\} \Rightarrow \|\alpha(a, b)\|_u < \tilde{\varepsilon}. \quad (9)$$

From Lin and Sontag [8], we have the following clf characterization.

Lemma 3.2. Let α be a universal stabilizing formula relative to $U \subseteq \mathbb{R}^m$ and let V be clf relative to (1) and U . Set $a(x) := \nabla V(x)f(x)$ and $b(x) := \nabla V(x)G(x)$. Then, $k(x) := \alpha(a(x), b(x))$ is smooth on $\mathbb{R}^n \setminus \{0\}$ and globally stabilizes (1). If, in addition, V has the scp, then k is almost smooth. Moreover, if the right-hand side of the system is real analytic in x and V is real analytic, then k is real analytic on $\mathbb{R}^n \setminus \{0\}$.

The proof uses condition 2 only to establish the almost smoothness of k when V has the scp. The proof of the second theorem makes use of the following ‘rotation lemma’.

Lemma 3.3. If $k(a, b)$ is a universal stabilizing formula with respect to the control set $U \subseteq \mathbb{R}^m$ and if $T : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is an invertible linear map, then $k_T(a, b) := Tk(a, T'b)$ is a universal stabilizing formula with respect to TU .

Proof. First note that

$$\mathcal{D}(TU) = (I \times (T')^{-1})\mathcal{D}(U).$$

Thus, if $(a, b) \in \mathcal{D}(TU)$, then $(a, T'b) \in \mathcal{D}(U)$, so

$$a + Tk(a, T'b)b = a + k(a, T'b)T'b < 0,$$

so k_T satisfies the first condition in Definition 3.1.

Now let $\varepsilon > 0$. Since $\|\cdot\|_2$ and $\|T'(\cdot)\|_2$ are equivalent, it remains to show that there is a $\delta > 0$ such that

$$\{(a, b) \in \mathcal{D}(TU)\} \wedge \{a < \delta \|T'b\|_2\} \wedge \{|a| < \delta\} \wedge \{\|T'b\|_2 < \delta\} \Rightarrow \|Tk(a, T'b)\|_2 < \varepsilon \quad (10)$$

be the equivalence of conditions 2 and 2' above. Pick $\hat{\delta}$ so that

$$\{(a, b) \in \mathcal{D}(U)\} \wedge \{a < \hat{\delta} \|b\|_2\} \wedge \{|a| < \hat{\delta}\} \wedge \{\|b\|_2 < \hat{\delta}\} \Rightarrow \|k(a, b)\|_2 < \varepsilon. \quad (11)$$

If (a, b) satisfies the hypothesis of (10) with $\delta = \hat{\delta}$, then $(a, T'b) \in \mathcal{D}(U)$ satisfies the hypothesis of (11), so we can pick $\delta = \hat{\delta}$ to satisfy condition (10). \square

This lemma will be used to reduce the stabilization problem for the control set $(-1, 1)^2$ to the problem of stabilizing with respect of $B_{m,1}$.

4. Usm's and their associated control sets

This section illustrates how the search for usf's for Minkowski balls can be viewed in the more general framework of a search for feedback formulas for control sets associated with universal stability maps (usm's).

These control sets have the form $U = \{x \in \mathbb{R}^m: \lambda(x) < 1\}$, where λ is scalar homogeneous (meaning $\lambda(\alpha x) = \alpha \lambda(x)$ for all $x \in \mathbb{R}^m$ and $\alpha \geq 0$), but in this more general framework, λ need not be continuous and U need not be precompact. The feedback laws obtained in this more general framework will be called *usm-based feedbacks*. We use the following definitions.

Definition 4.1. A *universal stability map (usm)* is a concave surjective function $\theta: [0, \infty) \rightarrow [0, \infty)$ which is real analytic on $(0, \infty)$ and satisfies $\theta(xy) \geq \theta(x)\theta(y)$ and $\theta'(x) > 0$ for all x and y in $(0, \infty)$. A usm is called *invertible* if its inverse extends to an even real analytic function on \mathbb{R} .

Definition 4.2. Let θ be an invertible usm, let $U \subseteq \mathbb{R}^m$, and let $q: \mathbb{R} \rightarrow \mathbb{R}$ be real analytic. We say that U is *q-associated with the usm θ* if

$$\mathcal{D}(U) \subseteq \left\{ (a, b) \in \mathbb{R} \times \mathbb{R}^m: a < \theta \left(\sum_{j=1}^m q(b_j) b_j \right) \right\},$$

where $\mathcal{D}(U)$ is as defined in Section 3 above.

4.1. Examples of usm's

Observe that $x^{1/2r}$ is an invertible usm for all $r \in \mathbb{N}$, and one easily checks that $B_{m, 2r/(2r-1)}$ is *q-associated* with this usm with the choice $q(x) := x^{2r-1}$. A totally different example is as follows. We show that $\theta(x) = (x \tanh x)^{1/4r}$ is a usm for each $r \in \mathbb{N}$. The concavity of these θ 's follows from the fact that

$$\frac{\sinh x}{x} \text{ increases on } (0, \infty) \quad \text{and} \quad \frac{\tanh x}{x} \text{ decreases on } (0, \infty) \tag{12}$$

and the calculation

$$\begin{aligned} \frac{d}{dx} (x \tanh x)^{1/4r} &= \frac{1}{4r} (x \tanh x)^{(1-4r)/4r} (\tanh x + x \operatorname{sech}^2 x) \\ &= \frac{1}{4r} \left\{ \frac{1}{x^{1-(1/2r)}} \left[\frac{\tanh x}{x} \right]^{1/4r} + \left[\frac{x}{\tanh^{4r-1} x \cosh^{8r} x} \right]^{1/4r} \right\} \\ &= \frac{1}{4r} \left\{ \frac{1}{x^{1-(1/2r)}} \left[\frac{\tanh x}{x} \right]^{1/4r} + \left[\frac{x}{\sinh^{4r-1} x \cosh^{4r+1} x} \right]^{1/4r} \right\} \\ &= \frac{1}{4r} \left\{ \frac{1}{x^{1-(1/2r)}} \left[\frac{\tanh x}{x} \right]^{1/4r} + \left[\frac{1}{x^{4r-2}} \left(\frac{x}{\sinh x} \right)^{4r-1} \frac{1}{\cosh^{4r+1} x} \right]^{1/4r} \right\}, \end{aligned}$$

and one easily verifies that $\theta' > 0$ on $(0, \infty)$ also. Since \tanh is real analytic, we conclude that the θ 's are usm's if $\tanh xy \geq \tanh x \tanh y$ for x and y in $(0, \infty)$. If $y \geq 1$, the monotonicity of \tanh implies that $\tanh y \leq 1 \leq \tanh xy / \tanh x$, so by symmetry in x and y we can assume that x and y are both smaller than one. Fix $y \in (0, 1)$, and define θ_y by $\theta_y(x) := \tanh xy - \tanh x \tanh y$. It remains to show that $\theta'_y(x) \geq 0$ for all $x \in (0, 1)$, i.e., that

$$\left(\frac{\cosh x}{\cosh xy} \right)^2 \geq \frac{\tanh y}{y} \quad \text{for all } x \in (0, 1). \tag{13}$$

But if $y < 1$, then $x > xy$, so the left of (13) is at least one, and one dominates the right side, since $\tanh y - y$ has the nonpositive derivative $1/\cosh^2(y) - 1$. The fact that $(x \tanh x)^{1/4r}$ is an invertible usm follows from the following general invertibility result.

Proposition 4.3. *Let θ be a usm, and assume there is an $\varepsilon > 0$, an $r \in \mathbb{N}$, and a real analytic function $h: (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ so that $h(0) > -1$ and*

$$\theta(x) = x^{1/2r}[1 + h(x)] \quad \text{for all } x \in (-\varepsilon, \varepsilon).$$

Then θ is an invertible usm.

Proof. Introduce $g(u) = u[1 + h(u^{2r})]$. We see that $g'(0) \neq 0$, so we can let G denote the local inverse of g near 0, the existence of which is guaranteed by the Inverse Function Theorem. For $x > 0$ near zero, we therefore get $G[x^{1/2r}(1 + h(x))] = x^{1/2r}$, so $x = [G(\theta(x))]^{2r}$. Put $K(y) = [G(y)]^{2r}$, which is real analytic in a neighborhood of 0. Since g is odd, G is odd, so K is even. About any positive number, the Inverse Function Theorem implies that θ is invertible. The uniqueness of inverse now gives a $\delta > 0$ and a well-defined $\tilde{\gamma}: (-\delta, \infty) \rightarrow \mathbb{R}$ which is even on $(-\delta, \delta)$ and which inverts θ on $[0, \infty)$. Extending $\tilde{\gamma}$ to \mathbb{R} by symmetry (i.e., evenness), we get a real analytic, even function γ which inverts θ , as desired. \square

To show that $[x \tanh x]^{1/4r}$ is an invertible usm, first write $x \tanh x = x^2(1 + \kappa(x))$, then set $h(x) = \sqrt[4r]{1 + \kappa(x)} - 1$ in the above proposition. Similarly, $[x^2 \tanh x^2]^{1/8r}$ is an invertible usm for all $r \in \mathbb{N}$, with the concavity following from (12) and the calculation

$$\begin{aligned} \frac{d}{dx}(\tanh x^2)^{1/4r} &= \frac{x}{2r} [\tanh x^2]^{(1-4r)/4r} \operatorname{sech}^2 x^2 \\ &= \frac{x(\cosh x^2)^{-1/4r}}{2r \cosh x^2 [\sinh x^2]^{(4r-1)/4r}} \\ &= \left(\frac{1}{2rx^{(2r-1)/2r} [\cosh x^2]^{(4r+1)/4r}} \right) \left(\frac{x^2}{\sinh x^2} \right)^{(4r-1)/4r}. \end{aligned}$$

4.2. Universal formulas for usm's

From Sontag [10], we know that the function defined on $S := \{(a, b) \in \mathbb{R}^2: b \leq 0 \Rightarrow a < 0\}$ by

$$(a, b) \mapsto \begin{cases} \frac{a + \sqrt{a^2 + b^4}}{b} & \text{if } b \neq 0, \\ 0 & \text{if } b = 0 \end{cases}$$

is real analytic. To get our general class of usm-based feedbacks, we generalize this result to the following.

Proposition 4.4. *Let S be as above, let θ be an invertible usm, and let θ^{-1} extend to the even analytic function γ on \mathbb{R} . Let $\zeta: \mathbb{R} \rightarrow \mathbb{R}$ be positive definite and real analytic. Then $\phi: S \rightarrow \mathbb{R}$ defined by*

$$\phi(a, b) = \begin{cases} \frac{\theta(\gamma(a) + \zeta(b)b^2) + a}{b} & \text{if } b \neq 0, \\ 0 & \text{if } b = 0 \end{cases}$$

is real analytic.

Proof. First note that if $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ is real analytic, then

$$D^f(a, t, y, b) := \begin{cases} \frac{f(a, t + b, y) - f(a, t, y)}{b} & \text{if } b \neq 0, \\ \frac{\partial f}{\partial t}(a, t, y) & \text{if } b = 0 \end{cases}$$

is real analytic on \mathbb{R}^4 , as is seen by writing it as $\int_0^1 (\partial f / \partial t)(a, t + \lambda b, y) d\lambda$. Choosing

$$f(a, t, y) := \gamma(ty - a)$$

and then evaluating the result at $t = 0$ allows us to conclude that

$$L(y, a, b) := \begin{cases} \frac{\gamma(by - a) - \gamma(a) - \zeta(b)b^2}{b} & \text{if } b \neq 0, \\ y\gamma'(-a) & \text{if } b = 0 \end{cases}$$

too is real analytic. But, $(\phi(a, b), a, b)$ is a root of L for all $(a, b) \in S$, so analyticity will follow from the Implicit Function Theorem if $(\partial L / \partial y)(\phi(a, b), a, b) \neq 0$ on S . But,

$$\frac{\partial L}{\partial y}(\phi(a, b), a, b) = \begin{cases} \gamma'(\theta[\gamma(a) + \zeta(b)b^2]) & \text{if } b \neq 0, \\ \gamma'(-a) & \text{if } b = 0, \end{cases}$$

For $b \neq 0$, this is nonzero, since $\gamma'(l) > 0$ when $l > 0$ and since $\theta(u)$ is positive for u positive; and for $b = 0$, this is again nonzero, since on S , a and b are not simultaneously zero and since $|\gamma'(\cdot)|$ is positive away from zero. This gives the needed analyticity of ϕ . \square

Note that the condition $\theta(xy) \geq \theta(x)\theta(y)$ from the definition of a usm was not needed in the proof. Equipped with Proposition 4.4, we are now ready to give the technique for constructing usm-based feedbacks.

Theorem 3. Assume that $\lambda: \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}$ is even and scalar homogeneous, that θ is an invertible usm, and that $U := \{x \in \mathbb{R}^m: \lambda(x) < 1\}$ is q -associated with θ . Let ζ and γ be as in Proposition 4.4, and set $\beta((b_1, \dots, b_m)') = \sum_{j=1}^m q(b_j)b_j$. If

$$\lambda\{(q(b_1), \dots, q(b_m))'\} \leq \frac{[1 + \theta\{1 + \zeta[\beta(b)]\beta(b)]\beta(b)}{\theta\{\beta(b)\} + \theta\{\beta(b) + \zeta[\beta(b)][\beta(b)]^2\}} \quad \text{for all } b \neq 0, \tag{14}$$

then the mapping $\mu = (\mu_1, \dots, \mu_m)$ defined by

$$\mu_j(a, b) := - \begin{cases} \frac{a + \theta(\gamma(a) + \zeta[\beta(b)][\beta(b)]^2)}{(1 + \theta\{1 + \zeta[\beta(b)]\beta(b)\})\beta(b)} q(b_j), & b \neq 0, \\ 0, & b = 0, \end{cases} \tag{15}$$

is real analytic on $\mathcal{D}(U)$, satisfies $a + \mu(a, b)b < 0$ for all $(a, b) \in \mathcal{D}(U)$, and is U -valued. Therefore, if V is a clf for (1) with controls in U , then $\mu(\nabla V(x)f(x), \nabla V(x)G(x))$ is smooth on $\mathbb{R}^n \setminus \{0\}$ and globally stabilizes the system with respect to $x = 0$ and controls in U . It is real analytic on $\mathbb{R}^n \setminus \{0\}$ when the right-hand side of (1) is real analytic in x and V is real analytic.

Proof. It suffices to show that the following two conditions hold on $\mathcal{D}(U)$:

$$a < \frac{a + \theta(\gamma(a) + \zeta(\sum_{j=1}^m q(b_j)b_j)(\sum_{j=1}^m q(b_j)b_j)^2)}{1 + \theta(1 + \zeta(\sum_{j=1}^m q(b_j)b_j) \sum_{j=1}^m q(b_j)b_j)} \quad \text{when } a > 0 \tag{16}$$

and

$$\frac{[a + \theta(\gamma(a) + \zeta(\sum_{j=1}^m q(b_j)b_j)(\sum_{j=1}^m q(b_j)b_j)^2)]\lambda\{(q(b_1), \dots, q(b_m))'\}}{[1 + \theta(1 + \zeta(\sum_{j=1}^m q(b_j)b_j) \sum_{j=1}^m q(b_j)b_j)] \sum_{j=1}^m q(b_j)b_j} < 1 \quad \text{when } b \neq 0. \tag{17}$$

These correspond to the negativity and boundedness requirements from Definition 3.1, respectively. The other assertions would then follow from the fact that compositions of analytic functions are analytic and the fact that, in the proof of Lemma 3.2 in [8], Condition 2 of Definition 3.1 is used only to show the almost smoothness of the feedback when V has the scp. When $a \geq 0$, condition (17) obtains as an immediate consequence of

(14), since $a < \theta(\sum_{j=1}^m q(b_j)b_j)$ implies $\gamma(a) < \beta(b)$; and for $a < 0$, (17) follows from (14) and subadditivity (which is consequence of the concavity of usm's).

It remains to verify condition (16). This follows if $\theta\{1 + \zeta(\sum_{j=1}^m q(b_j)b_j) \sum_{j=1}^m q(b_j)b_j\}$ is strictly majorized by

$$\frac{\theta(\gamma(a) + \zeta(\sum_{j=1}^m q(b_j)b_j)(\sum_{j=1}^m q(b_j)b_j)^2)}{\theta(\gamma(a))},$$

whenever $0 < a < \theta(\sum_{j=1}^m q(b_j)b_j)$. But by the concavity assumption, the mapping

$$y \mapsto \frac{\theta(y + \zeta(\sum_{j=1}^m q(b_j)b_j)(\sum_{j=1}^m q(b_j)b_j)^2)}{\theta(y)},$$

is strictly decreasing on $(0, \infty)$, so the majorization would follow from

$$\theta \left(1 + \zeta \left(\sum_{j=1}^m q(b_j)b_j \right) \sum_{j=1}^m q(b_j)b_j \right) \leq \frac{\theta(\sum_{j=1}^m q(b_j)b_j + \zeta(\sum_{j=1}^m q(b_j)b_j)(\sum_{j=1}^m q(b_j)b_j)^2)}{\theta(\sum_{j=1}^m q(b_j)b_j)}$$

which follows from the condition

$$\theta(xy) \geq \theta(x)\theta(y)$$

from the definition of a usm. \square

Remark 4.5. Note that condition (14) was used only to show that the feedback given by (15) was valued in U and to require that $\beta(\cdot)$ is positive except possibly at 0. Also, if one does not require μ to be U -valued, then we can relax the requirement that the usm θ is concave to the requirement that θ is log-concave (i.e., that $\ln \theta$ is concave on $(0, \infty)$).

5. Proof of Theorems 1 and 2

To prove Theorem 1, it suffices to show that for each fixed $r \in \mathbb{N}$, the function k_p given by

$$k_p = (k_{p,1}, \dots, k_{p,m}); \quad k_{p,j}(a, b) = \lambda_p(a, \|b\|_{2r}^{2r}) b_j^{2r-1} \quad \text{for } j = 1, \dots, m$$

is a universal stabilizing formula relative to $B_{m,p}$. The argument is an elementary application of Theorem 3. Fixing $p = 2r/(2r-1)$ for some $r \in \mathbb{N}$, we set

$$\begin{aligned} \lambda(x) &= \|x\|_{2r/(2r-1)}, & \theta(x) &= x^{1/2r}, & \gamma(x) &= x^{2r}, \\ U &= B_{m,2r/(2r-1)}, & \zeta(b) &= b^{2(r-1)} & \text{and } q(x) &= x^{2r-1}. \end{aligned}$$

Then θ is an invertible usm, and one easily checks that with this choice of λ , θ , ζ , and q , the feedback law (15) is exactly (6). Therefore, all but the last assertion of Theorem 1 follows once we check that $B_{m,2r/(2r-1)}$ is q -associated with the usm $x^{1/2r}$ and that condition (14) holds.

If $a + bu$ for some $u \in B_{m,p}$, then Hölder's inequality gives

$$a < |bu| \leq \|b\|_{p/(p-1)} \|u\|_p \leq \|b\|_{p/(p-1)} = \left[\sum_{j=1}^m |b_j|^{p/(p-1)} \right]^{(p-1)/p} = \left[\sum_{j=1}^m b_j^{2r-1} b_j \right]^{1/2r},$$

which gives the associatedness.

If $b = (b_1, \dots, b_m)' \neq 0$, then the right-hand side of (14) becomes

$$\begin{aligned} \frac{[1 + \{1 + (\|b\|_{2r}^{2r})^{2(r-1)}\|b\|_{2r}^{2r}\}^{1/2r}]\|b\|_{2r}^{2r}}{\|b\|_{2r}^{2r} + [\|b\|_{2r}^{2r} + (\|b\|_{2r}^{2r})^{2(r-1)}[\|b\|_{2r}^{2r}]^{1/2r}]^{2r}} &= \frac{1 + \{1 + (\|b\|_{2r}^{2r})^{2(r-1)}\}^{1/2r}\|b\|_{2r}^{2r}}{1 + \{1 + (\|b\|_{2r}^{2r})^{2(r-1)}\}^{1/2r}\|b\|_{2r}^{2r}} \\ &= \|b\|_{2r}^{2r-1} = \left[\sum_{j=1}^m b_j^{2r} \right]^{(2r-1)/2r} = \left\{ \sum_{j=1}^m (b_j^{2r-1})^{\frac{2r}{2r-1}} \right\}^{(2r-1)/2r} \\ &= \|(b_1^{2r-1}, \dots, b_m^{2r-1})\|_{2r/(2r-1)}, \end{aligned}$$

which is exactly $\lambda((q(b_1), \dots, q(b_m))')$, so (14) holds also.

The last assertion of Theorem 1 is also easily verified. Indeed, by the equivalence of conditions 2 and 2' above, we need only show that, given $\varepsilon > 0$, one could find a $\delta > 0$ such that

$$\{(a, b) \in \mathcal{D}(B_{m,p})\} \wedge \{a < \delta \|b\|_{2r}\} \wedge \{|a| < \delta\} \wedge \{\|b\|_{2r} < \delta\} \Rightarrow \|k_p\|_p < \varepsilon. \tag{18}$$

We show this next. (The fact that $(a, b) \in \mathcal{D}(B_{m,p})$ is redundant). Let (a, b) be as in the hypothesis of this implication, with $\delta > 0$ arbitrary for the moment. Then, $\|k_p\|_p$ is at most

$$\frac{\delta + \sqrt[2r]{\delta^{2r} + (\|b\|_{2r}^{2r})^{2r-1}}}{1 + \sqrt[2r]{1 + (\|b\|_{2r}^{2r})^{2r-1}}},$$

which can evidently be made as small as desired when $\|b\|_{2r}$ and δ are both taken to be small enough.

To prove our second theorem, let $\varepsilon > 0$ be given, then pick $p = 2r/(2r - 1)$ with r large enough so that $B_{m,p} \subset B_{m,1}^c$. Since k_p is a universal stabilizing formula for $B_{m,p}$, and since V is a clf for $B_{m,p}$ when it is one for $B_{m,1}$, we can pick $\phi_1 = k_p$. The result for the $B_{m,\infty}$ case with $m = 2$ now follows by rotating the $p = 1$ result. Let T be the rotation-dilation of \mathbb{R}^2 satisfying

$$T(B_{2,1}) = B_{2,\infty}$$

and let $\varepsilon > 0$ be given. Let ϕ_1 be the version associated with $\varepsilon/\sqrt{2}$. By Lemma 2.3, $T\phi_1(a, T'b)$ is a universal stabilizing formula for $T(B_{2,1})$, and therefore is a suitable ϕ_∞ . Indeed, $(a, b)' \in \mathcal{D}(B_{2,\infty})$ implies

$$d^{\|\cdot\|_2}(T\phi_1(a, T'b), B_{2,\infty}) \leq \sqrt{2}d^{\|\cdot\|_2}(\phi_1(a, T'b), B_{2,1}) < \sqrt{2}\frac{\varepsilon}{\sqrt{2}} = \varepsilon.$$

Remark (Maximality of the feedback family (6)).

For each $p > 1$, we let $q := p/(p - 1)$, and define $k_q : \mathcal{D}(B_{m,p}) \rightarrow \mathbb{R}^m$ by

$$k_q = (k_{q,1}, \dots, k_{q,m})',$$

where, for $j = 1, \dots, m$, we put

$$k_{q,j}(a, b) = - \begin{cases} \frac{a + \sqrt[q]{|a|^q + (\|b\|_q^q)^q}}{(1 + \sqrt[q]{1 + (\|b\|_q^q)^{q-1}})\|b\|_q^q} b_j^{q-1} & \text{if } b \neq 0, \\ 0 & \text{if } b = 0. \end{cases} \tag{19}$$

Then,

$$k_{q,1}(0, (1/2, b_2, 0, \dots, 0)') = \frac{-1/2^{q-1}}{1 + \sqrt[q]{1 + [(1/2^q) + |b_2|^q]^{q-1}}}.$$

If this function is real analytic on $\mathcal{D}(B_{m,p})$, then $|x|^q$ is real analytic near 0, so q is an even integer and $p = 2r/(2r - 1)$ for some $r \in \mathbb{N}$. This establishes the ‘maximality’ of the Minkowski ball construction (6).

6. Universal formulas for strong CLFs

As we saw in Section 2, if V is a clf with respect to (1) and controls in a set U , and if $\tilde{U} \subset U$, then it may or may not be the case that V is also a clf with respect to (1) and controls in \tilde{U} . If U is open and V is a clf for (1) with controls in U and also for (1) with controls in some set \tilde{U} with $\text{cl}(\tilde{U}) \subseteq U$, then we call V a *strong control-Lyapunov function (scLf)* with respect to (1) and controls in U . Now suppose that V is an scLf for (1) and controls in $U = B_{2,\infty}$, and set

$$\alpha B_{m,p} := \{x \in \mathbb{R}^m : \|x\|_p < \alpha\}$$

for all $m \in \mathbb{N}$, $p > 1$, and $\alpha \in (0, 1)$. Choose T as in the proof of Theorem 2, i.e., T is the linear transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

For $\alpha \in (0, 1) \cap \mathbb{Q}$ near enough to 1, it follows that V is also a clf for the control sets $\alpha B_{2,\infty}$ and $\hat{T}(B_{2,p})$ when $p = 2r/(2r - 1)$ and $r \in \mathbb{N}$ is chosen large enough so that $\hat{T}(B_{2,p}) \subset B_{2,\infty}$ and $\hat{T} := \alpha T$. A reapplication of Theorem 1 and Lemma 3.3 now gives a $\hat{T}(B_{2,p})$ -valued feedback which globally stabilizes (1) with respect to 0 and $B_{2,\infty}$. Similar reasoning gives the following variant of Theorem 2.

Theorem 4. *Let $j = 1$ or ∞ , assuming in the second case that $m = 2$. If V is an scLf for (1) with controls in $B_{m,j}$, then there is a $B_{m,j}$ -valued feedback ϕ_j which is algebraic in the Lie derivatives and smooth on $\mathbb{R}^n \setminus \{0\}$ which globally stabilizes the system with respect to $x = 0$. This feedback is real analytic on $\mathbb{R}^n \setminus \{0\}$ if the right-hand side of (1) is real analytic in x and V is real analytic.*

As in Theorems 1 and 2, we can take the ϕ_j 's of Theorem 4 to be almost smooth if V also satisfies the scp. To illustrate the stabilizing procedure of Theorem 4, consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + 10x_2, \\ \dot{x}_2 &= (10x_1 + x_2)u, \\ \dot{x}_3 &= -x_3 + 10x_4, \\ \dot{x}_4 &= (10x_3 + x_4)v \end{aligned} \tag{20}$$

with the controls

$$(u, v)' \in B_{2,\infty}$$

and set $x = (x_1, x_2, x_3, x_4)'$ in the sequel. We look for a feedback law $k : \mathbb{R}^4 \rightarrow B_{2,\infty}$ which is real analytic on $\mathbb{R}^4 \setminus \{0\}$, which is algebraic in the corresponding Lie derivatives $a(x) = \nabla V(x)f(x)$ and $b(x) = (b_1(x), b_2(x))' = \nabla V(x)G(x)$ for a suitable clf V , and which renders (20) g.a.s. Setting

$$V(x) = \frac{1}{2} \|x\|_2^2,$$

the Lyapunov condition (4) becomes

$$\inf_{(u,v)' \in B_{2,\infty}} \{10x_1x_2 - x_1^2 + 10x_3x_4 - x_3^2 + (10x_1x_2 + x_2^2)u + (10x_3x_4 + x_4^2)v\} < 0, \tag{21}$$

which is satisfied for $u = v = -1$ for all nonzero x . By continuity, it follows that V is a clf for (20) with controls in $B_{2,\infty}$. One obvious choice for the stabilizing feedback is

$$k_1(x) = - \begin{cases} \frac{(10x_1x_2 + x_2^2, 10x_3x_4 + x_4^2)'}{\max\{|10x_1x_2 + x_2^2|, |10x_3x_4 + x_4^2|\}}, & (10x_1x_2 + x_2^2)^2 + (10x_3x_4 + x_4^2)^2 \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

which evidently stabilizes the system, but this cannot be extended continuously to $\mathbb{R}^4 \setminus \{0\}$ (as is seen by setting $10x_3x_4 + x_4^2 = 0$, $x_2 = 1$, and examining the one-sided limits $x_1 \rightarrow -\frac{1}{10}^+$ and $x_1 \rightarrow -\frac{1}{10}^-$). We now use the method of Theorem 4 to find the desired feedback.

We first show that V is a strong clf for (20). The polynomial $\delta^2 - \frac{51}{25}\delta + 1$ is positive on $(0, 0.81)$ and negative on $(0.82, 1]$. Setting $u = v = -\delta$ in the infimand in (21) and completing squares gives

$$\sum_{j=1,3} \{10(1 - \delta)x_j x_{j+1} - x_j^2 - \delta x_{j+1}^2\} = \sum_{j=1,3} \{-[x_j + 5(\delta - 1)x_{j+1}]^2 + 25x_{j+1}^2 (\delta^2 - \frac{51}{25}\delta + 1)\},$$

which is negative for all x when $\delta \in (0.82, 1]$. If $x_1 = x_2 \neq 0$ and $x_3 = x_4 = 0$, then the infimand in (21) is only negative if $u < \frac{9}{11}$. Therefore, V is a clf for (20) with controls in $(\frac{-9}{10}, \frac{9}{10})^2$ but not for (20) with controls in $(-\delta, \delta)^2$ with $\delta < \frac{8}{10}$.

Let \tilde{T} be the linear transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{\tilde{T}} \frac{9}{10} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

so $\tilde{T}(B_{2,1}) = \frac{9}{10}B_{2,\infty}$. We look for a $p > 1$ so that

$$B_{2,1} \subset B_{2,p} \subsetneq \frac{10}{9}B_{2,1}.$$

The first inclusion, of course, holds by definition. The second inclusion is equivalent to

$$\frac{1}{2^{1/p}} < \frac{10}{18}, \tag{22}$$

i.e., the boundary points of $B_{2,p}$ along the line $y=x$ are inside $\frac{10}{9}B_{2,1}$. Notice that we *cannot* pick $p=2$, even if $\frac{9}{10}$ is replaced by some $\delta \in (0.82, 1)$, so even in conjunction with the rotation lemma, we cannot stabilize (20) for controls in $(-1, +1)^2$ using the formula in [7]. Condition (22) holds for $p=2r/(2r-1)$ when $r=5$, so we apply the rotation lemma to get a feedback stabilizer for $\tilde{T}(B_{2,10/9})$. Setting

$$\gamma(x) := \left(\frac{9}{10}\right)^{10} \{[x_2(10x_1 + x_2) + x_4(10x_3 + x_4)]^{10} + [x_2(10x_1 + x_2) - x_4(10x_3 + x_4)]^{10}\},$$

and

$$\eta_{\pm} := 10x_3x_4 + x_4^2 \pm (10x_1x_2 + x_2^2),$$

this gives

$$\frac{-(\frac{9}{10})^{10} \left(\sum_{j=1,3} (10x_j x_{j+1} - x_j^2) + \sqrt[10]{\left[\sum_{j=1,3} (10x_j x_{j+1} - x_j^2) \right]^{10} + \gamma^{10}(x)} \right)}{[1 + \sqrt[10]{1 + \gamma^9(x)}]\gamma(x)} \begin{pmatrix} \eta_+^9(x) - \eta_-^9(x) \\ \eta_+^9(x) + \eta_-^9(x) \end{pmatrix}$$

for $\gamma(x) \neq 0$ and 0 otherwise. By the proof of Theorem 4, this feedback is real analytic on $\mathbb{R}^4 \setminus \{0\}$, valued in $B_{2,\infty}$ and renders (20) g.a.s.

7. Conclusions

Let $\varepsilon > 0$ be given, and consider system (1) with controls in $(-1, +1)^2$. Suppose that V is a real-analytic control-Lyapunov function for this system which has the small control property and that the vector fields defining the system are also real analytic. Then one can find a feedback law $k : \mathbb{R}^n \rightarrow ((-1, +1)^2)^\varepsilon$ such that (2) has $x=0$ as a globally stable equilibrium. The control can be written as a simple algebraic function of the Lie-derivatives and is real-analytic on $\mathbb{R}^n \setminus \{0\}$. The same is true if $(-1, +1)^2$ is replaced by $B_{m,1}$ for any m in \mathbb{N} . If V is merely smooth, then k is almost smooth.

In both cases, the formulas are members of a family of algebraic universal stabilizing formulas having the feedback formula of Lin and Sontag [7] as one of its members, up to a rotation that reduces the $(-1, +1)^2$ case to the $B_{2,1}$ case. The formulas can also be viewed as members of a more general class of not-necessarily algebraic, usm-based feedback laws with similar properties. In this way, we can globally stabilize (1) relative

to $U = B_{m,1}$ or $U = B_{m,\infty}$ with an algebraic feedback k as long as we permit k to take its values in U^ε , where $\varepsilon > 0$ is as small as desired. We can also globally stabilize (1) with respect to the control set $U := B_{m,2r/(2r-1)}$, for any r and $m \in \mathbb{N}$ using universal stabilizing formulas, and for cases where a strong clf is known, this can be done for $B_{m,1}$ for each m and for $B_{2,\infty}$ as well. This generalizes the result of Lin and Sontag [7], which treats only the case of the control set $B_{m,2}$.

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