# New Characterizations of Input to State Stability


Eduardo D. Sontag
Yuan Wang



*Abstract*— **We present new characterizations of the Input to State Stability property. As a consequence of these results, we show the equivalence between the ISS property and several (apparent) variations proposed in the literature.**


## I. Introduction

This paper studies stability questions for systems of the general form

$$\Sigma: \qquad \dot{x} = f(x, u)\,, \tag{1}$$

with states $x(t)$ evolving in Euclidean space $\mathbb{R}^n$ and controls $u(\cdot)$ taking values $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m$, for some positive integers $n$ and $m$ (in all the main results, $\mathbb{U} = \mathbb{R}^m$). The questions to be addressed all concern the study of the size of each solution $x(t)$ — its asymptotic behavior as well as maximum value — as a function of the initial condition $x(0)$ and the magnitude of the control $u(\cdot)$.

One of the most important issues in the study of control systems is that of understanding the dependence of state trajectories on the magnitude of inputs. This is especially relevant when the inputs in question represent disturbances acting on a system. For linear systems, this leads to the consideration of gains and the operator-theoretic approach, including the formulation of $H^\infty$ control. For not necessarily linear systems, there is no complete agreement as yet regarding what are the most useful formulations of system stability with respect to input perturbations. One candidate for such a formulation is the property called "input to state stability" (ISS), introduced in [12]. Various authors, (see e.g. [4], [5], [6], [10], [17] have subsequently employed this property in studies ranging from robust control and highly nonlinear small-gain theorems to the design of observers and the study of parameterization issues; for expositions see [14] and most especially the textbooks [7], [8]. The ISS property is defined in terms of a decay estimate of solutions, and is known (cf. [15]) to be equivalent to the validity of a dissipation inequality

$$\frac{dV(x(t))}{dt} \;\leq\; \sigma(|u(t)|) - \alpha(|x(t)|)$$

holding along all possible trajectories (this is reviewed below), for an appropriate "energy storage" function $V$ and comparison functions $\sigma, \alpha$. (A dual notion of "output-to-state stability" (OSS) can also be introduced, and leads to the study of nonlinear detectability; see [16].)


E. Sontag is with SYCON - Rutgers Center for Systems and Control, Department of Mathematics, Rutgers University, New Brunswick, NJ 08903. e-mail: sontag@control.rutgers.edu.


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Y. Wang is with Department of Mathematics, Florida Atlantic University, Boca Raton, FL 33431. e-mail: ywang@control.math.fau.edu.


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In some cases, notably in [2], [6], [18], authors have suggested apparent variations of the ISS property, which are more natural when solving particular control problems. The main objective of this paper is to point out that such variations are in fact theoretically equivalent to the original ISS definition. (This does not in any way diminish the interest of these other authors' contributions; on the contrary, the alternative characterizations are of great interest, especially since the actual estimates obtained may be more useful in one form than another. For instance, the "small-gain theorems" given in [6], [2] depend, in their applicability, on having the ISS property expressed in a particular form. This paper merely states that from a theoretical point of view, the properties are equivalent. For an analogy, the notion of "convergence" in $\mathbb{R}^n$ is independent of the particular norm used — e.g. all $L^p$ norms are equivalent — but many problems are more naturally expressed in one norm than another.)

One of the main conclusions of this paper is that the ISS property is equivalent to the conjunction of the following two properties: (i) asymptotic stability of the equilibrium $x = 0$ of the unforced system (that is, of the system defined by Equation (1) with $u \equiv 0$) and (ii) every trajectory of (1) asymptotically approaches a ball around the origin whose radius is a function of the supremum norm of the control being applied. We prove this characterization along with many others. Since it is not harder to do so, the results are proved in slightly more generality, for notions relative to an arbitrary compact attractor rather than the equilibrium $x = 0$.

### A. Basic Definitions and Notations

Euclidean norm in $\mathbb{R}^n$ or $\mathbb{R}^m$ is denoted simply as $|\cdot|$. More generally, we will study notions relative to nonempty subsets $\mathcal{A}$ of $\mathbb{R}^n$; for such a set $\mathcal{A}$, $|\xi|_{\mathcal{A}} = d(\xi, \mathcal{A}) = \inf\{d(\eta, \xi),\ \eta \in \mathcal{A}\}$ denotes the point-to-set distance from $\xi \in \mathbb{R}^n$ to $\mathcal{A}$. (So for the special case $\mathcal{A} = \{0\}$, $|\xi|_{\{0\}} = |\xi|$.) We also let, for each $\varepsilon > 0$ and each set $\mathcal{A}$:

$$B(\mathcal{A}, \varepsilon) := \{\xi \mid |\xi|_{\mathcal{A}} < \varepsilon\}\,, \quad \overline{B}(\mathcal{A}, \varepsilon) := \{\xi \mid |\xi|_{\mathcal{A}} \leq \varepsilon\}\,.$$

*Most of the results to be given are new even for $\mathcal{A} = \{0\}$, so the reader may wish to assume this, and interpret $|\xi|_{\mathcal{A}}$ simply as the norm of $\xi$.* (We prefer to deal with arbitrary $\mathcal{A}$ because of potential applications to systems with parameters as well as the "practical stability" results given in Section VI.)

The map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ in (1) is assumed to be locally Lipschitz continuous. By a *control* or *input* we mean a measurable and locally essentially bounded function $u : \mathcal{I} \to \mathbb{R}^m$, where $\mathcal{I}$ is a subinterval of $\mathbb{R}$ which contains the origin, so that $u(t) \in \mathbb{U}$ for almost all $t$. Given a system with control-value set $\mathbb{U}$, we often consider the same system but with controls restricted to take values in some subset $\mathcal{O} \subseteq \mathbb{U}$; we use $\mathcal{M}_{\mathcal{O}}$ for the set of all such controls.

Given any control $u$ defined on an interval $\mathcal{I}$ and any $\xi \in \mathbb{R}^n$, there is a unique maximal solution of the initial value problem $\dot{x} = f(x,u)$, $x(0) = \xi$. This solution is defined on some maximal open subinterval of $\mathcal{I}$, and it is denoted by $x(\cdot, \xi, u)$. (For convenience, we allow negative times $t$ in the expression $x(t, \xi, u)$, even though the interest is in behavior for $t \geq 0$.) A *forward complete system* is one such that, for each $u$ defined on $\mathcal{I} = \mathbb{R}_{\geq 0}$, and each $\xi$, the solution $x(t, \xi, u)$ is defined on the entire interval $\mathbb{R}_{\geq 0}$. The $L^m_\infty$-norm (possibly infinite) of a control $u$ is denoted by $\|u\|_\infty$. That is, $\|u\|_\infty$ is the smallest number $c$ such that $|u(t)| \leq c$ for almost all $t \in \mathcal{I}$. Whenever the domain $\mathcal{I}$ of a control $u$ is not specified, it will be understood that $\mathcal{I} = \mathbb{R}_{\geq 0}$.

A function $F : S \to \mathbb{R}$ defined on a subset $S$ of $\mathbb{R}^n$ containing 0 is *positive definite* if $F(x) > 0$ for all $x \in S$, $x \neq 0$, and $F(0) = 0$. It is *proper* if the preimage $F^{-1}(-D, D)$ is bounded, for each $D > 0$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{N}$ (or an "$\mathcal{N}$ function") if it is continuous and nondecreasing; it is *of class* $\mathcal{N}_0$ (or an "$\mathcal{N}_0$ function") if in addition it satisfies $\gamma(0) = 0$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{K}$ (or a "$\mathcal{K}$ function") if it is continuous, positive definite, and strictly increasing, and is *of class* $\mathcal{K}_\infty$ if it is also unbounded (equivalently, it is proper, or $\gamma(s) \to +\infty$ as $s \to +\infty$). Finally, recall that $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a function of class $\mathcal{KL}$ if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \geq 0$, $\beta(s, t)$ decreases to zero as $t \to \infty$. (The notations $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ are fairly standard; the notations $\mathcal{N}$ and $\mathcal{N}_0$ are introduced here for convenience.)

### B. A Catalog of Properties

We catalog several properties of control systems which will be compared in this paper. Much of the terminology — except for "ISS" and the names for properties of unforced systems — is not standard, and should be considered tentative.

A *zero-invariant set* $\mathcal{A}$ for a system $\Sigma$ as in Equation (1) is a subset $\mathcal{A} \subseteq \mathbb{R}^n$ with the property that $x(t, \xi, \underline{0}) \in \mathcal{A}$ for all $t \geq 0$ and all $\xi \in \mathcal{A}$, where $\underline{0}$ denotes the control which is identically equal to zero on $\mathbb{R}_{\geq 0}$.

From now on, all definitions are with respect to a given *forward-complete* system $\Sigma$ as in Equation (1), and a given *compact* zero-invariant set $\mathcal{A}$ for this system. The main definitions follow.

We first recall the definition of the (ISS) property:

$$\exists\, \gamma \in \mathcal{K},\, \beta \in \mathcal{KL} \text{ st}: \forall\, \xi \in \mathbb{R}^n\, \forall\, u(\cdot)\, \forall\, t \geq 0$$
$$|x(t, \xi, u)|_{\mathcal{A}} \;\leq\; \beta(|\xi|_{\mathcal{A}}, t) \,+\, \gamma(\|u\|_\infty)\,. \qquad \text{(ISS)}$$

This was the form of the original definition of (ISS) given in [12]. It is known that a system is (ISS) if and only if it satisfies a dissipation inequality, that is to say, there exists a smooth $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and there are functions $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2, 3$ and $\sigma \in \mathcal{K}$ so that

$$\alpha_1(|\xi|_{\mathcal{A}}) \leq V(\xi) \leq \alpha_2(|\xi|_{\mathcal{A}}) \qquad (2)$$

and

$$\nabla V(\xi) f(\xi, v) \leq \sigma(|v|) - \alpha_3(|\xi|_{\mathcal{A}}) \qquad (3)$$

for each $\xi \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$. See [15], [14] for proofs and an exposition, respectively. A very useful modification of this characterization due to [11] is the fact that the (ISS) property is also equivalent to the existence of a smooth $V$ satisfying (2) and Equation (3) replaced by an estimate of the type $\nabla V(\xi) f(\xi, v) \leq -V(\xi) - \alpha_3(|\xi|_{\mathcal{A}})$. (This can be understood as: "for some positive definite and proper functions $y = V(x)$ and $v = W(u)$ of states and outputs respectively, along all trajectories of the system we have $\dot{y} = -y + v$".) The main purpose of this paper is to establish further equivalences for the (ISS) property.

It will be technically convenient to first introduce a local version of the property (ISS), by requiring only that the estimate hold if the initial state and the controls are small, as follows:

$$\exists\, \rho > 0, \gamma \in \mathcal{K},\, \beta \in \mathcal{KL} \text{ st}: \;\forall\, |\xi|_{\mathcal{A}} \leq \rho,\; \forall\, \|u\|_\infty \leq \rho$$
$$|x(t, \xi, u)|_{\mathcal{A}} \;\leq\; \beta(|\xi|_{\mathcal{A}}, t) \,+\, \gamma(\|u\|_\infty) \quad \forall\, t \geq 0\,. \qquad \text{(LISS)}$$

Several standard properties of the "unforced" system obtained when $u \equiv 0$ will appear as technical conditions. We review these now. The *0-global attraction property with respect to* $\mathcal{A}$ (0-GATT) holds if every trajectory $x(\cdot)$ of the zero-input system

$$(\Sigma_0): \qquad \dot{x} = f(x, 0) \qquad (4)$$

satisfies $\lim_{t\to\infty} |x(t, \xi, \underline{0})|_{\mathcal{A}} \to 0$; if this is merely required of trajectories with initial conditions satisfying $|x(0)|_{\mathcal{A}} < \rho$, for some $\rho > 0$, we have the *0-local attraction property with respect to* $\mathcal{A}$ (0-LATT). The *0-local stability property with respect to* $\mathcal{A}$ (0-LS) means that for each $\varepsilon > 0$ there is a $\delta > 0$ so that $|\xi|_{\mathcal{A}} < \delta$ implies that $|x(t, \xi, \underline{0})|_{\mathcal{A}} < \varepsilon$ for all $t \geq 0$. Finally, the *0-asymptotic stability property with respect to* $\mathcal{A}$ (0-AS) is the conjunction of (0-LATT) and (0-LS), and the *0-global asymptotic stability property with respect to* $\mathcal{A}$ (0-GAS) is the conjunction of (0-GATT) and (0-LS). Note that (0-GAS) is equivalent to the conjunction of (0-AS) and (0-GATT). It is useful (see e.g. [3], [12], [7]) to express these properties in terms of comparison functions:

$$\exists\, \beta \in \mathcal{KL} \text{ st}: \forall\, \xi \in \mathbb{R}^n\; \forall\, t \geq 0$$
$$|x(t, \xi, \underline{0})|_{\mathcal{A}} \;\leq\; \beta(|\xi|_{\mathcal{A}}, t)\,. \qquad \text{(0-GAS)}$$

and

$$\exists\, \rho > 0, \beta \in \mathcal{KL} \text{ st}: \forall\; |\xi|_{\mathcal{A}} < \rho\; \forall\, t \geq 0$$
$$|x(t, \xi, \underline{0})|_{\mathcal{A}} \;\leq\; \beta(|\xi|_{\mathcal{A}}, t) \qquad \text{(0-AS)}$$

respectively.

Next we introduce several new concepts. The *limit property with respect to* $\mathcal{A}$ holds if every trajectory must at some time get to within a distance of $\mathcal{A}$ which is a function of the magnitude of the input:

$$\exists\,\gamma \in \mathcal{N}_0 \text{ st} : \forall\,\xi \in \mathbb{R}^n \;\; \forall\,u(\cdot)$$
$$\inf_{t\geq 0} |x(t,\xi,u)|_{\mathcal{A}} \;\leq\; \gamma\,(\|u\|_\infty)\;. \tag{LIM}$$

Observe that, if this property holds, then it also holds with some $\gamma \in \mathcal{K}_\infty$. However, the case $\gamma \equiv 0$ will be of interest, since it corresponds to a notion of attraction for systems in which controls $u$ are viewed as disturbances.

The *asymptotic gain property with respect to* $\mathcal{A}$ holds if every trajectory must ultimately stay not far from $\mathcal{A}$, depending on the magnitude of the input:

$$\exists\,\gamma \in \mathcal{N}_0 \text{ st} : \forall\,\xi \in \mathbb{R}^n \;\; \forall\,u(\cdot)$$
$$\overline{\lim_{t\to+\infty}}\, |x(t,\xi,u)|_{\mathcal{A}} \;\leq\; \gamma\,(\|u\|_\infty)\;. \tag{AG}$$

Again, if the property holds, then it also holds with some $\gamma \in \mathcal{K}_\infty$, but the case $\gamma \equiv 0$ will be of interest later. The *uniform asymptotic gain property with respect to* $\mathcal{A}$ holds if the limsup in (AG) is attained uniformly with respect to initial states in compacts and all $u$:

$$\exists\,\gamma \in \mathcal{N}_0 \;\forall\,\varepsilon > 0 \;\; \forall\,\kappa > 0 \;\exists\,T = T(\varepsilon,\kappa) \geq 0 \text{ st} :\; \forall\,|\xi|_{\mathcal{A}} \leq \kappa$$
$$\sup_{t\geq T} |x(t,\xi,u)|_{\mathcal{A}} \;\leq\; \gamma\,(\|u\|_\infty) + \varepsilon \quad \forall u(\cdot)\;. \tag{UAG}$$

The *boundedness property with respect to* $\mathcal{A}$ holds if bounded initial states and controls produce uniformly bounded trajectories:

$$\exists\,\sigma_1,\sigma_2 \in \mathcal{N} \text{ st} :\; \forall\,\xi \in \mathbb{R}^n \;\; \forall\,u(\cdot)$$
$$\sup_{t\geq 0} |x(t,\xi,u)|_{\mathcal{A}} \;\leq\; \max\left\{\,\sigma_1(|\xi|_{\mathcal{A}})\,,\,\sigma_2(\|u\|_\infty)\,\right\}\;. \tag{BND}$$

(This is sometimes called the "UBIBS" or "uniform bounded-input bounded-state" property.) The *global stability property with respect to* $\mathcal{A}$ holds if in addition small initial states and controls produce uniformly small trajectories:

$$\exists\,\sigma_1,\sigma_2 \in \mathcal{N}_0 \text{ st} :\; \forall\,\xi \in \mathbb{R}^n \;\; \forall\,u(\cdot)$$
$$\sup_{t\geq 0} |x(t,\xi,u)|_{\mathcal{A}} \;\leq\; \max\left\{\,\sigma_1(|\xi|_{\mathcal{A}})\,,\,\sigma_2(\|u\|_\infty)\,\right\}\;. \tag{GS}$$

Observe that, if this property holds, then it also holds with both $\sigma_i \in \mathcal{K}_\infty$. The *local stability property with respect to* $\mathcal{A}$ holds if we merely require a local estimate of this type:

$$\exists\,\delta > 0,\; \alpha_1,\alpha_2 \in \mathcal{N}_0 \text{ st} :\; \forall\;|\xi|_{\mathcal{A}} \leq \delta \;\; \forall\;\|u\|_\infty \leq \delta$$
$$\sup_{t\geq 0} |x(t,\xi,u)|_{\mathcal{A}} \;\leq\; \max\left\{\,\alpha_1(|\xi|_{\mathcal{A}})\,,\,\alpha_2(\|u\|_\infty)\,\right\}\;. \tag{LS}$$

If this property holds, then it also holds with both $\alpha_i \in \mathcal{K}_\infty$, $i = 1,2$

*Theorem 1:* Assume given any forward-complete system $\Sigma$ as in Equation (1), with $\mathbb{U} = \mathbb{R}^m$, and a compact zero-invariant set $\mathcal{A}$ for this system. The following properties are equivalent:

A. (ISS)
B. (LIM) & (0-AS)
C. (UAG)
D. (LIM) & (0-GAS)
E. (AG) & (0-GAS)
F. (AG) & (LISS)
G. (AG) & (LS)
H. (LIM) & (LS)
I. (LIM) & (GS)
J. (AG) & (GS)

This theorem will follow from a several technical facts which are stated in the next section and proved later in the paper. These technical results are of interest in themselves.

### C. List of Main Technical Steps

We assume given a forward-complete system $\Sigma$ as in Equation (1), with $\mathbb{U} = \mathbb{R}^m$, and a compact zero-invariant set $\mathcal{A}$ for this system. For ease of reference, we first list several obvious implications:

$$(\text{UAG}) \implies (\text{AG})\,. \tag{5}$$
$$(\text{AG}) \implies (\text{LIM})\,. \tag{6}$$
$$(\text{ISS}) \implies (\text{0-GAS})\,. \tag{7}$$
$$(\text{LISS}) \implies (\text{0-AS})\,. \tag{8}$$
$$(\text{LISS}) \implies (\text{LS})\,. \tag{9}$$

Because (LIM) implies (0-GATT) and (0-GAS) is the same as (0-AS) plus (0-GATT), we have:

$$(\text{LIM})\,\&\,(\text{0-GAS}) \iff (\text{LIM})\,\&\,(\text{0-AS})\,. \tag{10}$$

It was shown in [15] that

$$(\text{ISS}) \quad\iff\quad (\text{UAG})\,\&\,(\text{LS})\,. \tag{11}$$

It turns out that (LS) is redundant, so (UAG) is in fact equivalent to (ISS):

*Proposition I.1:* (UAG) $\Rightarrow$ (LS).

This observation generalizes a result which is well-known for systems with no controls (for which see e.g. [1, Theorem 1.5.28] or [3, Theorem 38.1]). It should be noted that the standing hypothesis that $\mathcal{A}$ is compact is essential for this implication; in the general case of noncompact sets $\mathcal{A}$, the local stability property with respect to $\mathcal{A}$ is not redundant. From Proposition I.1 and Equation (7), we know then that:

$$(\text{UAG}) \implies (\text{0-GAS})\,. \tag{12}$$

We also prove these results:

*Lemma I.2:* (0-GAS) $\implies$ (LISS).
*Lemma I.3:* (BND) & (LS) $\iff$ (GS).
*Lemma I.4:* (LIM) & (GS) $\iff$ (AG) & (GS).
*Lemma I.5:* (LIM) $\Rightarrow$ (BND).

The converse of Lemma I.5 is of course false, as illustrated by the autonomous system $\dot{x} = 0$ (with $n{=}m{=}1$), which even satisfies (GS) but does not satisfy (LIM). From Lemmas I.3 and I.5, we have that:

$$(\text{LIM})\,\&\,(\text{LS}) \iff (\text{LIM})\,\&\,(\text{GS})\,. \tag{13}$$

The most interesting technical result will be this:

*Proposition I.6:* (LIM) & (LS) $\Rightarrow$ (UAG).

We now indicate how the proof of Theorem 1 follows from all these technical facts.

- (A $\iff$ C): by Proposition I.1 and Equation (11).
- (C$\Rightarrow$E): by (5) and (12).
- (E$\Rightarrow$F): by Lemma I.2.
- (F$\Rightarrow$G): by Equation (9).
- (G$\Rightarrow$H): by Equation (6).
- (H$\Rightarrow$I): by Equation (13).
- (I$\Rightarrow$J): by Lemma I.4.
- (J$\Rightarrow$G): obvious.
- (H$\Rightarrow$C): this is Proposition I.6.
- (E$\Rightarrow$D): by Equation (6).
- (B $\iff$ D): by Equation (10).
- (D$\Rightarrow$H): by Lemma I.2 and Equation (9).

A very particular consequence of the main Theorem is worth focusing upon: A $\iff$ J, i.e. (ISS) is equivalent to having both the global stability property with respect to $\mathcal{A}$ and the asymptotic gain property with respect to $\mathcal{A}$. Consider this property:

$$\exists\,\gamma\in\mathcal{N}_0 \text{ st} : \forall\,\xi\in\mathbb{R}^n \;\; \forall\,u(\cdot)$$
$$\overline{\lim_{t\to+\infty}}\,|x(t,\xi,u)|_{\mathcal{A}} \;\leq\; \gamma\left(\overline{\lim_{t\to+\infty}}\,|u(t)|\right) \tag{14}$$

(the limsup being understood in the "essential" sense, of holding up to a set of measure zero; note also that since $\gamma$ is continuous and nondecreasing, the right-hand term equals $\overline{\lim}_{t\to+\infty}\gamma(|u(t)|)$). It is easy to show (see Lemma (II.1)) that this is equivalent to (AG). The conjunction of (14) and (GS) is the "asymptotic $\mathcal{L}_\infty$ stability property" proposed by Teel and discussed in the survey paper [2] (in that paper, $\mathcal{A}=\{0\}$); it thus follows that asymptotic $\mathcal{L}_\infty$ stability is precisely the same as (ISS).

In [18], Tsinias considered the following property (in that paper, $\mathcal{A}=\{0\}$):

$$\exists\,\gamma\in\mathcal{K} \text{ st} : \forall\,\xi\in\mathbb{R}^n \;\; \forall\,u(\cdot)$$
$$[|x(t,\xi,u)|_{\mathcal{A}}\geq\gamma(|u(t)|)\;\forall\,t\geq 0] \;\Rightarrow\; \lim_{t\to\infty}|x(t,\xi,u)|_{\mathcal{A}}=0 \tag{15}$$

which obviously implies (LIM). The author considered the conjunction of (15) and (LS) (more precisely, the author also assumed a local stability property that implies (LS), namely $f(x,u)=Ax+Bu+o(x,u)$, with $A$ Hurwitz); because of the equivalence A $\iff$ H, this conjunction is also equivalent to (ISS).

The outline of the rest of the paper is as follows. In Section II we first prove Proposition I.1, Lemmas I.2, I.3, and I.4, and the equivalence between Property (14) and (AG), all of which are elementary. Section III contains the proof of the basic technical step needed to prove the main result, as well as a proof of Lemma I.5. After this, Section IV establishes a result showing that uniform global asymptotic stability of systems with disturbances (or equivalently, of an associated differential inclusion) follows from the non-uniform variant of the concept; this would appear to be a rather interesting result in itself, and in any case it is used in Section V to provide the proof of Proposition I.6. Finally, in Section VI we make some remarks characterizing so-called "practical" ISS stability in terms of ISS stability with respect to compact sets.

## II. Some Simple Implications

We start with the proof of Proposition I.1.

*Proof:* We will show the following property, which is equivalent to (LS):

$$\forall\,\varepsilon>0\;\exists\,\delta>0 \text{ st} : \forall\;|\xi|_{\mathcal{A}}\leq\delta\;\;\forall\;\|u\|_\infty\leq\delta$$
$$\sup_{t\geq 0}|x(t,\xi,u)|_{\mathcal{A}}\;\leq\;\varepsilon\,. \tag{16}$$

Indeed, assume given $\varepsilon>0$. Let $T=T(\varepsilon/2,1)$. Pick any $\delta_1>0$ so that $\gamma(\delta_1)<\varepsilon/2$. Then: for all $|\xi|_{\mathcal{A}}\leq 1$ and $\|u\|_\infty\leq\delta_1$

$$\sup_{t\geq T}|x(t,\xi,u)|_{\mathcal{A}}\;\leq\;\varepsilon/2+\gamma(\|u\|_\infty)<\varepsilon\,. \tag{17}$$

By continuity (at $u\equiv 0$ and states in $\mathcal{A}$) of solutions with respect to controls and initial conditions, and compactness and zero-invariance of $\mathcal{A}$, there is also some $\delta_2=\delta_2(\varepsilon,T)>0$ so that

$$|\eta|_{\mathcal{A}}\leq\delta_2 \text{ and } \|u\|_\infty\leq\delta_2 \;\Rightarrow\; \sup_{t\in[0,T]}|x(t,\eta,u)|_{\mathcal{A}}\;\leq\;\varepsilon\,.$$

Together with (17), this gives the desired property with $\delta:=\min\{1,\delta_1,\delta_2\}$. ■

We now prove Lemma I.2.

*Proof:* We first note that the 0-global asymptotic stability property with respect to $\mathcal{A}$ implies the existence of a smooth function $V$ such that

$$\alpha_1(|\xi|_{\mathcal{A}})\leq V(\xi)\leq\alpha_2(|\xi|_{\mathcal{A}}) \quad \forall\,\xi\in\mathbb{R}^n,$$

for some $\alpha_1,\alpha_2\in\mathcal{K}_\infty$, and

$$\nabla V(\xi)f(\xi,0)\leq-\alpha_3(|\xi|_{\mathcal{A}}) \quad \forall\,\xi\in\mathbb{R}^n,$$

for some $\alpha_3\in\mathcal{K}_\infty$ (this is well-known; see for instance, [9] for one such a converse Lyapunov theorem). Following exactly the same steps as in the proof of Lemma 3.2 in [13], one can show that there exists some function $\chi\in\mathcal{K}_\infty$ such that for all $\chi(|v|)\leq|\xi|_{\mathcal{A}}\leq 1$,

$$\nabla V(\xi)f(\xi,\,v)\leq-\alpha_3(|\xi|_{\mathcal{A}})/2. \tag{18}$$

(Here we note that in the proof of Lemma 3.2 in [13], the function $g(s)=1$ for $s\in[0,\,1]$.)

Using exactly the same arguments used on page 441 of [12], one can show that there exist a $\mathcal{KL}$-function $\beta$ and a $\mathcal{K}_\infty$-function $\gamma$ so that if $|x(t,\,\xi,\,u)|_{\mathcal{A}}\leq 1$ for all $t\in[0,\,T)$ for some $T>0$, then it holds that

$$|x(t,\,\xi,\,u)|_{\mathcal{A}}\leq\max\{\beta(|\xi|_{\mathcal{A}}\,,\,t),\;\gamma(\|u\|_\infty)\} \tag{19}$$

for all $t\in[0,\,T)$. Let $\rho=\min\{\kappa^{-1}(1/2),\;\gamma^{-1}(1/2)\}$, where $\kappa(r)=\beta(r,\,0)$ for $r\geq 0$. Note here that $\rho\leq\kappa^{-1}(1/2)\leq 1/2$. We now show that the (LISS) property holds with these $\beta$, $\gamma$, and $\rho$. Fix any $\xi$ and $u$ with $|\xi|_{\mathcal{A}}\leq\rho$ and

$\|u\|_\infty \le \rho$. First note that $|x(t,\xi,u)| < 1$ for $t$ small enough.

*Claim:* $|x(t,\xi,u)|_{\mathcal{A}} \le 1$ for all $t \ge 0$.

Assume the claim is false. Then with

$$t_1 = \inf\{t : \ |x(t,\xi,u)|_{\mathcal{A}} \ge 1\},$$

it holds that $0 < t_1 < \infty$. Note then that $|x(t,\xi,u)|_{\mathcal{A}} < 1$ for all $t \in [0, t_1)$. This then implies that

$$|x(t,\xi,u)|_{\mathcal{A}} \le \max\{\beta(\rho,0),\ \gamma(\rho)\} \ \le\ 1/2 \ \ \forall t \in [0, t_1).$$

By continuity, $|x(t_1,\xi,u)|_{\mathcal{A}} < 1$, contradicting to the definition of $t_1$. This shows that $t_1 = \infty$, i.e., $|x(t,\xi,u)|_{\mathcal{A}} \le 1$ for all $t \ge 0$. Thus the estimate in (19) holds for all $t$, as desired. ■

Next we prove Lemma I.3: boundedness property with respect to $\mathcal{A}$ and local stability property with respect to $\mathcal{A}$ implies global stability property with respect to $\mathcal{A}$ (the converse is obvious).

*Proof:* Assume that Equations (BND) and (LS) hold, for a given choice of $\delta, \sigma_1, \sigma_2, \alpha_1, \alpha_2$. Pick a constant $c \ge 0$ and two class-$\mathcal{K}$ functions $\beta_1$ and $\beta_2$ so that, for each $i = 1,2$, $\sigma_i(s) \le \beta_i(s) + c$ for all $s \ge 0$. Pick two class-$\mathcal{K}$ functions $\gamma_1$ and $\gamma_2$ so that, for each $i = 1,2$, it holds that:

$$\begin{array}{lcll} \gamma_i(s) & \ge & \alpha_i(s) & \forall\, 0 \le s \le \delta\,, \\ \gamma_i(s) & \ge & 2\beta_i(s) & \forall\, s \ge 0\,, \\ \gamma_i(s) & \ge & 2[\beta_i(s) + 2c] & \forall\, s \ge \delta\,. \end{array}$$

Consider any $\xi$ and $u$. Then Equation (GS) holds. Indeed, if both $|\xi|_{\mathcal{A}} \le \delta$ and $\|u\|_\infty \le \delta$ then this follows from Equation (LS). Assume now that $|\xi|_{\mathcal{A}} > \delta$. Thus Equation (BND) implies that, for all $t \ge 0$,

$$\begin{array}{rcl} |x(t,\xi,u)|_{\mathcal{A}} & \le & \sigma_1(|\xi|_{\mathcal{A}}) + \sigma_2(\|u\|_\infty) \\ & \le & \beta_1(|\xi|_{\mathcal{A}}) + c + \beta_2(\|u\|_\infty) + c \\ & \le & \beta_1(|\xi|_{\mathcal{A}}) + 2c + (1/2)\gamma_2(\|u\|_\infty) \\ & \le & (1/2)\ [\gamma_1(|\xi|_{\mathcal{A}}) + \gamma_2(\|u\|_\infty)] \\ & \le & \max\{\gamma_1(|\xi|_{\mathcal{A}}), \gamma_2(\|u\|_\infty)\}\ . \end{array}$$

The case $\|u\|_\infty > \delta$ is similar. ■

Lemma I.4 says that the limit property with respect to $\mathcal{A}$ plus the global stability property with respect to $\mathcal{A}$ imply the asymptotic gain property with respect to $\mathcal{A}$; it is shown as follows.

*Proof:* Let $\sigma_1, \sigma_2, \gamma \in \mathcal{N}_0$ be as in (LIM) and (GS). We claim that (AG) holds with:

$$\widetilde{\gamma}(s) \ :=\ \max\{(\sigma_1 \circ \gamma)(s), \sigma_2(s)\}\,.$$

Pick any $\xi, u$, and any $\varepsilon > 0$. By (LIM), there is some $T \ge 0$ so that $|x(T,\xi,u)|_{\mathcal{A}} \le \gamma(\|u\|_\infty) + \varepsilon$. Applying (GS) to the initial state $x(T,\xi,u)$ and the control $v(t) := u(t+T)$ we conclude that

$$\begin{array}{rl} \overline{\lim\limits_{t\to+\infty}} & |x(t,\xi,u)|_{\mathcal{A}} \ \le\ \sup\limits_{t\ge T} |x(t,\xi,u)|_{\mathcal{A}} \\ & \le \max\{\,\sigma_1(\gamma(\|u\|_\infty) + \varepsilon)\,,\ \sigma_2(\|u\|_\infty)\,\} \end{array}$$

and taking $\varepsilon \to 0$ provides the conclusion. ■

Finally, we show:

*Lemma II.1:* Property (14) is equivalent to (AG).

*Proof:* Since $\gamma(\overline{\lim}_{t\to+\infty} |u(t)|) \le \gamma(\|u\|_\infty)$, Property (14) implies (AG), with the same $\gamma$. Conversely, assume that (AG) holds; we next show that Property (14) holds with the same $\gamma$. Pick any $\xi \in \mathbb{R}^n$, control $u$, and $\varepsilon > 0$. Let $r := \overline{\lim}_{t\to+\infty} |u(t)|$. Let $h > 0$ be such that $\gamma(r+h) - \gamma(r) < \varepsilon$. Pick $T > 0$ so that $|u(t)| \le r + h$ for almost all $t \ge T$, and consider the functions $z(t) := x(t+T)$ and $v(t) := u(t+T)$ defined on $\mathbb{R}_{\ge 0}$. Note that $v$ is a control with $\|v\|_\infty \le r + h$ and that $z(t) = x(t,\zeta,v)$, where $\zeta = x(T,\xi,u)$. By the definition of the asymptotic gain property with respect to $\mathcal{A}$, applied with initial state $\zeta$ and control $v$,

$$\begin{array}{rcl} \overline{\lim\limits_{t\to+\infty}}\, |x(t,\xi,u)|_{\mathcal{A}} & = & \overline{\lim\limits_{t\to+\infty}}\, |z(t,\zeta,v)|_{\mathcal{A}} \\ & \le & \gamma(\|v\|_\infty) \ \le\ \gamma(r+h) < \gamma(r) + \varepsilon\,. \end{array}$$

Letting $\varepsilon \to 0$ gives Property (14). ■

## III. Uniform Reachability Time

Let (1) be a forward-complete system. For each subset $\mathcal{O}$ of the input-value space $\mathbb{U}$, each $T \ge 0$, and each subset $C \subseteq \mathbb{R}^n$, we denote

$$\mathcal{R}_{\mathcal{O}}^T(C) \ :=\ \{x(t,\xi,u)\,|\, 0 \le t \le T, u \in \mathcal{M}_{\mathcal{O}}, \xi \in C\}$$

and

$$\begin{array}{rcl} \mathcal{R}_{\mathcal{O}}(C) & := & \{x(t,\xi,u)\,|\, t \ge 0, u \in \mathcal{M}_{\mathcal{O}}, \xi \in C\} \\ & = & \bigcup\limits_{T\ge 0} \mathcal{R}_{\mathcal{O}}^T(C)\,. \end{array}$$

In [9, Proposition 5.1], it is shown that:

*Fact III.1:* Let (1) be a forward-complete system. For each bounded subset $\mathcal{O}$ of the input-value space $\mathbb{U}$, each $T \ge 0$, and each bounded subset $C \subseteq \mathbb{R}^n$, $\mathcal{R}_{\mathcal{O}}^T(C)$ is bounded. □

Given a fixed system (1) which is forward-complete, a point $\xi \in \mathbb{R}^n$, a subset $S \subseteq \mathbb{R}^n$, and a control $u$, one may consider the "first crossing time"

$$\tau(\xi, S, u) \ :=\ \inf\{t \ge 0\,|\, x(t,\xi,u) \in S\}$$

with the convention that $\tau(\xi,S,u) = +\infty$ if $x(t,\xi,u) \notin S$ for all $t \ge 0$.

The following result and its corollary are central. They state in essence that, for bounded controls, if $\tau(\xi,S,u)$ is finite for all $u$ then this quantity is uniformly bounded over $u$, up to small perturbations of $\xi$ and $S$, and (the Corollary) uniformly on compact sets of initial states as well. (Observe that we are not making the assumption that $f$ is convex on control values and that the set of such values is compact and convex, which would make the result far simpler, by means of a routine weak-$\star$ compactness argument.) The result will be mainly applied in the following special case: $\mathcal{O}$ is a closed ball in $\mathbb{R}^m$, $W = \mathbb{R}^n$, and for a given compact set $\mathcal{A}$, $C$ (in the Corollary) is a closed ball of the type

$\overline{B}(\mathcal{A}, 2s)$, $p \in C$, $\Omega = B(\mathcal{A}, 2s)$, and $K = \overline{B}(\mathcal{A}, (3/2)s)$. But the general case is not harder to prove, and it is of independent interest.

*Lemma III.2:* Let (1) be a forward-complete system. Assume given:

- an open subset $\Omega$ of the state-space $\mathbb{R}^n$,
- a compact subset $K \subset \Omega$,
- a bounded subset $\mathcal{O}$ of the input-value space $\mathbb{U}$,
- a point $p \in \mathbb{R}^n$, and
- a neighborhood $W$ of $p$,

so that

$$\sup_{u \in \mathcal{M}_{\mathcal{O}}} \tau(p, \Omega, u) \;=\; +\infty \,. \tag{20}$$

Then there is some point $q \in W$ and some $v \in \mathcal{M}_{\mathcal{O}}$ such that

$$\tau(q, K, v) \;=\; +\infty \,. \tag{21}$$

*Proof:* Let $p_0 = p$ be as in the hypotheses. Thus for each integer $k \geq 1$ we may pick some $d_k \in \mathcal{M}_{\mathcal{O}}$ so that $x(t,\, p_0,\, d_k) \notin \Omega$ for all $0 \leq t \leq k$. For each $j \geq 1$, we let $\theta_j(t) = x(t,\, p_0,\, d_j)$, $t \geq 0$.

Consider first $\{\theta_j(t)\}_{j \geq 1}$ as a sequence of functions defined on [0, 1]. From Fact III.1 we know that there exists some compact subset $S_1$ of $\mathbb{R}^n$ such that $x(t,\, p_0,\, d_j) \in S_1$ for all $0 \leq t \leq 1$, for all $j \geq 1$. Let $M = \sup\{|f(\xi, \lambda)| : \xi \in S_1, \lambda \in \mathcal{O}\}$. Then $\left|\frac{d}{dt}\theta_j(t)\right| \leq M$ for all $j$ and almost all $0 \leq t \leq 1$. Thus the sequence $\{\theta_j(t)\}_{j \geq 1}$ is uniformly bounded and equicontinuous on [0, 1], so by the Arzela-Ascoli Theorem, we may pick a subsequence $\{\sigma_1(j)\}_{j \geq 1}$ of $\{j\}_{j \geq 1}$ with the property that $\{\theta_{\sigma_1(j)}(t)\}_{j \geq 1}$ converges to a continuous function $\kappa_1(t)$, uniformly on [0, 1]. Now we consider $\{\theta_{\sigma_1(j)}(t)\}_{j \geq 1}$ as a sequence of functions defined on [1, 2]. Using the same argument as above, one proves that there exists a subsequence $\{\sigma_2(j)\}_{j \geq 1}$ of $\{\sigma_1(j)\}_{j \geq 1}$ such that $\{\theta_{\sigma_2(j)}(t)\}_{j \geq 1}$ converges uniformly to a function $\kappa_2(t)$ for $t \in [1, 2]$. Since $\{\sigma_2(j)\}$ is a subsequence of $\{\sigma_1(j)\}$, it follows that $\kappa_2(1) = \kappa_1(1)$. Repeating the above procedure, one obtains inductively on $k \geq 1$ a subsequence $\{\sigma_{k+1}(j)\}_{j \geq 1}$ of $\{\sigma_k(j)\}_{j \geq 1}$ such that the sequence $\{\theta_{\sigma_{k+1}(j)}(t)\}_{j \geq 1}$ converges uniformly to a continuous function $\kappa_{k+1}$ on $[k,\, k+1]$. Clearly, $\kappa_k(k) = \kappa_{k+1}(k)$ for all $k \geq 1$. Let $\kappa$ be the continuous function defined by $\kappa(t) = \kappa_k(t)$ for $t \in [k-1,\, k)$ for each $k \geq 1$. Then on each interval $[k-1,\, k]$, $\kappa(t)$ is the uniform limit of $\{\theta_{\sigma_k(j)}(t)\}$.

Since the complement of $\Omega$ is closed and the $\theta_j$'s have images there, it is clear that $\kappa$ remains outside $\Omega$, and hence outside $K$. If $\kappa$ would be a trajectory of the system corresponding to some control $v$, the result would be proved (with $q = p_0$). The difficulty lies, of course, in the fact that there is no reason for $\kappa$ to be a trajectory. However, $\kappa$ can be well approximated by trajectories, and the rest of the proof consists of carrying out such an approximation.

Some more notations are needed. For each control $d$ with values in $\mathcal{O}$, we will denote by $\Delta d$ the control given by $\Delta d(t) = d(t+1)$ for each $t$ in the domain of $d$ (so, for instance, the domain of $\Delta d$ is $[-1, +\infty)$ if the domain of $d$ was $\mathbb{R}_{\geq 0}$). We will also consider iterates of the $\Delta$ operator, $\Delta^k d$, corresponding to a shift by $k$. Since $K$ is compact and $\Omega$ is open, we may pick an $r > 0$ such that $B(K,\, r) \subseteq \Omega$. We pick an $r_0$ smaller than $r/2$ and so that the closed ball of radius $r_0$ around $p_0$ is included in the neighborhood $W$ in which $q$ must be found. Finally, let $p_k = \kappa(k)$ for each $k \geq 1$. Observe that both $p_0$ and $p_1$ are in $S_1$ by construction.

Next, for each $j \geq 1$, we wish to study the trajectory $x(-t,\, p_1,\, \Delta d_{\sigma_1(j)})$ for $t \in [0, 1]$. This may be a priori undefined for all such $t$. However, since $S_1$ is compact, we may pick another compact set $\widetilde{S}_1$ containing $B(S_1, r)$ in its interior, and we may also pick a function $\widetilde{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ which is equal to $f$ for all $(x, u) \in \widetilde{S}_1 \times \mathcal{O}$ and has compact support; now the system $\dot{x} = \widetilde{f}(x, u)$ is complete, meaning that solutions exist for all $t \in (-\infty, \infty)$. We use $\widetilde{x}(t, \xi, u)$ to denote solutions of this new system. Observe that for each trajectory $\widetilde{x}(t, \xi, u)$ which remains in $\widetilde{S}_1$, $x(t, \xi, u)$ is also defined and coincides with $\widetilde{x}(t, \xi, u)$. In particular, $\widetilde{x}(-t,\, \theta_{\sigma_1(j)}(1),\, \Delta d_{\sigma_1(j)}) \;=\; x(-t,\, \theta_{\sigma_1(j)}(1),\, \Delta d_{\sigma_1(j)})$, for each $j$, since these both equal $x(1-t, p_0, d_{\sigma_1(j)})$, for each $t \in [0, 1]$. The set of states reached from $S_1$, using the modified system, in negative times $t \in [-1, 0]$, is included in some compact set (because the modified system is complete, and again appealing to Fact III.1). Thus, by Gronwall's estimate, there is some $L \geq 0$ so that, for all $j \geq 1$ and all $t \in [0, 1]$,

$$\begin{aligned}&\left|\widetilde{x}(-t,\, p_1,\, \Delta d_{\sigma_1(j)}) - x(-t,\, \theta_{\sigma_1(j)}(1),\, \Delta d_{\sigma_1(j)})\right| \\ &\quad \leq\; L\left|p_1 - \theta_{\sigma_1(j)}(1)\right| \,,\end{aligned}$$

(no "$\sim$" needed in the second solution, since it is also a solution of the original system). Since $\theta_{\sigma_1(j)}(1) \to p_1$, it follows that there exists some $j_1$ such that for all $j \geq j_1$,

$$\left|\widetilde{x}(-t,\, p_1,\, \Delta d_{\sigma_1(j)}) - x(-t,\, \theta_{\sigma_1(j)}(1),\, \Delta d_{\sigma_1(j)})\right| < \frac{r_0}{2} \tag{22}$$

for all $t \in [0,\, 1]$. Note that this means in particular that $\widetilde{x}(-t,\, p_1,\, \Delta d_{\sigma_1(j)}) \in B(S_1, r/4) \subseteq \widetilde{S}_1$ for all such $t$, for all $j \geq j_1$, so "$\sim$"can be dropped in Equation (22) for all $j \geq j_1$. Now let $0 < r_1 < r_0$ be such that

$$\left|\widetilde{x}(-t,\, p,\, \Delta d_{\sigma_1(j_1)}) - x(-t,\, p_1,\, \Delta d_{\sigma_1(j_1)})\right| \;<\; \frac{r_0}{2} \tag{23}$$

for all $t \in [0,\, 1]$, for all $p \in B(p_1,\, r_1)$. As this implies in particular that $\widetilde{x}(-t,\, p,\, \Delta d_{\sigma_1(j_1)}) \in B(S_1, r/2) \subseteq \widetilde{S}_1$, again tildes can be dropped. Combining (22) and (23), it follows that for each $p \in B(p_1,\, r_1)$, $x(-t,\, p,\, \Delta d_{\sigma_1(j_1)})$ is defined for all $t \in [0, 1]$ and

$$\left|x(-t,\, p,\, \Delta d_{\sigma_1(j_1)}) - x(-t,\, \theta_{\sigma_1(j_1)}(1),\, \Delta d_{\sigma_1(j_1)})\right| \;<\; r_0 \tag{24}$$

for all $t \in [0,\, 1]$. Let $w_1(t) = d_{\sigma_1(j_1)}(t)$. Then (24) implies that for each $p \in B(p_1,\, r_1)$ it holds that $x(-1,\, p_1,\, \Delta w_1) \in B(p_0,\, r_0)$, and, since $x(-t,\, \theta_{\sigma_1(j_1)},\, \Delta d_{\sigma_1(j_1)}) \notin \Omega$ for all $t \in [0,\, 1]$,

$$x(-t,\, p,\, \Delta w_1) \notin B(K,\, r/2) \qquad \forall\, t \in [0, 1].$$

In what follows we will prove, by induction, that for each $i \geq 1$, there exist $0 < r_i < r_{i-1}$ and $w_i$ of the form

$w_i = \Delta^{i-1} d_{\sigma_i(j)}$ for some $j$, so that, for all $p \in B(p_i, r_i)$, $x(-t, p, \Delta w_i)$ is defined for all $t \in [0, 1]$,

$$x(-t, p, \Delta w_i) \notin B(K, r/2) \qquad \forall\, t \in [0, 1],$$

and

$$x(-1, p, \Delta w_i) \in B(p_{i-1}, r_{i-1}).$$

The case $i = 1$ has already been shown in the above argument.

Assume now that the above conclusion is true for $i = 1, 2, \ldots, k$. Consider, for each $j$, the trajectory $x(-t, p_{k+1}, \Delta^{k+1} d_{\sigma_{k+1}(j)})$. Again, this may be a priori undefined for all such $t$. But, again, by modifying the system in a compact set $\tilde{S}_{k+1}$ containing a neighborhood of $\mathcal{R}_{\mathcal{O}}^{k+1}(p_0)$, one can show that there exists some $\bar{j}_{k+1} \geq k+1$ so that, for all $j \geq \bar{j}_{k+1}$, $x(-t, p_{k+1}, \Delta^{k+1} d_{\sigma_{k+1}(j)})$ is defined for all $t \in [0, 1]$, and

$$\begin{aligned} &|x(-t, p_{k+1}, \Delta^{k+1} d_{\sigma_{k+1}(j)}) \\ &\quad - x(-t, \theta_{\sigma_{k+1}(j)}(k+1), \Delta^{k+1} d_{\sigma_{k+1}(j)})| < r_k/4 \end{aligned} \tag{25}$$

for all $0 \leq t \leq 1$. We note here that, for any $j$, $x(-t, \theta_{\sigma_{k+1}(j)}, \Delta^{k+1} d_{\sigma_{k+1}(j_{k+1})})$ is defined for all $t \in [0, k+1]$ and

$$x(-1, \theta_{\sigma_{k+1}(j)}(k+1), \Delta^{k+1} d_{\sigma_{k+1}(j)}) = \theta_{\sigma_{k+1}(j)}(k).$$

Since $\theta_{\sigma_k(j)}(k) \to p_k$ as $j \to \infty$, and $\{\sigma_{k+1}(j)\}_{j \geq 1}$ is a subsequence of $\{\sigma_k(j)\}_{j \geq 1}$, it follows that there exists some $\tilde{j}_{k+1} \geq 0$ such that

$$\left|x(t, \theta_{\sigma_{k+1}(j)}(k), \Delta^k d_{\sigma_k(j)}) - x(t, p_k, \Delta^k d_{\sigma_k(j)})\right| < \frac{r_k}{4} \tag{26}$$

for all $t \in [0, 1]$, for all $j \geq \tilde{j}_{k+1}$. Let $j_{k+1} = \max\{\bar{j}_{k+1}, \tilde{j}_{k+1}\}$, and let $w_{k+1}(t) = \Delta^k d_{\sigma_{k+1}(j_{k+1})}(t)$ for $t \in [0, 1]$. Then, (25), applied with $j = j_{k+1}$, says that:

$$\begin{aligned} &|x(-t, p_{k+1}, \Delta w_{k+1}) \\ &\quad - x(-t, \theta_{\sigma_{k+1}(j_{k+1})}(k+1), \Delta w_{k+1})| < r_k/4 \end{aligned} \tag{27}$$

for all $t \in [0, 1]$. Using the case $t = 1$ of this equation, as well as (26) applied with $j = j_{k+1}$ and $t = 0$, and the equality $x(-1, \theta_{\sigma_{k+1}(j_{k+1})}(k+1), \Delta w_{k+1}) = \theta_{\sigma_{k+1}(j_{k+1})}(k)$, we conclude that:

$$x(-1, p_{k+1}, \Delta w_{k+1}) \in B(p_k, r_k/2). \tag{28}$$

Pick any $r_{k+1}$ so that $0 < r_{k+1} < r_k$ and for every $p \in B(p_{k+1}, r_{k+1})$, $x(-t, p, \Delta w_{k+1})$ is defined for all $t \in [0, 1]$, and

$$|x(-t, p, \Delta w_{k+1}) - x(-t, p_{k+1}, \Delta w_{k+1})| < r_k/2 \tag{29}$$

for all $t \in [0, 1]$. Then for such a choice of $r_{k+1}$, it holds, for each $p \in B(p_{k+1}, r_{k+1})$, by equation (27), that

$$\begin{aligned} &\left|x(-t, p, \Delta w_{k+1}) - x(-t, \theta_{\sigma_{k+1}(j_{k+1})}(k+1), \Delta w_{k+1})\right| \\ &< \frac{3r_k}{4} < r/2 \end{aligned}$$

for all $t \in [0, 1]$, which implies that, for each $p \in B(p_{k+1}, r_{k+1})$,

$$x(-t, p, \Delta w_{k+1}) \notin B(K, r/2) \qquad \forall\, t \in [0, 1].$$

Moreover, it follows from (28) and (29) that

$$x(-1, p, \Delta w_{k+1}) \in B(p_k, r_k)$$

for all $p \in B(p_{k+1}, r_{k+1})$. This completes the induction step.

Finally, we define a control $v$ on $\mathbb{R}_{\geq 0}$ as follows:

$$v(t) \;=\; w_k(t - k + 1) \quad \text{if } \; t \in [k-1, k)$$

for each integer $k \geq 1$. Then $v \in \mathcal{M}_{\mathcal{O}}$.

For each $k$, we consider the trajectory $x(-t, p_k, \Delta^k v)$ for $t \in [0, k]$. Inductively, $x(-i, p_k, \Delta^k v) \in B(p_{k-i}, r_{k-i})$ for each $i \leq k$, so this trajectory is indeed well-defined. We now let $z_k = x(-k, p_k, \Delta^k v)$. Note that also from $x(-i, p_k, \Delta^k v) \in B(p_{k-i}, r_{k-i})$ we can conclude that

$$x(-t, p_k, \Delta^k v) \notin B(K, r/2) \qquad \forall t \in [0, k], \tag{30}$$

which is equivalent to

$$x(t, z_k, v) \notin B(K, r/2) \qquad \forall t \in [0, k]. \tag{31}$$

As $z_k \in B(p_0, r_0)$ for all $k$, there exists a subsequence of $\{z_k\}$ that converges to some point $q \in W$. To simplify notation, we still use $\{z_k\}$ to denote one such convergent subsequence. We will finish our proof by showing that $x(t, q, v) \notin K$ for all $t \geq 0$.

Fix any integer $N > 0$. Since the system is forward-complete, and using uniform Lipschitz continuity of solutions as a function of initial states in $B(p_0, r_0)$ and using the control $v$ on the interval $[0, N]$, we know that there exists some $L_1 > 0$ such that

$$|x(t, q, v) - x(t, z_k, v)| \leq L_1 \,|q - z_k|\,, \quad \forall\, t \in [0, N]\,, \forall\, k\,.$$

Hence, there exists some $k_0 > 0$ such that

$$|x(t, q, v) - x(t, z_{k_0}, v)| < \frac{r}{4}\,, \qquad \forall\, t \in [0, N]. \tag{32}$$

Without loss of generality, we assume that $k_0 \geq N$. Combining (32) with (31), it follows that $x(t, q, v) \notin B(K, \frac{r}{4})$ for all $t \in [0, N]$. As $N$ was arbitrary, it follows that $x(t, q, v) \notin K$ for all $t \geq 0$. ■

The contrapositive of Lemma III.2 gives that, if from each point we can reach $\Omega$ in finite time, then the set $K$ can be reached in uniform time from each state. This can in turn be made stronger to provide in addition uniformity on compacts for initial states, as follows.

*Corollary III.3:* Let (1) be a forward-complete system. Assume given

- a compact subset $C$ of the state-space $\mathbb{R}^n$,
- an open subset $\Omega$ of the state-space $\mathbb{R}^n$,
- a compact subset $K \subset \Omega$, and
- a bounded subset $\mathcal{O}$ of the input-value space $\mathbb{U}$,

so that

$$\forall \xi \in \mathbb{R}^n \;\; \forall u \in \mathcal{M}_\mathcal{O} \;\; \exists t \geq 0 \quad \text{s.t. } \; x(t,\xi,u) \in K \; .$$

Then

$$\sup \{\tau(p,\Omega,u) \mid p \in C, u \in \mathcal{M}_\mathcal{O}\} \; < \; +\infty \; .$$

*Proof:* Pick an open set $\Omega_0$ and an $\varepsilon > 0$ so that $K \subset \Omega_0 \subset B(\Omega_0,\varepsilon) \subseteq \Omega$. By Lemma III.2, applied with $\Omega_0$ in place of $\Omega$ and $W = \mathbb{R}^n$, we know that for each $p \in C$ there is some $T = T_p$ so that, for each $u \in \mathcal{M}_\mathcal{O}$ there is some $t \in [0,T]$ so that $x(t,p,u) \in \Omega_0$. (Otherwise, we would have $\sup_{u\in\mathcal{M}_\mathcal{O}} \tau(p,\Omega_0,u) = +\infty$, and thus by the Lemma there is some $q \in W = \mathbb{R}^n$ and some $v \in \mathcal{M}_\mathcal{O}$ such that $\tau(q,K,v) = +\infty$, that is so that $x(t,q,v) \notin K$ for all $t \geq 0$, contradicting the assumption.) By Fact III.1 and Gronwall's Lemma, there is also some $L = L_p > 0$ ($L_p$ is obtained from the Lipschitz constant for $f$ on $\mathcal{R}^T_\mathcal{O}(C)$) so that

$$|x(t,\xi,u) - x(t,p,u)| \; \leq \; L\,|\xi - p|$$

for all $t \in [0,T]$ and all $\xi \in C$. Denote $\delta_p := L_p/\varepsilon$, pick a finite cover of $C$ by sets of the form $B(p,\delta_p)$, and let $T$ be the largest of the $T_p$'s in this cover. Now given any $\xi \in C$ and any $u \in \mathcal{M}_\mathcal{O}$, choose $p$ so that $\xi \in B(p,\delta_p)$ and $t \in [0,T_p]$ so that $x(t,p,u) \in \Omega_0$; then also $x(t,\xi,u) \in \Omega$, and $t \leq T$. ■

*Corollary III.4:* Under the hypotheses of Corollary III.3, and if in addition $\Omega \subset C$, then there is some $T \geq 0$ so that

$$\mathcal{R}_\mathcal{O}(C) = \mathcal{R}^T_\mathcal{O}(C)\,,$$

and in particular $\mathcal{R}_\mathcal{O}(C)$ is bounded.

*Proof:* Let $T = \sup\{\tau(p,\Omega,u) \mid p \in C, u \in \mathcal{M}_\mathcal{O}\}$. Pick any $y \in \mathcal{R}_\mathcal{O}(C)$; thus $y = x(t_0,\xi,u)$, for some $\xi \in C$, $u \in \mathcal{M}_\mathcal{O}$, and $t_0 \geq 0$. Let $t_1 := \max\{t \leq t_0 \,|\, x(t,\xi,u) \in C\}$, and $p := x(t_1,\xi,u)$. By maximality of $t_1$, $x(t,p,v) \notin C$ for any $t \in [0,t_0-t_1]$, where $v(t) = v(t+t_1)$. Thus, from the definition of $T$, $t_0-t_1 < T$. So $y = x(t_0-t_1,p,v) \in \mathcal{R}^T_\mathcal{O}(C)$. ■

As a simple application, we now prove Lemma I.5: if $\Sigma$ is forward-complete and satisfies the limit property with respect to $\mathcal{A}$, then it satisfies the boundedness property with respect to $\mathcal{A}$.

*Proof:* Assume without loss of generality that $\gamma \in \mathcal{K}_\infty$. For each $s > 0$, define

$$\sigma_1(s) := \sup\{|x(t,\xi,u)|_\mathcal{A} \mid \xi \in \overline{B}(\mathcal{A},2s), u \in \mathcal{M}_\mathcal{O}, t \geq 0, \mathcal{O} = \overline{B}(0,\gamma^{-1}(s))\} \;\; .$$

We claim that $\sigma_1(s) < \infty$ for all $s > 0$. To see this, pick an $s > 0$ and let $C = \overline{B}(\mathcal{A},2s)$, $\Omega = B(\mathcal{A},2s)$, and $K = \overline{B}(\mathcal{A},3s/2)$. Pick any $u \in \mathcal{M}_\mathcal{O}$, $\mathcal{O} = \overline{B}(0,\gamma^{-1}(s))$. Note that $\sigma_1(s)$ is the largest possible value of $|\nu|_\mathcal{A}$, for $\nu \in \mathcal{R}_\mathcal{O}(C)$. Property (LIM) implies that, for each $\xi \in \mathbb{R}^n$, there is some $t \geq 0$ so that

$$|x(t,\xi,u)|_\mathcal{A} \; \leq \; (3/2)\gamma\left(\|u\|_\infty\right) \; \leq \; (3/2)\gamma(\gamma^{-1}(s)) = 3s/2\,,$$

that is, $x(t,\xi,u) \in K$. We may then apply Corollary III.4, to conclude that $\mathcal{R}_\mathcal{O}(C)$ is bounded, so $\sigma_1(s)$ is indeed finite. We also define, for each $r > 0$, the (finite) number:

$$\sigma_2(r) \; := \; \sigma_1(\gamma(r))\;.$$

Finally, we define $\sigma_i(0) := 0$ for $i = 1,2$. Note that both $\sigma_1$ and $\sigma_2$ are nondecreasing functions. Assume without loss of generality that both $\sigma_i \in \mathcal{N}$ (if this does not hold, one may pick a larger $\widetilde{\sigma}_i \geq \sigma_i$). We next show that (BND) holds with these definitions.

Pick $\xi \in \mathbb{R}^n$ and a control $u$, and let $s := |\xi|_\mathcal{A}$ and $r := \|u\|_\infty$. Let $x(t) = x(t,\xi,u)$ for all $t \geq 0$. If $s{=}r{=}0$ then $\xi \in \mathcal{A}$, so zero-invariance of $\mathcal{A}$ means that the right hand side in the estimate (BND) vanishes. So assume $s > 0$ or $r > 0$. Consider two cases:

**Case 1:** Assume that $s \geq \gamma(r)$. Note that necessarily $s > 0$. We have a trajectory with $\xi \in \overline{B}(\mathcal{A},2s)$ and with $\|u\|_\infty = r \leq \gamma^{-1}(s)$, so by definition of $\sigma_1(s)$ it holds that $|x(t)|_\mathcal{A} \leq \sigma_1(s) = \sigma_1(|\xi|_\mathcal{A})$ for all $t$.

**Case 2:** Assume that $s \leq \gamma(r)$. Note that necessarily $r > 0$. Define $\widetilde{s} := \gamma(r)$. We have that $|\xi|_\mathcal{A} = s \leq \widetilde{s} < 2\widetilde{s}$ and that $\|u\|_\infty = r = \gamma^{-1}(\widetilde{s})$. Thus for all $t$ it holds that $|x(t)|_\mathcal{A} \leq \sigma_1(\widetilde{s}) = \sigma_1(\gamma(r)) = \sigma_2(\|u\|_\infty)$, as wanted. ■

## IV. Uniform Stability

The notions introduced next are motivated by thinking of inputs not as controls but as "time varying parameters" or "multiplicative uncertainties." Again we assume given a forward-complete system $\Sigma$ as in Equation (1), and a compact zero-invariant set $\mathcal{A}$ for this system.

The *global asymptotic stability property with respect to* $\mathcal{A}$ (GAS) holds if the system is uniformly stable with respect to $\mathcal{A}$:

$$\begin{array}{l}\forall\, \varepsilon > 0\; \exists\, \delta > 0 \quad \text{s.t.:} \quad \forall\; |\xi|_\mathcal{A} \leq \delta \\ \quad \sup\limits_{t\geq 0} |x(t,\xi,u)|_\mathcal{A} \; \leq \; \varepsilon \;\; \forall u(\cdot)\;. \end{array} \tag{33}$$

and is attractive with respect to $\mathcal{A}$:

$$\forall\, \xi \in \mathbb{R}^n \;\; \forall\, u(\cdot) \qquad \lim_{t\geq 0} |x(t,\xi,u)|_\mathcal{A} \; = \; 0\;. \tag{34}$$

Note that this last condition is just as in (AG) with $\gamma = 0$. Observe also that if (33) holds then the set $\mathcal{A}$ must be invariant in the strong sense that $x(t,\xi,u) \in \mathcal{A}$ whenever $\xi \in \mathcal{A}$, for all inputs $u$ (not just if $u \equiv 0$).

The *uniform global asymptotic stability property with respect to* $\mathcal{A}$ (UGAS) holds if the system is uniformly stable with respect to $\mathcal{A}$ and is *uniformly* attractive with respect to $\mathcal{A}$:

$$\begin{array}{c}\forall\, \varepsilon > 0\; \forall\, \kappa > 0\; \exists\, T = T(\varepsilon,\kappa) \geq 0 \quad \text{s.t.:} \\ |\xi|_\mathcal{A} \leq \kappa \;\; \Rightarrow \;\; \sup\limits_{t\geq T} |x(t,\xi,u)|_\mathcal{A} \; \leq \; \varepsilon \;\; \forall u(\cdot)\;. \end{array}$$

This is the same as asking that (UAG) hold with $\gamma = 0$. The main result in this Section is as follows.

*Theorem 2:* Assume given a forward-complete system $\Sigma$ as in Equation (1), and a compact zero-invariant set $\mathcal{A}$ for this system. Furthermore, assume that the set of control values $\mathbb{U}$ is compact. Then the system satisfies (GAS) if and only if it satisfies (UGAS).

*Proof:* Given $\varepsilon > 0$ and $\kappa > 0$ first find $\delta$ as in (33). Let $\mathcal{O} = \mathbb{U}$, $K = \overline{B}(\mathcal{A}, \delta/2)$, $\Omega = B(\mathcal{A}, \delta)$, and $C = \overline{B}(\mathcal{A}, \kappa)$. By (34), the hypotheses of Corollary III.3 hold, so there is some $T \geq 0$ such that, whenever $|\xi|_{\mathcal{A}} \leq \kappa$, and for each control $u$, there is some $t_0 = t_0(\xi, u) \leq T$ so that $|x(t_0, \xi, u)|_{\mathcal{A}} \leq \delta$. From the choice of $\delta$, it follows that $|x(t, \xi, u)|_{\mathcal{A}} \leq \varepsilon$ for all $t \geq t_0$, and hence for all $t \geq T$. ∎

## V. Proof of Main Proposition

We now prove Proposition I.6. We must show that (LIM) and (LS) together imply (UAG) (the converse is obvious). By Equation 13, we may assume that (GS) holds, so by Lemma I.4, (AG) is also true. So from now on we assume both (AG) and (GS). The proof is based on first introducing a new system — which appears also in an argument used in [15] — having compact input-value set and which satisfies (GAS), and then using the equivalence (UGAS) = (GAS) for this auxiliary system to conclude that the original system is (UAG). (An intuitive interpretation is that this allows to restrict attention to input values that are subject to a state-dependent constraint of the type $\gamma(|u(t)|) \leq |x(t)|_{\mathcal{A}}$; inputs not satisfying this constraint do not matter, since for them already $x(t)$ is bounded by a function of the input magnitude.)

We start with a general construction. Take any locally Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ which vanishes on the set $\mathcal{A}$. Consider the auxiliary system with the same state space $\mathbb{R}^n$, input-value set $\mathbb{U}_0$ equal to the closed unit ball $\overline{B}(0, 1)$ in $\mathbb{U} = \mathbb{R}^m$, and equations as follows:

$$(\Sigma_\varphi) : \qquad \dot{x} \;=\; f(x,\, \varphi(x)\, d) \;=\; f_\varphi(x, d)\,.$$

We use "$d(\cdot)$" to denote inputs to $(\Sigma_\varphi)$ in order to avoid confusion with inputs to the original system, and for each $\xi \in \mathbb{R}^n$ and each $d$, we use $x_\varphi(t, \xi, d)$ to denote the trajectory of $(\Sigma_\varphi)$ with initial state $\xi$ and input $d$. The system $(\Sigma_\varphi)$ may not be complete even if the original system $\Sigma$ is (example: $\dot{x} = u$ and $\varphi(x) = x^2$), but on the domain of definition of the solution one has that $x_\varphi(t, \xi, d) = x(t, \xi, u)$ where $u(t) := \varphi(x_\varphi(t, \xi, d))d(t)$. Note that, for $(\Sigma_\varphi)$, $\mathcal{A}$ is invariant in the strong sense that all trajectories starting in $\mathcal{A}$ remain there (since for each $\xi$ the solution of $\dot{x} = f(x, 0)$, $x(0) = \xi$, satisfies $\varphi(x(t)) \equiv 0$, by zero-invariance of $\mathcal{A}$, and hence also satisfies $\dot{x}(t) = f(x(t), \varphi(x(t))d(t))$ for all $d$).

Let $\sigma_1$, $\sigma_2$, and $\gamma$ be so that (GS) and (14) (already shown to be equivalent to (AG)) both hold. Without loss of generality, we assume that these are of class $\mathcal{K}_\infty$ and that $\gamma = \sigma_2$. Pick any smooth $\mathcal{K}_\infty$-function $\rho$ such that

$$\sigma_2(\rho(s)) \;\leq\; \frac{s}{2} \qquad \forall\, s \geq 0,$$

and let $\varphi(\xi) = \rho(|\xi|_{\mathcal{A}})$. We claim, with this choice of $\varphi$, the system $(\Sigma_\varphi)$ is complete and satisfies the global asymptotic stability property with respect to $\mathcal{A}$.

We first show that, for each $\xi \in \mathbb{R}^n$ and each input $d$, and denoting $x_\varphi(t, \xi, d)$ simply as $x_\varphi(t)$,

$$\sigma_2\left(\varphi(x_\varphi(t))\right) \;\leq\; \frac{2}{3}\sigma_1(|\xi|_{\mathcal{A}}) \tag{35}$$

for each $t$ in the maximal interval $[0, T_{\max})$ of definition of $x_\varphi$. Once this is established, it will follow that $T_{\max} = +\infty$ (completeness) and that every trajectory is bounded. If $\xi \in \mathcal{A}$, then (35) holds for all $t \geq 0$, since both sides vanish. So assume that $\xi \in \mathbb{R}^n \setminus \mathcal{A}$ and $d$ is an input. Denote $x_\varphi(t, \xi, d)$ simply as $x_\varphi(t)$. For each $t$ small enough, $\sigma_2(\varphi(x(t))) < (2/3)\sigma_1(|\xi|_{\mathcal{A}})$ since

$$\sigma_2(\varphi(x(0))) \;=\; \sigma_2(\rho(|\xi|_{\mathcal{A}})) \;\leq\; (1/2)\,|\xi|_{\mathcal{A}} \;\leq\; (1/2)\sigma_1(|\xi|_{\mathcal{A}})\,.$$

Now let

$$t_1 \;:=\; \inf\,\{t \in [0, T_{\max}) \mid \sigma_2(\varphi(x_\varphi(t))) \geq (2/3)\sigma_1(|\xi|_{\mathcal{A}})\}$$

with $t_1 := T_{\max}$ if the set is empty. Then Equation (35) holds on $[0, t_1)$. The restriction to the interval $[0, t_1)$ of $x_\phi$ is the same as the restriction to that interval of $x(t, \xi, u)$, where $u(t) = \varphi(x_\varphi(t))d(t)$ for $t \in [0, t_1)$ and $u(t) = 0$ for $t \geq t_1$. Note that $|u(t)| = |\varphi(x_\varphi(t))d(t)| \leq \varphi(x_\varphi(t))$ for $t \in [0, t_1)$, so $\sigma_2(\|u\|_\infty) \leq \sup_{t\in[0,t_1)} \sigma_2(\varphi(x_\varphi(t)))$, and (35) on $t \in [0, t_1)$ implies that $\sigma_2(\|u\|_\infty) \leq (2/3)\sigma_1(|\xi|_{\mathcal{A}})$. Together with (GS), we conclude that

$$|x_\varphi(t)|_{\mathcal{A}} \leq \sigma_1(|\xi|_{\mathcal{A}}) \tag{36}$$

whenever $t \in [0, t_1)$. From this it follows that

$$\sigma_2(\varphi(x_\varphi(t))) \leq \sigma_2(\rho(\sigma_1(|\xi|_{\mathcal{A}}))) \leq \frac{1}{2}\sigma_1(|\xi|_{\mathcal{A}})$$

for all $t \in [0, t_1)$. If $t_1 < T_{\max}$, then by continuity as $t \to t_1^-$, we conclude that $\sigma_2(\varphi(x_\varphi(t_1))) \leq \frac{1}{2}\sigma_1(|\xi|_{\mathcal{A}})$, which contradicts the definition of $t_1$. Thus $t_1 = T_{\max}$, and in particular (35) and (36) hold for all $t \geq 0$, and $T_{\max} = +\infty$. Equation (36) shows uniform stability, and also that $\overline{\lim}_{t\to\infty} |x_\varphi(t)|_{\mathcal{A}} < \infty$ for each trajectory. Attraction follows because, for any trajectory $x_\varphi$, and using the fact that, for all $\zeta$,

$$\sigma_2(|\varphi(\zeta)d(t)|) \;\leq\; \sigma_2(\varphi(\zeta)) \;=\; \sigma_2(\rho(|\zeta|_{\mathcal{A}})) \;<\; (1/2)\,|\zeta|_{\mathcal{A}}\,,$$

Property (14) implies

$$\begin{aligned}\overline{\lim_{t\to+\infty}}\, |x_\varphi(t)|_{\mathcal{A}} \;&\leq\; \overline{\lim_{t\to+\infty}}\, \sigma_2(|\varphi(x_\varphi(t))d(t)|) \\ &\leq\; (1/2)\, \overline{\lim_{t\to+\infty}}\, |x_\varphi(t, \xi, d)|_{\mathcal{A}}\,,\end{aligned}$$

from which it follows that

$$\lim_{t\to+\infty} |x_\varphi(t, \xi, d)|_{\mathcal{A}} = 0\,, \tag{37}$$

and so the system $(\Sigma_\varphi)$ is (GAS) as claimed.

It then follows from Theorem 2 that $(\Sigma_\varphi)$ is (UGAS). To complete the proof of Proposition I.6, we need to show that this implies that the original system $\Sigma$ is (UAG). There are at least two ways to prove this fact. The first is very short but uses a converse Lyapunov theorem; the second is self-contained but longer. Each proof is of interest, so we provide both.

By the main result in [9], $(\Sigma_\varphi)$ being (UGAS) implies that there exists a smooth, proper, positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ with the property that $\nabla V(\xi) f_\varphi(\xi, \nu) < 0$

for all $\xi \neq 0$ and all vectors $\nu \in \overline{B}(0,1)$ in $\mathbb{R}^m$. This means that $\nabla V(\xi) f(\xi,\mu) < 0$ whenever $\mu \in \mathbb{R}^m$ and $\rho(|\xi|_{\mathcal{A}}) \geq |\mu|$, which, since $\rho \in \mathcal{K}_\infty$, implies the uniform asymptotic gain property with respect to $\mathcal{A}$ (see e.g. [12]). Another proof, not using the existence of $V$, is as follows.

Pick any compact subset $C \subset \mathbb{R}^n$ and any $\varepsilon > 0$. Since $(\Sigma_\varphi)$ is (UGAS), there is some $T \geq 0$ so that

$$\sup_{\xi \in C, d} \sup_{t \geq T} |x_\varphi(t,\xi,d)|_{\mathcal{A}} \leq \varepsilon \,.$$

Pick any $\xi \in C$ and any control $u$. Consider $x(t) = x(t,\xi,u)$. Let

$$t_0 \;:=\; \inf\left\{t \;\middle|\; \left\|u^t\right\|_\infty \geq \rho(|x(t)|_{\mathcal{A}})\right\}$$

($t_0 = \infty$ if the set is empty), where $u^t$ denotes the restriction of $u$ to the interval $[t,\infty)$. Define

$$d(t) \;:=\; \frac{u(t)}{\rho(|x(t)|_{\mathcal{A}})}$$

for each $t \in [0,t_0)$ and $d(t) := 0$ for $t \geq t_0$. Note that $\|d\|_\infty \leq 1$, i.e. $d$ is an input for the system $(\Sigma_\varphi)$. Thus

$$\begin{aligned} \dot{x}(t) &= f(x(t),u(t)) \\ &= f(x(t),d(t)\varphi(x(t))) = f_\varphi(x(t),d(t))\,, \end{aligned}$$

so $x(t) = x_\varphi(t,\xi,d)$ for all $0 \leq t < t_0$. From the choice of $T$, it follows that $|x_\varphi(t,\xi,d)|_{\mathcal{A}} \leq \varepsilon$ for all $t \geq T$. Thus either

$$t_0 \leq T \tag{38}$$

or

$$|x(t)|_{\mathcal{A}} \leq \varepsilon \text{ for all } t \in [T,t_0)\,. \tag{39}$$

If $t_0 = \infty$ then this means that $|x(t)|_{\mathcal{A}} \leq \varepsilon$ for all $t \geq T$. Otherwise, there is some sequence $t_n \to t_0^+$ such that

$$\|v\|_\infty \geq \left\|u^{t_n}\right\|_\infty \geq \rho(|x(t_n)|_{\mathcal{A}})$$

by definition of $t_0$, where we denote $v := u^t$. Therefore $\rho(|x(t_0)|_{\mathcal{A}}) \leq \|v\|_\infty$ as well, by continuity of $|x(t)|_{\mathcal{A}}$. So, for each $t \geq t_0$, using (GS),

$$\begin{aligned} |x(t)|_{\mathcal{A}} &\leq \max\{\sigma_1(|x(t_0)|_{\mathcal{A}}), \sigma_2(\|v\|_\infty)\} \\ &\leq \max\left\{\left(\sigma_1 \circ \rho^{-1}\right)(\|v\|_\infty), \sigma_2(\|v\|_\infty)\right\} \\ &= \alpha(\|v\|_\infty) \leq \alpha(\|u\|_\infty) \end{aligned}$$

where we defined

$$\alpha(s) \;:=\; \max\{(\sigma_1 \circ \rho^{-1})(s), \sigma_2(s)\}\,.$$

We conclude that, for each $t \geq T$: in case (38), since $t \geq t_0$ then $|x(t)|_{\mathcal{A}} \leq \alpha(\|u\|_\infty)$, and in case (39) we have that $|x(t)|_{\mathcal{A}} \leq \varepsilon$ when $t \in [T,t_0)$ and $|x(t)|_{\mathcal{A}} \leq \alpha(\|u\|_\infty)$ when $t \geq t_0$. In either case, $t \geq T$ implies

$$|x(t)|_{\mathcal{A}} \;\leq\; \max\{\varepsilon, \alpha(\|u\|_\infty)\} \;\leq\; \alpha(\|u\|_\infty) + \varepsilon\,,$$

completing the proof of condition (UAG) (with "$\gamma$" function $\alpha$).

## VI. A Remark on "Practical" ISS

Assume given a forward-complete system $\Sigma$ as in Equation (1), with $\mathbb{U} = \mathbb{R}^m$. In the paper [6], the system $\Sigma$ is said to be "input-to-state practically stable" (ISpS) if there exist a $\mathcal{KL}$-function $\beta$, a $\mathcal{K}$-function $\gamma$ and a constant $c \geq 0$ such that

$$|x(t,\,\xi,\,u)| \;\leq\; \beta(|\xi|\,,\,t) \,+\, \gamma(\|u\|_\infty) \,+\, c \tag{40}$$

holds for each control $u$ and each $\xi \in \mathbb{R}^n$. Compared with the definition of the ISS property, the difference is in the possibly nonzero constant $c$. In this section, we remark that this property can be rephrased in terms of the plain ISS property.

*Definition VI.1:* The system $\Sigma$ has the *compact ISS property* if there is some compact zero-invariant set $\mathcal{A}$ such that $\Sigma$ has the uniform asymptotic gain property with respect to $\mathcal{A}$. □

For any subset $\mathcal{A} \subseteq \mathbb{R}^n$, not necessarily zero-invariant, we may consider the zero-input orbit from $\mathcal{A}$:

$$O(\mathcal{A}) \;:=\; \{\eta :\; \eta = x(t,\,\xi,\,0),\; t \geq 0, \xi \in \mathcal{A}\}$$

and let $\mathbf{O}(\mathcal{A})$ be the closure of $O(\mathcal{A})$.

*Lemma VI.2:* If property (UAG) holds for a set $\mathcal{A}$, not necessarily zero-invariant, then $O(\mathcal{A})$ is bounded and the system is ISS with respect to $\mathbf{O}(\mathcal{A})$.

*Proof:* Since $\mathbf{O}(\mathcal{A})$ includes $\mathcal{A}$, the system satisfies (UAG) with respect to $\mathbf{O}(\mathcal{A})$, and the latter set is 0-invariant (since $O(\mathcal{A})$ is). Thus we only need to prove that $\mathbf{O}(\mathcal{A})$ is compact, or equivalently, that $O(\mathcal{A})$ is bounded. From property (UAG), we know that there exists some $T \geq 0$ so that, for each $\xi \in \mathcal{A}$, $|x(t,\xi,0)|_{\mathcal{A}} \leq 1$ for all $t \geq T$. By continuity at states in $\mathcal{A}$ of solutions of the system with $u \equiv 0$ with respect to initial conditions, and compactness of $\mathcal{A}$, there is some constant $c$ so that $|x(t,\xi,0)|_{\mathcal{A}} \leq c$ for all $t \in [0,T]$. Thus $O(\mathcal{A})$ is included in a ball of radius $\max\{1,c\}$. ■

*Proposition VI.3:* The system $\Sigma$ is ISpS if and only if it is compact-ISS.

*Proof:* If Equation (40) holds, then the property (UAG) holds with respect to the set $\mathcal{A} := \overline{B}(0,c)$, and hence by Lemma VI.2, the system is ISS with respect to $\mathbf{O}(\mathcal{A})$. Conversely, assume that $\Sigma$ satisfies the input to state stability property with respect to some compact (zero-invariant) set $\mathcal{A}$. Without loss of generality, we assume that $0 \in \mathcal{A}$ (otherwise, we enlarge $\mathcal{A}$; zero-invariance is not required). Then

$$\begin{aligned} |x(t,\xi,u)| \;&\leq\; |x(t,\xi,u)|_{\mathcal{A}} + c \;\leq\; \beta(|\xi|_{\mathcal{A}}\,,t) \\ &+\; \gamma(\|u\|_\infty) + c \;\leq\; \beta(|\xi|\,,t) + \gamma(\|u\|_\infty) + c \end{aligned}$$

with $c := \sup\{|\xi|\,, \xi \in \mathcal{A}\}$. ■

As a consequence of the equivalence between these concepts, one may apply the general theory of ISS systems to the study of the ISpS property. As a simple illustration, we show next that the latter property can be characterized in terms of Lyapunov functions.

*Proposition VI.4:* The system $\Sigma$ is compact-ISS if and only if there are a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2, 3$ and $\sigma \in \mathcal{N}$ so that

$$\alpha_1(|\xi|) \;\leq\; V(\xi) \;\leq\; \alpha_2(|\xi|)$$

and

$$\nabla V(\xi) f(\xi, v) \;\leq\; \sigma(|v|) - \alpha_3(|\xi|)\,. \tag{41}$$

*Proof:* The sufficiency part is routine: If $V$ is as stated, then $\nabla V(\xi)f(\xi,v) \leq -\alpha_3(|\xi|)/2$ whenever $|\xi| \geq \chi(|v|)$, where $\chi(r) = \alpha_3^{-1} \circ 2\sigma(r)$. Observe that $\chi$ is a nondecreasing function. Using exactly the same arguments as used on page 441 of [12], one can then show that there exist some $\mathcal{KL}$-function $\beta$ and some nondecreasing function $\gamma$ such that

$$|x(t, \xi, u)| \;\leq\; \beta(|\xi|, t) + \gamma(\|u\|_\infty)$$

for each input $u$ and each initial state $\xi$, and it follows from here that the system is indeed compact-ISS.

To prove the converse, namely the existence of such a function $V$, we first appeal to the converse Lyapunov result in [9]. Assuming that the uniform asymptotic gain property with respect to $\mathcal{A}$ holds, there is some smooth $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and there are $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2, 3$ and $\sigma \in \mathcal{K}$ so that both Equations (2) and (3) hold. As earlier, we may assume without loss of generality that $\mathcal{A}$ contains the origin. Thus the case $\widetilde{\mathcal{A}} = \{0\}$ of the following claim then provides the desired conclusion:

**Claim:** Let $\widetilde{\mathcal{A}}$ be any nonempty compact subset of $\mathcal{A}$. Then there is a smooth function $\widetilde{V} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ which satisfies the following conditions:

- $\widetilde{V}$ is proper and positive definite with respect to the set $\widetilde{\mathcal{A}}$, that is, there exist $\widetilde{\alpha}_1, \widetilde{\alpha}_2 \in \mathcal{K}_\infty$ such that for all $\xi \in \mathbb{R}^n$,

$$\widetilde{\alpha}_1(|\xi|_{\widetilde{\mathcal{A}}}) \leq \widetilde{V}(\xi) \leq \widetilde{\alpha}_2(|\xi|_{\widetilde{\mathcal{A}}})\,, \tag{42}$$

- there exist a function $\widetilde{\alpha}_3 \in \mathcal{K}_\infty$ and a nondecreasing continuous function $\widetilde{\sigma}$ such that

$$\nabla \widetilde{V}(\xi) f(\xi, v) \;\leq\; \widetilde{\sigma}(|v|) - \widetilde{\alpha}_3(|\xi|_{\widetilde{\mathcal{A}}}) \tag{43}$$

  for all $\xi \in \mathbb{R}^n$ and for all $v \in \mathbb{R}^m$.

Indeed, consider the set $\mathcal{A}_1 = \{\xi : \; |\xi|_{\mathcal{A}} \geq 1\}$. Since this is disjoint from $\mathcal{A}$, there is a smooth function $\varphi : \mathbb{R}^n \to [0, 1]$ so that

$$\varphi(\xi) = \begin{cases} 0\,, & \text{if } \xi \in \mathcal{A}, \\ 1\,, & \text{if } \xi \in \mathcal{A}_1. \end{cases}$$

Similarly, there is some smooth, nonnegative function $\lambda$ defined on $\mathbb{R}^n$ which vanishes exactly on $\widetilde{\mathcal{A}}$. Now we define

$$\widetilde{V}(\xi) \;:=\; \lambda(\xi)(1 - \varphi(\xi)) \;+\; V(\xi)\varphi(\xi)\,.$$

By construction, $\widetilde{V}$ is smooth and is proper and positive definite with respect to $\widetilde{\mathcal{A}}$, that is, there are comparison functions as in (42). Furthermore, since $V(\xi) = \widetilde{V}(\xi)$ for $|\xi|_{\mathcal{A}} > 1$, also

$$\nabla \widetilde{V}(\xi)\, f(\xi, v) \leq -\alpha_3(|\xi|_{\mathcal{A}}) + \gamma(|v|)\,,$$

(where $\alpha_3$ and $\gamma$ are the comparison functions associated to $V$) for all $v \in \mathbb{R}^m$ and all $|\xi|_{\mathcal{A}} > 1$. Since both $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ are compact, there exists some $s_0 \geq 0$ such that $|\xi|_{\widetilde{\mathcal{A}}} \leq |\xi|_{\mathcal{A}} + s_0$; thus

$$\nabla \widetilde{V}(\xi)\, f(\xi, v) \leq -\tilde{\alpha}_3(|\xi|_{\widetilde{\mathcal{A}}}) + \gamma(|v|), \tag{44}$$

whenever $|\xi|_{\widetilde{\mathcal{A}}} \geq 1 + s_0$, where $\tilde{\alpha}_3$ is any $\mathcal{K}_\infty$ function which satisfies $\tilde{\alpha}_3(r) \leq \alpha_3(r - s_0)$ for all $r \geq s_0 + 1$. The proof of the claim, and hence the Proposition, is completed by taking any nondecreasing continuous function $\widetilde{\sigma}$ which majorizes both $\gamma(r)$ and the maximum of $\nabla \widetilde{V}(\xi)\, f(\xi, v)$ over all $|\xi|_{\widetilde{\mathcal{A}}} \leq 1 + s_0, |v| \leq r$. ■

Note that this result, as opposed to the Lyapunov characterization of the ISS property with respect to $\mathcal{A} = \{0\}$, does not require $\sigma \in \mathcal{K}$. However, one may always write $\sigma \leq c + \widetilde{\sigma}$, for some class-$\mathcal{K}$ function $\widetilde{\sigma}$ and some positive constant $c$.

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